Definition (28) Let X and Y be any two discrete r.v's. The real-valued function $F: \mathbb{R}^2 \to \mathbb{R}$ is called the joint cumulative probability distribution function of X and Y iff:

$$F(x,y) = P(X \leq x, Y \leq y)$$

c.d.f

From this definition it can be shown that for any real numbers a, b, c and C:

$$F(a \leq X \leq b, c \leq Y \leq d) = F(a,b) + F(a,c) - F(a,d) - F(c$$

(2.) Bivariate Continuous Random Variables:

Definition (29) The joint probability density function of the r.v's X and Y is an integrable function $f(x,y)$ such that

(a) $f(x,y) \geq 0$ for all $(x,y) \in \mathbb{R}^2$; and

(b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$.

Example (22) Let the joint density of the continuous r.v's X and Y be:

$$f(x,y) = \begin{cases} \frac{6}{5}(x^2+2xy), & 0 \leq x \leq 1;\ 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

what is the probability of the event $(X \leq Y)$?

S/**Solution:** Let $A = (X \leq Y)$, we want to find:

$$P(A) = \iint_A f(x,y)\,dx\,dy$$

$$= \int_0^1 \left[\int_0^y \frac{6}{5}(x^2+2xy)\,dx\right] dy$$

$$= \frac{6}{5}\int_0^1 \left[\frac{x^3}{3} + x^2 y\right]_0^y dy = \frac{6}{5}\int_0^1 \frac{4}{3}y^3\,dy$$

$$= \frac{8}{5}\left[\frac{y^4}{4}\right]_0^1 = \frac{2}{5}$$

Definition (30): Let (X, Y) be a continuous bivariate r.v's. Let $f(x, y)$ be the j.p.d.f. of X and Y. The function

$$f_1(x) = \int_{\forall y} f(x, y)\, dy$$

is called the m.p.d.f. of X. Similarly, the function

$$f_2(y) = \int_{\forall x} f(x, y)\, dx$$

is called the m.p.d.f. of Y.

Example (23):

If the j.p.d.f. for X and Y is given by:

$$f(x, y) = \begin{cases} \frac{3}{4}, & 0 < y^2 < x < 1 \\ 0, & o.w \end{cases}$$

then what is the m.p.d.f. of X for $0 < x < 1$? $y^2 < x$

Solution:

$$f_1(x) = \int_{\forall y} f(x, y)\, dy = \int_{-\sqrt{x}}^{\sqrt{x}} \frac{3}{4}\, dy \qquad -\sqrt{x} < y < \sqrt{x}$$

$$= \left[\frac{3y}{4}\right]_{-\sqrt{x}}^{\sqrt{x}} = \frac{3}{4}\left[\sqrt{x} + \sqrt{x}\right] = \frac{3}{2}\sqrt{x}$$

Example (24). Let (X, Y) be distributed uniformly on the circular disk centrered at $(0,0)$ with radius $\frac{2}{\sqrt{\pi}}$. what is the m.p.d.f. of X where nonzero?

Solution: معادلة

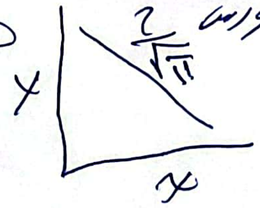
The equation of a circular (الدائرة) with radius (نصف) $\frac{2}{\sqrt{2\pi}}$ at Center at the origin is

$$x^2 + y^2 = \frac{4}{\pi} \qquad \left(\frac{2}{\sqrt{\pi}}\right)^2$$

$\Rightarrow \quad y = \pm\sqrt{\frac{4}{\pi} - x^2}$; then the m.p.d.f. of X is

$$f_1(x) = \int_{-\sqrt{\frac{4}{\pi}-x^2}}^{\sqrt{\frac{4}{\pi}-x^2}} f(x, y)\, dy = \int_{-\sqrt{\frac{4}{\pi}-x^2}}^{\sqrt{\frac{4}{\pi}-x^2}} \frac{1}{\text{area of circle}}\, dy$$

