Review

Definition ( ): Consider a random experiment whose sample space is S. A random variable X is a function from the sample space S into the set of real numbers $\mathbb{R}$ such that for each interval I in $\mathbb{R}$, the set $\{s \in S \mid X(s) \in I\}$ is an event in S.

متقطع

There are three types of random variables: discrete, continuous, and mixed.
مستمر مختلط

هناك ثلاث انواع من المتغيرات العشوائية

Definition (1):

If the space of r.v. X is countable, then X is called a discrete r.v.
متقطع

Definition (2):

Let R_X be the space of r.v. X. The function:

$$f: R_X \to \mathbb{R} \text{ defined by}$$

$$p(x) = P(X = x)$$

is called the probability mass function of X (pmf).

Theorem (1) If X is a discrete r.v. with space R_X and pmf $p(x)$, then:

(a) $p(x) \geq 0$ for all x in R_X, and

(b) $\sum_{x \in R_X} p(x) = 1$

c d f

Definition (3) : The cumulative distribution function $F(x)$ of a r.v. X is defined as

$$F(x) = P(X \leq x)$$

Theorem (2) : If X is a r.v. with the space R_X, then

$$F(x) = \sum_{u \leq x} f(u)$$

for $x \in R_X$

$$= \int^{x} f(u)\, du$$

Theorem (3): Let X be a r.v. with c.d.f. $F(x)$. Then the c.d.f. satifies the followings:

(a) $F(-\infty) = 0$

(b) $F(\infty) = 1$ تقرأ اسير

(c) $F(x)$ is an increasing function, that is if $x < y$, then $F(x) \le F(y)$ for all reals x, y.

Theorem (4): If the space R_X of the r.v. X is given by $R_X = \{x_1 < x_2 < \cdots < x_n\}$, then:

$$p(x_1) = F(x_1)$$ ايجاد p.m.f

$$p(x_2) = F(x_2) - F(x_1)$$ من بيانات = متغير من c.d.f

$$\vdots$$

$$p(x_n) = F(x_n) - F(x_{n-1}).$$ نحصل على p.m.f

Definition (4): A r.v. X is said to be continuous r.v. X if there exists a continuous function $f : \mathbb{R} \to [0, \infty)$ such that for every set of real numbers A

$$P(X \in A) = \int_A f(x)\,dx \quad \ldots (1)$$

Definition (5) The function f in (1) is called the probability density function of the continuous r.v. X, and

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$ شرط التحقق من p.d.f

Definition (6): Let $f(x)$ be the probability density function (pdf) of a continuous r.v. X. The c.d.f. $F(x)$ of X is defined by:

$$F(x) = Pr(X \le x) = \int_{-\infty}^{x} f(u)\,du$$ ايجاد c.d.f

Theorem (5): If $F(x)$ is the c.d.f. of continuous r.v. X, the pdf $[f(x)]$ of X is the derivative of $F(x)$, that is

$$f(x) = \frac{d}{dx} F(x)$$

نشتق الـ c.d.f

نحصل على ~~f~~ p.d.f

Theorem (6) Let X be a continuous r.v. whose c.d.f. is $F(x)$. The the following are true.

(a) $P(X \leq x) = F(x)$

(b) $P(X > x) = 1 - F(x)$

(c) $P(X = x) = 0$, and

(d) $P(a < X < b) = F(b) - F(a)$.

خصائص

الـ c.d.f

Definition (7) Let P be a real number between 0 and 1. A $100p^{th}$ percentile of the distribution of a r.v. X is any real number q satisfying:

$$P(X \leq q) \leq p \text{ and } P(X > q) \leq 1 - p.$$

A $100p^{th}$ percentile is a measure of location for the probability distribution in the sense that (q) divides the distribution of the probability mass into two parts, one having probability mass (p) and other having probability mass $(1-p)$.

Example (1) If the r.v. X has the density function:

$-\infty < x < 2$

$$f(x) = \begin{cases} e^{x-2} & , x < 2 \\ 0 & , \text{o.w} \end{cases}$$

75%

تعني

then, what is the 75 th percentile of X?

75% لا

Solution: since $100p^{th} = 75$, we get $p = 0.75$.

then:

$$0.75 = \int_{-\infty}^{q} f(x)\,dx = P$$

$$\Rightarrow \quad = \int_{-\infty}^{q} e^{x-2}\,dx = \left[e^{x-2} \right]_{-\infty}^{q} = e^{q-2}$$

$$0.75 = e^{q-2}$$

$$\text{Ln } 0.75 = q - 2$$

$$\Rightarrow \quad q = 2 + \text{Ln}\left(\tfrac{3}{4}\right)$$

Example (2) what is the 87.5 percentile for the distribution with density function:

$$f(x) = \tfrac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$$

Sol: Note that this density function is symmetric about the y-axis, that is $f(x) = f(-x)$.

$$\frac{87.5}{100} = \int_{-\infty}^{q} f(x)\,dx \qquad |x| = \mp x$$

$$= \int_{-\infty}^{0} \tfrac{1}{2} e^{-|x|}\,dx + \int_{0}^{q} \tfrac{1}{2} e^{-|x|}\,dx$$

$$= \int_{-\infty}^{0} \tfrac{1}{2} e^{x}\,dx + \int_{0}^{q} \tfrac{1}{2} e^{-x}\,dx$$

$$= \tfrac{1}{2} + \int_{0}^{q} \tfrac{1}{2} e^{-x}\,dx = \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{2} e^{-q}$$

$$0.125 = \tfrac{1}{2} e^{-q} \qquad 0.875 = 1 - \tfrac{1}{2} e^{-q}$$

$$\therefore \; q = -\text{Ln}\left(\tfrac{25}{100}\right) = \text{Ln } 4. \qquad \tfrac{1}{2} e^{-q} = 1 - 0.875$$

$$\tfrac{1}{2} e^{-q} = 0.125$$

05%

Definition (8) The 50th percentile of any dist. is called the <u>median</u> of the dist.

Definition (9) the 25th and 75th percentiles of any dist. are called the <u>first</u> and <u>third</u> quartiles respectively.

Definition (10): A mode of the dist. of continuous r.v. X is the value of x where the pdf attains a relative maximum.

Example (3) A c.r.v. has density function:

$$f(x) = \begin{cases} \dfrac{3x^2}{\theta^3} & , \quad 0 \le x \le \theta \quad , \theta > 0 \\ 0 & , \quad \text{o.w} \end{cases}$$

what is the ratio of the mode to the median for this distribution? بما أن الدالة أو الاقتران $\theta > 0$ الرقم الأصغر أو ...

Sol: For fixed $\theta > 0$ the $f(x)$ is an increasing function. Thus, $f(x)$ has maximum at the right end point of the interval $[0, \theta]$. Hence the mode of this dist. is θ. The median of this dist. is:

$$\int_0^q f(x)\, dx = \frac{1}{2}$$

$$\int_0^q \frac{3x^2}{\theta^3}\, dx = \frac{1}{2}$$

$$\therefore \quad q = 2^{-\frac{1}{3}} \theta$$

إذا كانت θ أكبر من الصفر وهي الأعلى للفترة ويكون بين $[0, \theta]$ إذن تكون الدالة أعلى شيء عند θ هو mode

$$\frac{3}{\theta^3}\left\{\frac{x^3}{3}\right\}_0^q = \frac{1}{2}$$

$$\frac{q^3}{\theta^3} = \frac{1}{2}$$

Then, $$\frac{\text{mode}}{\text{median}} = \frac{\theta}{2^{-1/3}\,\theta} = \sqrt[3]{2}.$$

$$\frac{q}{\theta} = \frac{1}{2^{\frac{1}{3}}} \rightarrow q = 2^{-\frac{1}{3}} \cdot \theta$$

Definition (11) The n^{th} moment about the origin of a r.v. X as denoted by $E(X^n)$, is defined to be:

العزوم

$$E(X^n) = \begin{cases} \sum\limits_{x \in R_X} x^n\, p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x^n f(x)\, dx & \text{if } X \text{ is Cont.} \end{cases}$$

العزوم حول نقطة الصفر

for $n = 0, 1, 2, 3, \ldots$ provide the right side converges absolutely.

Definition (12) Let X be a r.v. with space R_X.
The mean μ_X of the r.v. X is defined as:

$$\mu_X = \begin{cases} \sum_{x \in R_X} x\, p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x\, f(x)\, dx & \text{if } X \text{ is continuous} \end{cases}$$

Theorem (7) Let X be a r.v. with pdf (pmf). If a and b are any two real numbers, then:

$$E(aX + b) = a\,E(X) + b$$

Definition (13) Let X be a r.v. with mean μ_X. The variance of X, denoted by $Var(X)$ is defined by:

$$\begin{aligned} Var(X) &= E(|X - \mu_X|^2) \\ &= E(X^2) - (EX)^2 \\ &= E(X^2) - (\mu_X)^2 = \sigma_X^2 \end{aligned}$$

Theorem (8) If X is a r.v. with mean μ_X and variance σ_X^2, then:

$$var(aX + b) = a^2\, var(X).$$

Theorem (9): $\boxed{\text{Chebychev Inequality.}}$
Let X be a r.v. with probability density function $f(x)$. If μ and $\sigma > 0$ are the mean and Standard deviation of X, then

$$P(|X - \mu| \leq K\sigma) \geq 1 - \frac{1}{K^2}$$

for any nonzero real positive constant K.

Example (4) Let the pdf of a r.v. X be:

$$f(x) = \begin{cases} 630x^4(1-x)^4 & , 0 < x < 1 \\ 0 & , o.w \end{cases}$$

what is the exact value of $P(|X-\mu| \leq 2\sigma)$? what is the approximate value of $P(|X-\mu| \leq 2\sigma)$ when we uses the chebychev inequality?

Sol:

$$E(X) = \int_0^1 x f(x)\, dx$$

$$= \int_0^1 630 x^{5+1-1}(1-x)^{4+1-1} dx = 630 \cdot \frac{5!\,4!}{(5+4+1)!}$$

$$= \frac{1}{2}$$

$$630 \frac{(5\cdot4\cdot3\cdot2\cdot1)(4\cdot3\cdot2\cdot1)}{(10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1)} = \frac{1}{2}$$

$$630 \frac{\Gamma(5+1)\,\Gamma(4+1)}{\Gamma(5+4+2)}$$

توزيع بيتا

$$var(X) = \int_0^1 x^2 f(x)\, dx - \mu_x^2$$

$$= \int_0^1 630\, x^6 (1-x)^4 dx - \left(\frac{1}{2}\right)^2 = 630 \frac{(5+1-1)!\,(4+1-1)!}{(5+4+2-1)!}$$

$$= 630 \cdot \frac{6!\,4!}{(6+4+1)!} - \frac{1}{4}$$

$$= \frac{1}{44}$$

$$\therefore \sigma = \sqrt{\frac{1}{44}} = 0.15$$

Thus $P(|X-\mu| \leq 2\sigma) = P(|X - \frac{1}{2}| \leq 0.3)$

$$= P(-0.3 \leq X - 0.5 \leq 0.3) \quad +0.5$$

$$= P(0.2 < X < 0.8)$$

$$= \int_{0.2}^{0.8} 630 x^4 (1-x)^4 dx$$

$$= 0.96$$

نستخدمها في حالة التوزيع غير معروف

By chebychev inequality:

$$P(|X-\mu| \leq 2\sigma) \geq 1 - \frac{1}{4} = 0.75$$

Hence, chebychev inequality tells us that if we do not Know the dist. of X, then $P(|X-\mu| \leq 2\sigma)$ is at least 0.75.

$$\frac{1}{K^2} = \frac{1}{(2)^2} = \frac{1}{4}$$

Definition (14) Let X be a r.v. whose probability density function is $f(x)$. A real valued function $M: \mathbb{R} \to \mathbb{R}$ defined by:

$$M(t) = E(e^{tX})$$

is called the moment generating function (mgf) of the r.v. X.

using the definition of expected value of a r.v., one obtain the explicit representation for $M(t)$ as:

$$M(t) = \begin{cases} \sum_{x \in R_X} e^{tx} p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & \text{if } X \text{ is continuous.} \end{cases}$$