نطرح فيثاغورس



$$x^2 + y^2 = \frac{2}{\sqrt{\pi}}$$

$$\therefore f_1(x) = \int_{-\sqrt{\frac{4}{\pi}-x^2}}^{+\sqrt{\frac{4}{\pi}-x^2}} \frac{1}{4}\, dy$$

$$= \left[\frac{1}{4}y\right]_{-\sqrt{\frac{4}{\pi}-x^2}}^{+\sqrt{\frac{4}{\pi}-x^2}} = \frac{1}{2}\sqrt{\frac{4}{\pi}-x^2}$$

Definition (31) Let X and Y be the continuous r.v's with j.p.d.f. $f(x,y)$. The joint cumulative distribution function $F(x,y)$ of X and Y is defined as

$$F(x,y) = P(X \le x;\ Y \le y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)\, du\, dv$$

From the fundemental theorem of calculus, we get:

مشتقة $f(x,y) = \dfrac{\partial^2}{\partial x \partial y} F(x,y)$.

Example (25) If the j.c.d.f. of X and Y is given by:

$$F(x,y) = \begin{cases} \frac{1}{5}(2x^3y + 3x^2y^2) & , 0 < x, y < 1 \\ 0 & , o.w \end{cases}$$

then what is the j.p.d.f. of X and Y?

Solution:

$$f(x,y) = \frac{1}{5}\frac{\partial}{\partial x}\cdot\frac{\partial}{\partial y}(2x^3y + 3x^2y^2)$$

$$= \frac{1}{5}\cdot\frac{\partial}{\partial x}(2x^3 + 6x^2y)$$

$$= \frac{1}{5}(6x^2 + 12xy) = \frac{6}{5}(x^2 + 2xy)$$

then $$f(x,y) = \begin{cases} \frac{6}{5}(x^2 + 2xy) & , 0 < x, y < 1 \\ 0 & , o.w \end{cases}$$

Example (26). Let X and Y have the joint density function

$$f(x,y) = \begin{cases} x+y & , \ 0 \le x \le 1 ; \ 0 \le y < 1 \\ 0 & , \ \text{o.w} \end{cases}$$

what is the $P(2X \le 1 / X+Y \le 1)$?

Solution:

$$P(2X \le 1 / X+Y \le 1) = \frac{P[(X \le \frac{1}{2}) \cap (X+Y \le 1)]}{P(X+Y \le 1)}$$

$$P(X+Y \le 1) = \int_0^1 \left[\int_0^{1-x} (x+y)\,dy\right] dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \int_0^1 \left[x(1-x) + \frac{(1-x)^2}{2} \right] dx$$

$$= \int_0^1 \left[x - x^2 + \frac{(1-x)^2}{2} \right] dx$$

مشتقة داخل القوس قبل التكامل فطلع (-)

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} - \frac{(1-x)^3}{6} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$P[(X \le \tfrac{1}{2}) \cap (X+Y \le 1)] = \int_0^{1/2} \int_0^{1-x} (x+y)\,dy\,dx$$

$$= \int_0^{1/2} \left[xy + \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \int_0^{1/2} \left[x - x^2 + \frac{(1-x)^2}{2} \right] dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} - \frac{(1-x)^3}{6} \right]_0^{1/2}$$

$$= \frac{1}{8} - \frac{1}{24} - \frac{1}{48} = \frac{3}{48} = \frac{1}{16}$$

$$\therefore P(2X \le 1 / X+Y \le 1) = \frac{\frac{1}{16}}{\frac{1}{6}} = \frac{1}{16} \cdot \frac{6}{1}$$

$$= \frac{3}{8}$$

Definition (32) Let X and Y be any two r.v's with joint density $f(x,y)$ and marginals $f_1(x)$ and $f_2(y)$. The Conditional probability density function g of X, given $Y=y$ is defined as

$$g(x/y) = \frac{f(x,y)}{f_2(y)}, \quad f_2(y) > 0.$$

Similarly, the Conditional probability density Function h of Y, given $X=x$, is defined as:

$$h(y/x) = \frac{f(x,y)}{f_1(x)}, \quad f_1(x) > 0$$

Example (27) Let X and Y be r.v's such that X has density function

$$f_1(x) = \begin{cases} 24x^2 & , 0 < x < 1/2 \\ 0 & , o.w \end{cases}$$

and the conditional density of Y given $X=x$ is

$$h(y/x) = \begin{cases} \frac{y}{2x^2} & , 0 < y < 2x \\ 0 & , o.w \end{cases}$$

what is the conditional density of X given $Y=y$ over the appropriate domain?.