Theorem (10): Let $M(t)$ be the m.g.f. of the r.v. X if:

$$a_n = \frac{E(X^n)}{n!}$$

$$M(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n + \dots \quad \dots (2)$$

is the Tayler Series expansion of $M(t)$, then

$$E(X^n) = (n!)\, a_n$$

for all natural number n.

proof: Let $M(t)$ be the m.g.f. of the r.v. X. The Tayer series expansion of $M(t)$ about 0 is given by

$$M(t) = M(0) + \frac{M'(0)}{1!} t + \frac{M''(0)}{2!} t^2 + \dots + \frac{M^{(n)}(0)}{n!} t^n + \dots \quad \dots (3)$$

since $E(X^n) = M^{(n)}(0)$ for $n \geq 1$ and $M(0) = 1$, we have

$$M(t) = 1 + \frac{E(X)}{1!} t + \frac{E(X^2)}{2!} t^2 + \dots + \frac{E(X^n)}{n!} t^n + \dots$$

From (2) and (3), equating the coefficients of the like powers of (t), we obtain:

$$a_n = \frac{E(X^n)}{n!} \Rightarrow E(X^n) = (n!)\, a_n$$

Example (5) If the m.g.f. of a r.v. X is

$$M(t) = \sum_{j=0}^{\infty} \frac{e^{tj-1}}{j!}$$

then what is the probability of $X = 2$?

Sol:

$$M(t) = \sum_{j=0}^{\infty} \frac{e^{tj-1}}{j!} = \sum_{j=0}^{\infty} \frac{e^{-1}}{j!} e^{tj}$$

$$\Rightarrow f(j) = \frac{e^{-1}}{j!}, \quad j = 0, 1, \dots, \infty$$

$$\therefore p(X=2) = f(2) = \frac{e^{-1}}{2!} = \frac{1}{2e}$$

$M(0) = \frac{e^{-1}}{j!}$

$f(j) = \frac{e^{-1}}{j!}$

Example (6) Let X be a r.v. with

$$E(X^n) = 0.8, \quad n = 1, 2, \dots, \infty$$

what is the m.g.f. and th pmf of X?

Sol:

$$M(t) = M(0) + \sum_{n=1}^{\infty} M^{(n)}(0)\left(\frac{t^n}{n!}\right)$$

$$= M(0) + \sum_{n=1}^{\infty} E(X^n)\left(\frac{t^n}{n!}\right)$$

$$= M(0) + 0.8 \sum_{n=1}^{\infty}\left(\frac{t^n}{n!}\right)$$

$$= 1 + 0.8 \sum_{n=1}^{\infty}\left(\frac{t^n}{n!}\right)$$

$$= 0.2 + 0.8 + 0.8 \sum_{n=1}^{\infty}\left(\frac{t^n}{n!}\right) = 0.2 + 0.8 \sum_{n=0}^{\infty}\left(\frac{t^n}{n!}\right)$$

$$= 0.2e^{0t} + 0.8e^{1t}$$

$\because M(0) = 1$, $M^{(n)}(0) = EX^n$

Therefore, we get $f(0) = p(X=0) = 0.2$ and $f(1) = p(X=1) = 0.8$. Hence the m.g.f. of X is:

$$M(t) = \underset{f(0)}{0.2} + \underset{f(1)}{0.8}e^{t}$$

and the pmf of X is:

$$p(x) = \begin{cases} |x - 0.2| & , x = 0, 1 \\ 0 & \text{o.w} \end{cases}$$

$|0 - 0.2|$, $|1 - 0.2| = 0.8$

Theorem (11): Let X be a r.v. with m.g.f. $M_X(t)$. If a and b are any two real constants, then:

(i) $M_{X+a}(t) = e^{at} M_X(t)$

(ii) $M_{bX}(t) = M_X(bt)$

(iii) $M_{\frac{X+a}{b}}(t) = e^{\frac{a}{b}t} M_X\left(\frac{t}{b}\right)$

Definition (15) The factorial moment generating function (FMGF) of X is denoted by $G(t)$ and defined as:
$$G(t) = E(t^X)$$

The relationship between m.g.f. and (FMGF): اثبات

$$G(t) = E(t^X)$$
$$= E\left(e^{\ln t^X}\right) = E\left(e^{X \ln t}\right) = M_X(\ln t)$$

مهم

Definition (16): The n^{th} factorial moment of a r.v. X is:
$$E[X(X-1)(X-2)\cdots(X-n+1)]$$

Definition (17) Let X be a r.v. The characteristic function $\phi(t)$ of X is defined as:
$$\phi(t) = E(e^{itX})$$
$$= E[\cos(tX) + i\sin(tX)]$$
$$= E[\cos(tX)] + i\,E[\sin(tX)]$$

The probability density function can be recovered from the characteristic function by using the following formula.
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\,\phi(t)\,dt$$

if $X \sim$ Cauchy $(0,1)$
$$\phi(t) = E(e^{itX}) = \int_{-\infty}^{\infty} \frac{e^{itx}}{\pi(1+x^2)}\,dx$$
$$= e^{-|t|}$$

مهم

التعريفات المهمة بعض

Some Important Definitions

Definitin (18) A nonempty collection $\mathcal{F}$ of subsets of a set Ω is called a σ-field of subsets of Ω provided that the following properties hold:

(i) the empty set $\phi \in \mathcal{F}$.

(ii) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

(iii) if A_n, $n = 1, 2, \ldots$, then $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$ are both in $\mathcal{F}$.

Example (7).

we have an experiment with $\Omega = \{1, 2\}$, then:

$\mathcal{F} = \{\{\phi\}, \{1\}, \{2\}, \Omega\}$, each of the elements of $\mathcal{F}$ is an event. And, the smallest field is $\{\phi, \Omega\}$.

Definition (19).

Borel σ-field $\mathcal{B}^k$: the smallest σ-field containing all open subsets of R^k.

Definition (20)

Borel function: A function $\mathcal{F}$ from Ω to R^k is Borel with respect to a σ-field $\mathcal{F}$ on Ω iff $f^{-1}(B) \in \mathcal{F}$ for any $B \in \mathcal{B}^k$.

Definition (21): A probability measure P on a σ-field of subsets $\mathcal{F}$ of a set Ω is a real-valued function having domain $\mathcal{F}$ satisfying the following conditions:

(i) $P(\Omega) = 1$

(ii) $P(A) \geq 0$, for all $A \in \mathcal{F}$.

(iii) If $A_n, n=1,2,\dots$ are mutually disjoint sets in $\mathcal{F}$ then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n).$$

Definition (22):

probability space denoted by $(\Omega, \mathcal{F}, P)$ is a set Ω, a σ-field of subsets $\mathcal{F}$ and a probability measure P defined on $\mathcal{F}$.

Definition (23)

Let X be a random K-vector with a Lebesgue density $f_X(x)$ and let $Y = g(X)$, where g is a Borel function from (R^K, B^K) to (R^k, B^k). Let $A_1, \dots, A_m$ be disjoint sets in B^K such that $R^K - (A_1 \cup \dots \cup A_m)$ has Lebesgue measure (0) and (g) on A_j is one-to-one with a nonvanishing Jacobian, i.e., the determinant $\text{Det}\left(\frac{\partial g(x)}{\partial x}\right) \neq 0$ on A_j, $j=1,\dots,m$.

Then, Y has the following Lebesgue density:

$$f_Y(x) = \sum_{j=1}^{m} \left|\text{Det}\left(\frac{\partial h_j(x)}{\partial x}\right)\right| f_X(h_j(x)),$$

where h_j is the inverse function of g on A_j, $j=1,2,\dots,m$.

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Example (8):

Let X_1 and X_2 be independent random variables having the standard Normal distribution. Obtain the joint Lebesgue density of (Y_1, Y_2), where $Y_1 = \sqrt{X_1^2 + X_2^2}$ and $Y_2 = \frac{X_1}{X_2}$. Are Y_1 and Y_2 independent?

Solution: Let $A_1 = \{(x_1, x_2);\ x_1 > 0,\ x_2 > 0\}$,
$A_2 = \{(x_1, x_2);\ x_1 > 0,\ x_2 < 0\}$,
$A_3 = \{(x_1, x_2);\ x_1 < 0,\ x_2 > 0\}$, and
$A_4 = \{(x_1, x_2);\ x_1 < 0,\ x_2 < 0\}$.

Then the Lebesgue measure $R^2 - (A_1 \cup A_2 \cup A_3 \cup A_4)$ is (0).
On each A_i, the function $(y_1, y_2) = \left(\sqrt{x_1^2 + x_2^2},\ \frac{x_1}{x_2}\right)$ is one-to-one with

$$\operatorname{Det}\left(\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)}\right) = \begin{vmatrix} \frac{y_2}{\sqrt{1+y_2^2}} & \frac{y_1}{\sqrt{1+y_2^2}} - \frac{y_1 y_2^2}{(1+y_2^2)^{3/2}} \\ \frac{1}{\sqrt{1+y_2^2}} & -\frac{y_1 y_2}{(1+y_2^2)^{3/2}} \end{vmatrix}$$

$$= \frac{y_1}{1+y_2^2}.$$

Since the joint Lebesgue density of (X_1, X_2) is

$$\frac{1}{2\pi} e^{-(x_1^2 + x_2^2)/2}$$

and $x_1^2 + x_2^2 = y_1^2$, the joint Lebesgue density of (Y_1, Y_2) is

$$\sum_{i=1}^{4} \frac{1}{2\pi} e^{-(x_1^2+x_2^2)/2} \left|\operatorname{Det}\left(\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)}\right)\right|$$

$$= \frac{2}{\pi} e^{-y_1^2} \frac{y_1}{1+y_2^2}$$

Since the joint Lebesgue density of (Y_1, Y_2) is a product of two functions that are functions of one variable, Y_1 and Y_2 are independent.

Example (9): Let (X, Y, Z) be a random 3-vector with the following Lebesgue density:

$$f(x,y,z) = \begin{cases} \dfrac{1-\sin x \sin y \sin z}{8\pi^3} & , 0 \le x, y, z \le 2\pi \\ 0 & , \text{o.w} \end{cases}$$

show that X, Y, Z are pairwise independent, but not independent.

Solution: The Lebesgue density for (X, Y) is:

$$f(x,y) = \int_0^{2\pi} f(x,y,z)\, dz$$

$$= \int_0^{2\pi} \frac{1-\sin x \sin y \sin z}{8\pi^3}\, dz = \frac{1-0}{4\pi^2}$$

$$\Rightarrow f(x,y) = \begin{cases} \dfrac{1}{4\pi^2} & 0 \le x, y \le 2\pi \\ 0 & \text{o.w} \end{cases}$$

The Lebesgue density for X is:

$$f(x) = \int_0^{2\pi}\int_0^{2\pi} f(x,y,z)\, dy\, dz = \int_0^{2\pi} \frac{1}{4\pi^2}\, dy$$

$$\Rightarrow f(x) = \begin{cases} \dfrac{1}{2\pi} & , 0 \le x \le 2\pi \\ 0 & , \text{o.w} \end{cases}$$

$$f(y) = \int_0^{2\pi}\int_0^{2\pi} f(x,y,z)\, dx\, dz = \int_0^{2\pi} \frac{1}{4\pi^2}\, dx$$

$$f(y) = \begin{cases} \dfrac{1}{2\pi} & , 0 \le y \le 2\pi \\ 0 & , \text{o.w} \end{cases}$$

$$\Rightarrow f(x,y) = f(x)\, f(y)$$

Hence, X and Y are independent. Similarly, X and Z are independent and Y and Z are independent.

Note that:

$$P(X \le \pi) = P(Y \le \pi) = P(Z \le \pi) = \int_0^{\pi} \frac{1}{2\pi}\, dx = \frac{1}{2}.$$

Hence, $P(X \le \pi)\, P(Y \le \pi)\, P(Z \le \pi) = \frac{1}{8}$. On the other hand,

$$P(X \le \pi, Y \le \pi, Z \le \pi) = \int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \frac{1 - \sin x \sin y \sin z}{8\pi^3}\, dx\, dy\, dz$$

$$= \frac{1}{8} - \frac{1}{8\pi^3}\left(\int_0^{\pi} \sin x\, dx\right)^2$$

$$= \frac{1}{8} - \frac{1}{\pi^3}.$$

Hence X, Y and Z are not independent.