Solution:

$0 < x < \frac{1}{2}$]×2

$0 < 2x < 1$

$0 < y < 2x$

$$f(x,y) = h(y/x)\, f_1(x)$$
$$= \frac{y}{2x^2} \cdot 24x^2$$
$$= \begin{cases} 12y & , 0 < y < 2x < 1 \\ 0 & , o.w \end{cases}$$

The m.p.d.f. of X is given by

$\frac{y}{2} < x < \frac{1}{2}$

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{y/2}^{1/2} 12y\,dx$$

$$= \begin{cases} 6y(1-y) & , 0 < y < 1 \\ 0 & , o.w \end{cases}$$

Hence, the conditional density of X given $Y=y$ is

$$g(x/y) = \frac{f(x,y)}{f_2(y)} = \frac{12y}{6y(1-y)}$$

$$= \begin{cases} \frac{2}{1-y} & , \quad 0<y<2x<1 \\ 0 & , \quad o.w \end{cases}$$

Definition (33): Let X and Y be any two r.v's with joint density $f(x,y)$ and marginals $f_1(x)$ and $f_2(y)$. The random variables X and Y are stochastically independent iff

$$f(x,y) = f(x)\, f(y)$$ مستقلة

for all $(x,y) \in R_X \times R_Y$.

مثال

Example (28): Let X and Y be two independent r.v's with identical p.d.f. given by

$$f(x) = \begin{cases} e^{-x} & , \quad x>0 \\ 0 & , \quad o.w \end{cases}$$

what is the probability density function of $W = \min(X,Y)$?

Solution: Let $G(w)$ be the c.d.f. of W. Then

$$\therefore G(w) = P(W \le w)$$

$$= 1 - P(W > w)$$

$$= 1 - P(\min\{X,Y\} > w)$$

$$= 1 - P(X > w \text{ and } Y > w)$$

$$= 1 - P(X>w)\,P(Y>w) \quad \text{(since } X \text{ and } Y \text{ are independent).}$$

$$= 1 - \left(\int_w^\infty e^{-x}\,dx\right)\left(\int_w^\infty e^{-y}\,dy\right)$$ c.d.f

$$= 1 - e^{-2w}$$

Thus, the p.d.f. of W is

$$g(w) = \frac{d}{dw} G(w) = \frac{d}{dw}(1 - e^{-2w}) = 2e^{-2w}$$

$$\therefore g(w) = \begin{cases} 2e^{-2w} & , \quad w>0 \\ 0 & , \quad o.w \end{cases}$$

Definition(34): Let X and Y be any two r.v's with j.p.d.f $f(x,y)$. The product moment of X and Y, denoted by $E(XY)$, is defined as:

$$E(XY) = \begin{cases} \sum\limits_{x \in R_X} \sum\limits_{y \in R_Y} xy\, p(x,y) & \text{if X and Y are discrete r.v's} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy & \text{if X and Y are continuous r.v's} \end{cases}$$

Definition(35) Let X and Y be any two r.v's with j.p.d.f $f(x,y)$. The covariance between X and Y, denoted by $\mathrm{cov}(X,Y)$ or (σ_{XY}), is defined as

$$\mathrm{cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

where μ_X and μ_Y are mean of X and Y, respectively.
the mean of X is given by:

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x\, f_1(x)\, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f(x,y)\, dx\, dy.$$

and similarly, the mean of Y is given by:

$$\mu_Y = E(Y) = \int_{-\infty}^{\infty} y\, f_2(y)\, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y\, f(x,y)\, dy\, dx.$$

Theorem (15) Let X and Y be any two r.v's. Then:

$$\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y)$$

Example (29): Let X and Y be continuous r.v's with j.p.d.f:

$$f(x,y) = \begin{cases} 2 & , 0<y<1-x, \quad 0<x<1 \\ 0 & \text{o.w} \end{cases}$$

what is the covariance between X and Y ?