Example (10): Let X_i, $i = 1, 2, 3$, be independent random variables having the same Lebesgue density: $f(x) = e^{-x} I_{(0,\infty)}(x)$. Obtain the joint Lebesgue density of (Y_1, Y_2, Y_3), where $Y_1 = X_1 + X_2 + X_3$, $Y_2 = X_1/(X_1 + X_2)$, and $Y_3 = (X_1 + X_2)/(X_1 + X_2 + X_3)$. Are Y_i's independent?

Solution: Let $x_1 = y_1 y_2 y_3$, $x_2 = y_1 y_3 - y_1 y_2 y_3$, and $x_3 = y_1 - y_1 y_3$. Then,

$$\operatorname{Det}\left(\frac{\partial(x_1, x_2, x_3)}{\partial(y_1, y_2, y_3)}\right) = y_1^2 y_3$$

we obtain the joint Lebesgue density of (Y_1, Y_2, Y_3) as:

$$e^{-y_1} y_1^2 I_{(0,\infty)}(y_1)\, I_{(0,1)}(y_2)\, y_3 I_{(0,1)}(y_3)$$

Because this function is a product of three functions $e^{-y_1} y_1^2 I_{(0,\infty)}(y_1)$, $I_{(0,1)}(y_2)$ and $y_3 I_{(0,1)}(y_3)$, Y_1, Y_2 and Y_3 are independent.

Definition (24): Let X be a random variable having a cumulative distribution function F. Show that if $E(X)$ exists, then:

$$E(X) = \int_0^{\infty} [1 - F(x)]\, dx - \int_{-\infty}^{0} F(x)\, dx$$

Example (11): Let $X \sim U(-1, 4)$

$$f(x) = \begin{cases} \frac{1}{5} & , -1 < x < 4 \\ 0 & , \text{o.w} \end{cases}$$

$$F(x) = Pr(X \le x) = \int_{-1}^{x} \frac{1}{5}\, du = \frac{u}{5}\Big]_{-1}^{x} = \frac{x+1}{5}$$

$$1 - F(x) = 1 - \frac{x+1}{5} = \frac{5-x-1}{5} = \frac{4-x}{5}$$

$F(b) - F(a)$

$-\infty \quad -1 \quad 0 \quad 4 \quad \infty$

$$\therefore\ E(X) = \int_0^4 \left(\frac{4-x}{5}\right) dx - \int_{-1}^{0} \left(\frac{x+1}{5}\right) dx$$

$$= \frac{1}{5}\left[4x - \frac{x^2}{2}\right]_0^4 - \frac{1}{5}\left[\frac{x^2}{2} + x\right]_{-1}^{0}$$

$$= \frac{1}{5}\left[\left(16 - \frac{16}{2}\right) - (0)\right] - \frac{1}{5}\left[(0) - \left(\frac{(-1)^2}{2} + (-1)\right)\right]$$

$$= \frac{1}{5}[8] - \frac{1}{5}\left[-\frac{1}{2} + 1\right] = \frac{1}{5}\left[8 - \frac{1}{2}\right]$$

$$= \frac{1}{5}\left(\frac{16-1}{2}\right) = \frac{15}{10} = \frac{3}{2}$$

Now; if $X \sim U(-1, 4)$

$$EX = \frac{4-1}{2} = \frac{3}{2}$$

او موضحة

مميزة   دالة

Characteristic function:

Definition (25)

Let X be a r.v. with p.d.f. (pmf). The characteristic function of X (ch.f. of X) denoted by ϕ_X is a function defined on $\mathbb{R}$, taking complex values, in general, and defined as follows:

$$\phi_X(t) = E(e^{itX}) = \begin{cases} \sum_x e^{itx} p(x) & X \text{ is d.r.v.} \\ \int_{-\infty}^{\infty} e^{itx} f(x)\, dx & X \text{ is c.r.v.} \end{cases}$$

$$= \begin{cases} \sum_x [\cos(tx) p(x) + i \sum_x \sin(tx) p(x) \\ \int_{-\infty}^{\infty} \cos(tx) f(x)\, dx + i \int_{-\infty} \sin(tx) f(x)\, dx \end{cases}$$

Theorem (12) Some Properties of ch.f's.

(i) $\phi_X(0) = 1$.

(ii) $|\phi_X(t)| \le 1$.

(iii) $\phi_{X+d}(t) = e^{itd} \phi_X(t)$, where d is a constant.

(iv) $\phi_{cX}(t) = \phi_X(ct)$, where c is a constant.

(v) $\phi_{cX+d}(t) = e^{itd} \phi_X(ct)$.

(vi) $\frac{d^n}{dt^n} \phi_X(t) \Big|_{t=0} = i^n E(X^n)$, $n = 1, \cdots$, if $E|X^n| < \infty$.

proof:

(i) $\phi_X(t) = E e^{itX}$, Thus $\phi_X(0) = E e^{i0X} = E(1) = 1$

(ii) $|\phi_X(t)| = |E e^{itX}| \le E|e^{itX}| = E(1) = 1$

(iii) $\phi_{X+d}(t) = E e^{it(X+d)} = E(e^{itX} e^{itd}) = e^{itd} E(e^{itX})$

$$= e^{itd} \phi_X(t)$$

(iv) $\phi_{cX}(t) = E e^{it(cX)} = E e^{i(ct)X} = \phi_X(ct)$

Example (12) Let X be $B(n, p)$.

$$\phi_x(t) = Ee^{itx} = \sum_{x=0}^{n} e^{itx} C_x^n p^x q^{n-x}$$

$$= \sum_{x=0}^{n} C_x^n (pe^{it})^x q^{n-x} = (pe^{it} + q)^n$$

Hence;

$$\frac{d}{dt}\phi_x(t)\Big|_{t=0} = n(pe^{it}+q)^{n-1} i pe^{it}\Big|_{t=0}$$

$$= inp \leq E(X)$$

So that $E(X) = np$. Also,

$$\frac{d^2}{dt^2}\phi(x)\Big|_{t=0} = inp \frac{d}{dt}\left[(pe^{it}+q)^{n-1} e^{it}\right]\Big|_{t=0}$$

$$= inp\left[(n-1)(pe^{it}+q)^{n-2} i pe^{it} e^{it} + (pe^{it}+q)^{n-1} i e^{it}\right]\Big|_{t=0}$$

$$= i^2 np\left[(n-1)p+1\right]$$

$\because i$ is $\sqrt{-1}$

i^2 is -1

$$= -np\left[(n-1)p+1\right] = -E(X^2),$$

So that

$$E(X^2) = i^2 np\left[(n-1)p+1\right],$$

and $\sigma^2(X) = E(X^2) - [E(X)]^2$

$$= n^2p^2 - np^2 + np - n^2p^2 = npq;$$

that is, $\sigma^2(X) = npq$. $\sigma(x) = i^2 n^2 p^2$

Example (13) Let $X \sim P(\lambda)$.

$$\phi_x(t) = Ee^{itx} = \sum_{x=0}^{\infty} e^{itx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{it})^x}{x!} = e^{-\lambda} e^{\lambda e^{it}} = e^{\lambda e^{it} - \lambda}$$

Hence; $\frac{d}{dt}\phi_x(t)\Big|_{t=0} = \frac{d}{dt}\left(e^{\lambda e^{it}-\lambda}\right)\Big|_{t=0} = i\lambda$ $\quad$ $s\, e^{\lambda e^{it}-\lambda} + i\lambda e^{it}\Big]_{t=0} = e^{0} + i\lambda e^{0} \quad s\, i\lambda$

So that, $E(X) = \lambda$. Also,

$$\frac{d^2}{dt^2}\phi_x(t)\Big|_{t=0} = \frac{d}{dt}\left(i\lambda e^{-\lambda} e^{\lambda e^{it}+it}\right)\Big|_{t=0}$$

$$= i\lambda e^{-\lambda} \frac{d}{dt}\left(e^{\lambda e^{it}+it}\right)\Big|_{t=0}$$

$$= i\lambda e^{-\lambda} \cdot e^{\lambda}(\lambda i + i)$$

$$= i^2\lambda(\lambda+1) = -\lambda(\lambda+1) = -EX^2$$

So that; $\sigma^2(X) = E(X^2) - [E(X)]^2$

$$= -\lambda(\lambda+1) - (i\lambda)^2 = \lambda.$$

Example (14): Let X be $N(\mu, \sigma^2)$. Then $\phi_x(t) = e^{i\mu t - \frac{\sigma^2 t^2}{2}}$, and, in particular, if $X \sim N(0,1)$, then $\phi_x(t) = e^{-t^2/2}$. If $X \sim N(\mu, \sigma^2)$, then $\left(\frac{X-\mu}{\sigma}\right) \sim N(0,1)$, Thus

$$\phi_{\left(\frac{x-\mu}{\sigma}\right)}(t) = \phi_{\left(\frac{1}{\sigma}x - \frac{\mu}{\sigma}\right)}(t) = e^{-i\mu t/\sigma}\, \phi_x\left(\frac{t}{\sigma}\right), \text{ and}$$

$$\phi_x\left(\frac{t}{\sigma}\right) = e^{i\mu t/\sigma}\, \phi_{\left(\frac{x-\mu}{\sigma}\right)}(t)$$

So it suffices to find the ch.f. of an $N(0,1)$ r.v. Y, say. Now:

صحيح

$$\phi_Y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ity}\, e^{-y^2/2}\, dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(y^2 - 2ity)}{2}}\, dy$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2} \underbrace{\int_{-\infty}^{\infty} e^{-(y-it)^2/2}\, dy}_{\sqrt{2\pi}} = e^{-t^2/2}$$ كويس

Hence $\phi_X(t/\sigma) = e^{it\mu/\sigma}\, e^{-t^2/2}$ and replacing t/σ by t, we get, finally:

$$\phi_X(t) = \exp\left[it\mu - \frac{\sigma^2 t^2}{2}\right]$$

$$\left.\frac{d}{dt}\phi_X(t)\right|_{t=0} = \left.\exp\left(i\mu t - \frac{\sigma^2 t^2}{2}\right)(i\mu - \sigma^2 t)\right|_{t=0} = i\mu$$

So that; $E(X) = \mu$

$$\left.\frac{d^2}{dt^2}\phi_X(t)\right|_{t=0} = \left.\exp\left(i\mu t - \frac{\sigma^2 t^2}{2}\right)(i\mu - \sigma^2 t)^2 - \sigma^2 \exp\left(i\mu t - \frac{\sigma^2 t^2}{2}\right)\right|_{t=0}$$

$$= i^2\mu^2 - \sigma^2 = i^2(\mu^2 + \sigma^2)$$

Then $E(X^2) = \mu^2 + \sigma^2$ and $\sigma^2(X) = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

Example (15): Let $X \sim Gam(\alpha, \beta)$.

$$\phi_X(t) = \frac{1}{\Gamma\alpha\, \beta^\alpha} \int_0^\infty e^{itx}\, x^{\alpha-1}\, e^{-x/\beta}\, dx$$

$$= \frac{1}{\Gamma\alpha\, \beta^\alpha} \int_0^\infty x^{\alpha-1}\, e^{-x(1-i\beta t)/\beta}\, dx$$

Setting $x(1-i\beta t)~~/\beta~~ = y$, we get

$$x = \frac{y}{1-i\beta t}, \quad dx = \frac{dy}{1-i\beta t}, \quad y \in [0,\infty)$$

Hence;

$$\phi_x(t) = \frac{1}{\Gamma\alpha\,\beta^\alpha}\int_0^\infty \frac{1}{(1-i\beta t)^{\alpha-1}}\, y^{\alpha-1} e^{-y/\beta}\cdot\frac{dy}{(1-i\beta t)}$$

$$= (1-i\beta t)^{-\alpha}\cdot\frac{1}{\Gamma\alpha\,\beta^\alpha}\underbrace{\int_0^\infty y^{\alpha-1} e^{-y/\beta}\,dy}_{\Gamma\alpha\,\beta^\alpha}$$

$$= (1-i\beta t)^{-\alpha}.$$

Therefore;

$$\frac{d}{dt}\phi_x(t)\Big|_{t=0} = \frac{i\alpha\beta}{(1-i\beta t)^{\alpha+1}}\Big|_{t=0} = i\,\alpha\beta$$

So that $E(X) = \alpha\beta$, and

$$\frac{d^2}{dt^2}\phi_x(t)\Big|_{t=0} = i^2\frac{\alpha(\alpha+1)\beta^2}{(1-i\beta t)^{\alpha+2}}\Big|_{t=0} = i^2\alpha(\alpha+1)\beta^2.$$

So that $E(X^2) = \alpha(\alpha+1)\beta^2$. Thus $\sigma^2(X) = \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2$
$= \alpha\beta^2$.