Solution: The marginal density function of X is given by

$$f_1(x) = \int_0^{1-x} 2\, dy = 2(1-x)$$

Hence, the expected value of X is

$$\mu_X = E(X) = \int_0^1 x\, f_1(x)\, dx = \int_0^1 2(1-x) \cdot x\, dx = \frac{1}{3}$$

Similarly, the m.p.d.f. of Y is:

$$f_2(y) = \int_0^{1-y} 2\, dx = 2(1-y)$$

عوض تطلع

Hence, the expected value of Y is:

$$\mu_Y = E(Y) = \int_0^1 y\, f_2(y)\, dy = \int_0^1 2y(1-y)\, dy = \frac{1}{3}$$

The product moment of X and Y is given by:

$$E(XY) = \int_0^1 \int_0^{1-x} xy\, f(x,y)\, dy\, dx$$

تكامل ثنائي سهل

$$= \int_0^1 \int_0^{1-x} 2xy\, dy\, dx$$

$$= 2\int_0^1 x \left[\frac{y^2}{2}\right]_0^{1-x} dx = 2 \cdot \frac{1}{2} \int_0^1 x(1-x)^2\, dx$$

$$= \int_0^1 (x - 2x^2 + x^3)\, dx$$

$$= \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{1}{12}$$

Then; $\quad \operatorname{cov}(X,Y) = E(XY) - E(X)E(Y)$

$$= \frac{1}{12} - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = -\frac{1}{36}$$

Theorem (16)

If X and Y are any two random variables and a, b, c and d are real constants, then

$$\operatorname{cov}(aX + b, cY + d) = ac\, \operatorname{cov}(X,Y).$$

انتبه على التكامل للتعويض

$E(x) = \int x f(x)\, dx$ 35-

دعو
نجوم
$E(XY)$

Example (30) If the product moment of X and Y is 3, and the mean of X and Y are both equal to 2, then what is the covariance of the random variables $(2X+10)$ and $(-\frac{5}{2}Y+3)$?

Solution: since $E(XY)=3$ and $E(X)=2=E(Y)$, the covariance of X and Y is given by:

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y) = 3-4 = -1$$

Then the covariance of $(2X+10)$ and $(-\frac{5}{2}Y+3)$ is given by:

$$\operatorname{cov}(2X+10, -\tfrac{5}{2}Y+3) = (2)(-\tfrac{5}{2})\operatorname{cov}(X,Y)$$
$$= (-5)(-1)$$
$$= 5$$

Theorem (17) If X and Y are independent r.v's, then
$$E(XY) = E(X)\,E(Y).$$

دعو

Example (31) Let X and Y be two independent r.v's with respective density functions:

$$f(x) = \begin{cases} 3x^2, & 0<x<1 \\ 0, & o.w \end{cases}$$

and;

$$f(y) = \begin{cases} 4y^3, & 0<y<1 \\ 0, & o.w \end{cases}$$

what is $E(\frac{X}{Y})$?

solution: since X and Y are independent r.v's, then the j.p.d.f. of X and Y is given by:

$$h(x,y) = f(x)\,f(y)$$

Therefore; $$E\left(\frac{X}{Y}\right) = \int_{-\infty}^{\infty}\int \frac{x}{y}\, h(x,y)\,dx\,dy$$
$$= \int_0^1\int_0^1 \frac{x}{y}\cdot f(x)\,f(y)\,dx\,dy$$

③ $$E\left(\frac{X}{Y}\right) = \int_0^1\int_0^1 \frac{x}{y}\cdot 3x^2\cdot 4y^3\,dx\,dy$$
$$= \left(\int_0^1 3x^3\,dx\right)\left(\int_0^1 4y^2\,dy\right)$$
$$= \left(\frac{3}{4}\right)\left(\frac{4}{3}\right) = 1$$

Remark:

1. $E\left(\frac{X}{Y}\right) \neq \frac{E(X)}{E(Y)}$, but only implies $E\left(\frac{X}{Y}\right) = E(X)\,E(Y^{-1})$.

2. $E(Y^{-1}) \neq \frac{1}{E(Y)}$.