Example (16) $X \sim$ Cauchy $(\mu=0, \sigma=1)$. Then $\phi_x(t) = e^{-|t|}$.

$$\phi_x(t) = \int_{-\infty}^\infty e^{itx}\cdot\frac{1}{\pi(1+x^2)}\,dx = \frac{1}{\pi}\int_{-\infty}^\infty \frac{e^{itx}}{1+x^2}\,dx$$

$$= \frac{1}{\pi}\int_{-\infty}^\infty \frac{\cos(tx)}{1+x^2}\,dx + \frac{i}{\pi}\int_{-\infty}^\infty \frac{\sin(tx)}{1+x^2}\,dx$$

because; $\int_{-\infty}^\infty \frac{\sin(tx)}{1+x^2}\,dx = 0$, since $\sin(tx)$ is an odd function, and $\cos(tx)$ is an even function. Further, it can be shown by complex variables theory that:

$$\int_0^\infty \frac{\cos(tx)}{1+x^2}\,dx = \frac{\pi}{2}e^{-|t|},$$

Hence;

$$\phi_x(t) = e^{-|t|}$$

Now; $\frac{d}{dt}\phi_x(t)\Big|_{t=0} = \frac{d}{dt}e^{-|t|}$

does not exist for $t=0$. This is consistent with fact of nonexistence of $E(X)$.

Theorem (13): (Inverse Formula)

Let X be a r.v. with p.d.f. (p.m.f.) $f(x)$ and $p(x)$ and ch.f. ϕ. Then if X is of the discrete type, taking on the (distinct) values x_j, $j \geq 1$, one has:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt$$

حفظ قانون

دليل

Example (17): Let $X \sim B(n, p)$. So that

$$\phi_x(t) = (pe^{it} + q)^n.$$

According to Theorem (13):

$$\frac{1}{2T}\int_{-T}^{+T} e^{-itx} \phi(t)\, dt = \frac{1}{2T}\int_{-T}^{T} e^{-itx} (pe^{it} + q)^n\, dt$$

$$= \frac{1}{2T}\int_{-T}^{T}\left[\sum_{r=0}^{n} C_r^n (pe^{it})^r q^{n-r} e^{-itx}\right] dt$$

$$= \frac{1}{2T}\sum_{r=0}^{n} C_r^n p^r q^{n-r} \int_{-T}^{T} e^{i(r-x)t}\, dt$$

$$= \frac{1}{2T}\sum_{\substack{r=0\\ r\neq x}}^{n} C_r^n p^r q^{n-r} \frac{1}{i(r-x)}\int_{-T}^{T} e^{i(r-x)t}\, i(r-x)\, dt$$

$$+ \frac{1}{2T} C_x^n p^x q^{n-x} \int_{-T}^{T} dt$$

$$= \sum_{\substack{r=0\\ r\neq x}}^{n} C_r^n p^r q^{n-r} \frac{e^{i(r-x)T} - e^{-i(r-x)T}}{2Ti(r-x)}$$

$$+ \frac{1}{2T} C_x^n p^x q^{n-x} (2T)$$

$$= \sum_{\substack{r=0\\ r\neq x}}^{n} C_r^n p^r q^{n-r} \frac{\sin(r-x)T}{(r-x)T} + C_x^n p^x q^{n-x}$$

Taking the limit as $T \to \infty$, we get the desired result, namely: $f(x) = C_x^n p^x q^{n-x}$

Example (18): For continuous type, Let $X \sim N(0,1)$.

$\phi_x(t) = e^{-t^2/2}$. Since $|\phi(t)| = e^{-t^2/2}$, we know that $\int_{-\infty}^{\infty} |\phi(t)|\, dt < \infty$ Thus, we have:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx}\, \phi(t)\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx}\, e^{-t^2/2}\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t^2+2itx)}\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[t^2+2t(ix)+(ix)^2]}\, e^{-\frac{1}{2}(ix)^2}\, dt$$

$$= \frac{e^{-\frac{1}{2}x^2}}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+ix)^2}\, dt$$

$$\frac{1}{\sqrt{\pi 2}} \cdot \frac{1}{\sqrt{\pi 2}} = \frac{1}{2\pi}$$

$$= \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}u^2}\, du}_{=1}$$

since, $u = t + ix$

$\Rightarrow du = dt$, $u \in (-\infty, \infty)$

$$\therefore f(x) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} * 1$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \quad x \in \mathbb{R}$$

امثلة مهمة

important Examples

Example (19): prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Sol:

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty \frac{1}{\sqrt{x}} e^{-x}\, dx$$

$\therefore y^2 = x$

Let $y = \sqrt{x} \Rightarrow dy = \frac{1}{2\sqrt{x}}\, dx = \frac{1}{2y}\, dx$

$dx = 2y\,dy$

$\therefore dx = 2y\, dy$

$$\Rightarrow \Gamma(\tfrac{1}{2}) = \int_0^\infty \frac{1}{\cancel{y}} \cdot e^{-y^2} \cdot 2\cancel{y}\, dy = 2\int_0^\infty e^{-y^2}\, dy$$

Hence; $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-u^2}\, du$ and

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-v^2}\, dv$$

Multiplying above two expressions, we get

$$\left(\Gamma(\tfrac{1}{2})\right)^2 = 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\, du\, dv$$

Now we change the integral into polar form by the Transformation $u = r\cos\theta$, $v = r\sin\theta$