Theorem (18). If X and Y are independent r.v's, then the covariance between X and Y is always zero, that is:
$$\operatorname{cov}(X,Y) = 0$$
x , y إذا كانا مستقلين فالتباين المشترك = صفر

Theorem (19) Let X and Y be any two r.v's and let a and b be any two real numbers. Then
$$\operatorname{var}(aX \mp bY) = a^2\operatorname{var}(X) + b^2\operatorname{var}(Y) \mp 2ab\,\operatorname{cov}(X,Y).$$
حفظ

Example (32)
If $\operatorname{var}(X+Y) = 3$, $\operatorname{var}(X-Y) = 1$, $E(X) = 1$ and $E(Y) = 2$. What is $E(XY)$?

$$\operatorname{var}(X+Y) = \sigma_X^2 + \sigma_Y^2 + 2\operatorname{cov}(X,Y)$$
$$\operatorname{var}(X-Y) = \sigma_X^2 + \sigma_Y^2 - 2\operatorname{cov}(X,Y)$$
$$\Rightarrow \operatorname{cov}(X,Y) = \frac{1}{4}[\operatorname{var}(X+Y) - \operatorname{var}(X-Y)]$$
$$= \frac{1}{4}[3-1] = \frac{1}{2}$$

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y)$$
$$\frac{1}{2} = E(XY) - (1)(2)$$
$$\Rightarrow E(XY) = \frac{1}{2} + 2 = 5/2$$

Theorem (20) If X and Y are independent r.v's with $E(X) = 0 = E(Y)$, then:

مستقلة

$$var(XY) = var(X)\,var(Y)$$

Definition (36) Let X and Y be two r.v's. Then the correlation coefficient (ρ) between X and Y is given by

$$\rho = \frac{cov(X,Y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

And, if X and Y are independent r.v's, then:

$$\rho = 0 \rightarrow$$ أي صابر
لا استقلالية

Moment Generating Functions: الدالة المولدة للعزوم للمتغيرين معاً

Defintion (37) Let X and Y be two r.v's with joint density function $f(x,y)$. A real valued function $M: \mathbb{R}^2 \to \mathbb{R}$, defined by

$$M(t_1, t_2) = E(e^{t_1 X + t_2 Y})$$

is called the joint moment generating function of X and Y if the expected values exists for all t_1 and t_2 where $-h < t_1 < h$ and $-K < t_2 < K$ for some positive h and K.

ملاحظات **Remarks**

1. $M(t_1, 0) = M(t_1) = E e^{t_1 X}$, and
   $M(0, t_2) = M(t_2) = E e^{t_2 Y}$.

2. $E(X^K) = \left.\dfrac{\partial^K M(t_1, t_2)}{\partial t_1^K}\right|_{(t_1=0,\, t_2=0)}$ for $K = 1, 2, \ldots$

   $E(Y^K) = \left.\dfrac{\partial^K M(t_1, t_2)}{\partial t_2^K}\right|_{(t_1=0,\, t_2=0)}$ for $K = 1, 2, \ldots$;

3. $E(XY) = \left.\dfrac{\partial^2 M(t_1, t_2)}{\partial t_1 \partial t_2}\right|_{(t_1=0,\, t_2=0)}$.

4. If X and Y are independent r.v's, then
   $$M_{aX+bY}(t) = M_X(at)\, M_Y(bt)$$

Example (33) If the joint moment generating function of X and Y is:

$$M(t_1, t_2) = e^{(t_1 + 3t_2 + 2t_1^2 + 18t_2^2 + 12t_1t_2)}$$

what is the covariance of X and Y ?

Solution :

$$M(t_1, t_2) = e^{(t_1 + 3t_2 + 2t_1^2 + 18t_2^2 + 12t_1t_2)}$$

$$\frac{\partial M}{\partial t_1} = (1 + 4t_1 + 12t_2)\, M(t_1, t_2)$$

$$\left.\frac{\partial M}{\partial t_1}\right|_{(0,0)} = M(0,0) = 1$$

$$\frac{\partial M}{\partial t_2} = (3 + 36t_2 + 12t_1)\, M(t_1, t_2)$$

$$\left.\frac{\partial M}{\partial t_2}\right|_{(0,0)} = 3\, M(0,0) = 3$$

Hence; $\mu_x = 1$ and $\mu_y = 3$

$$\frac{\partial^2 M(t_1,t_2)}{\partial t_1 \partial t_2} = \frac{\partial}{\partial t_2}\left(\frac{\partial M}{\partial t_1}\right)$$