$du = dr$, $1 = \cos\theta$

The Jacobian is: $J(r,\theta) = \begin{vmatrix} \frac{du}{dr} & \frac{du}{d\theta} \\ \frac{dv}{dr} & \frac{dv}{d\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$u^2 + v^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(1) = r^2$

$$= r\cos^2\theta + r\sin^2\theta = r \qquad \therefore u^2+v^2 = r^2$$

and; $\{(u,v);\ 0<u<\infty,\ 0<v<\infty\} \Rightarrow \{(r,\theta);\ 0<r<\infty,\ 0<\theta<\frac{\pi}{2}\}$

$$\left(\Gamma(\tfrac{1}{2})\right)^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} J(r,\theta)\, dr\, d\theta$$

$$= 4\int_0^{\pi/2}\left[\int_0^\infty e^{-r^2}\cdot r\, dr\right] d\theta = -2\int_0^{\frac{\pi}{2}} \left[e^{-r^2}\right]_0^\infty d\theta$$

$$= 2\int_0^{\pi/2} 1\, d\theta = 2\left[\theta\right]_0^{\pi/2} = 2\left[\frac{\pi}{2} - 0\right]$$

$$\left(\Gamma(\tfrac{1}{2})\right)^2 = \pi$$

$$\therefore \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

مهم

Example (20) prove that if $X \sim N(\mu, \sigma^2)$, the $f(x)$ is a pdf.

Sol: $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

$-\infty \to \infty$ تقسم العشرة

$= 2\int_{0}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$ لأن الشيء متماثل

Let $z = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \Rightarrow 2z = \left(\frac{x-\mu}{\sigma}\right)^2$

$\Rightarrow \sqrt{2}\sqrt{z} = \frac{x-\mu}{\sigma} \Rightarrow x = \sigma\sqrt{2}\sqrt{z} + \mu$

$\therefore dx = \sigma\sqrt{2}\cdot\frac{1}{2\sqrt{z}}\,dz = \frac{\sigma}{\sqrt{2z}}\,dz$

$2 = \sqrt{2}\cdot\sqrt{2}$

$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 2\int_{0}^{\infty} \frac{1}{\cancel{\sigma}\sqrt{2\pi}}\cdot e^{-z}\cdot\frac{\cancel{\sigma}}{\sqrt{2z}}\,dz$

$= \frac{1}{\sqrt{\pi}}\int_{0}^{\infty}\frac{1}{\sqrt{z}}e^{-z}\,dz$

but; $\int_{0}^{\infty}\frac{1}{\sqrt{z}}e^{-z}\,dz = \Gamma\frac{1}{2}$

$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{\pi}}\cdot\Gamma\frac{1}{2} = \frac{1}{\sqrt{\pi}}\cdot\sqrt{\pi} = 1$

مهم

Example (25) prove that $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$.

Recall that; $\Gamma z = (z-1)\,\Gamma(z-1)$, $z \in \mathbb{R}$

then let $z = \frac{1}{2}$

$$\Rightarrow \Gamma(\tfrac{1}{2}) = (\tfrac{1}{2}-1)\,\Gamma(\tfrac{1}{2}-1)$$

$$\Gamma(\tfrac{1}{2}) = -\tfrac{1}{2}\,\Gamma(-\tfrac{1}{2})$$

$$\Rightarrow \Gamma(-\tfrac{1}{2}) = -2\,\Gamma(\tfrac{1}{2}) = -2\sqrt{\pi}$$

Example (21): Evaluate $\Gamma(-\frac{7}{2})$.

Sol: consider: $\Gamma z = (z-1)\,\Gamma(z-1)$

$$\Gamma(-\tfrac{1}{2}) = (-\tfrac{1}{2}-1)\,\Gamma(-\tfrac{1}{2}-1) = (-\tfrac{3}{2})\,\Gamma(-\tfrac{3}{2})$$

$$= (-\tfrac{3}{2}-1)\,\Gamma(-\tfrac{3}{2}-1) = -\tfrac{5}{2}\,\Gamma(-\tfrac{5}{2})$$

$$= (-\tfrac{5}{2}-1)\,\Gamma(-\tfrac{5}{2}-1) = -\tfrac{7}{2}\,\Gamma(-\tfrac{7}{2})$$

$$\Gamma(-\tfrac{1}{2}) = (-\tfrac{3}{2})\,\Gamma(-\tfrac{3}{2})$$

$$= (-\tfrac{3}{2})(-\tfrac{5}{2})\,\Gamma(-\tfrac{5}{2})$$

$$= (-\tfrac{3}{2})(-\tfrac{5}{2})(-\tfrac{7}{2})\,\Gamma(-\tfrac{7}{2})$$

$$-2\sqrt{\pi} = -\frac{105}{8}\,\Gamma(-\tfrac{7}{2})$$

$$\therefore\ \Gamma(-\tfrac{7}{2}) = (-2\sqrt{\pi})/(-105/8) = \frac{16}{105}\sqrt{\pi}$$

Joint, marginal and Conditional Random variables

1. Bivariate Discrete random variables

Definition (26) Let (X, Y) be a bivariate r.vs. and let R_X and R_Y be the range spaces of X and Y, respectively. A real-valued function $f: R_X \times R_Y \to \mathbb{R}$ is called a joint probability density function for X and Y iff

$$p(x, y) = p(X=x, Y, y)$$

for all $(x, y) \in R_X \times R_Y$.

Definition (27) Let (X, Y) be a discrete bivariate r.vs. Let R_X and R_Y be the range spaces of X and Y. Let $p(x, y)$ be the joint density function of X and Y. The function:

$$P_1(x) = \sum_{y \in R_y} p(x, y)$$

is called the marginal probability density function of X (m.p.m.f). Similarly, the function:

$$P_2(y) = \sum_{y \in R_y} p(x, y)$$

is called the marginal probability density function of Y.

Theorem (14) A real valued function P of two variables is a joint probability density function of a pair of discrete r.vs X and Y iff:

(a) $p(x, y) \geq 0$ for all $(x, y) \in R_X \times R_y$.

(b) $\sum_{x \in R_X} \sum_{y \in R_y} p(x, y) = 1$.