$$= \frac{\partial}{\partial t_2}\left[M(t_1, t_2)\,(1 + 4t_1 + 12t_2)\right]$$

$$= (1 + 4t_1 + 12t_2)\frac{dM}{dt_2} + M(t_1, t_2).(12)$$

$$\frac{\partial^2 M(t_1,t_2)}{\partial t_1 , \partial t_2} = (1)(3) + (1)(12) = 15 = E(XY)$$

$$\begin{aligned} cov(X,Y) &= E(XY) - E(X)E(Y) \\ &= 15 - (1)(3) \\ &= 12 . \end{aligned}$$

التوقع الشرطي

Conditional Expectation

Definition (38). The conditional mean of X given $Y=y$ is defined as

$$\mu_{x/y} = E(X/y).$$

where

$$E(X/y) = \begin{cases} \sum_{x \in R_x} x\, g(x/y) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x\, g(x/y)\, dx & \text{if } X \text{ is continuous} \end{cases}$$

Similarly, the conditional mean of Y given $X=x$ is defined

$$E(Y/x) = \begin{cases} \sum_{y \in R_y} y\, h(y/x) & \text{if } Y \text{ is discrete} \\ \int_{-\infty}^{\infty} y\, h(y/x)\, dy & \text{if } Y \text{ is continuous} \end{cases}$$

Definition (39). The expected value of the random variable $E(Y/X)$ is equal to the expected value of Y, that is

$$E_x(E_{y/x}(Y/X)) = E_y(Y).$$

مثال

Example (34). An insect lays Y number of eggs, where Y has a Poisson distribution with parameter (λ). If the probability of each egg surviving is (P), then on the average how many eggs will survive?

Sol: Let X denote the number of surviving eggs. Then, given that $Y=y$ (that is given that the insect has laid y eggs) the number variable X has a binomil (y, P). Thus

$$X/Y \sim Bin(Y, P)$$
$$Y \sim Poi(\lambda)$$

Therefore, the expected number of survivors is given by:

$$E_x(X) = E_y[E_{x/y}(X/Y)]$$
$$= E_y[\rho Y]$$
$$= \rho E_y(Y)$$
$$= \rho\lambda$$

Theorem (21)

Let X and Y be two r.v's with mean μ_x and μ_y, and standard deviation σ_x and σ_y, respectively. If the conditional expectation of Y given $X=x$ is Linear in x, then

$$E(Y/X=x) = \mu_y + \rho \frac{\sigma_y}{\sigma_x}(x-\mu_x).$$

where ρ denotes the correlation Coefficient of X and Y.

مثال

Example (35) suppose X and Y are random variables with $E(Y/X=x) = -x+3$ and $E(X/Y=y) = -\frac{1}{4}y+5$. what is the correlation Coefficient of X and Y?

Sol:
$$\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x-\mu_x) = -x+3$$

Therefore, equating the coefficients of x terms, we get:

$$\rho\frac{\sigma_y}{\sigma_x} = -1 \quad \text{--- (1)}$$

similarly, since

$$\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y) = -\frac{1}{4}y+5$$

we have
$$\rho\frac{\sigma_x}{\sigma_y} = -\frac{1}{4} \quad \text{--- (2)}$$

multiplying (1) with (2), we get:

$$\rho\frac{\cancel{\sigma_y}}{\cancel{\sigma_x}} \cdot \rho\frac{\cancel{\sigma_x}}{\cancel{\sigma_y}} = (-1)\left(-\frac{1}{4}\right) \Rightarrow \rho^2$$

$$\rho^2 = \frac{1}{4} \Rightarrow \rho = \pm\frac{1}{2}$$

since $\rho\frac{\sigma_y}{\sigma_x} = -1$ and $\frac{\sigma_y}{\sigma_x} > 0$, we get $\rho = -\frac{1}{2}$

Definition (40) Let X and Y be two r.v's with j.p.d.f. $f(x,y)$ and $f(y/x)$ be the conditional density of Y given $X=x$. The conditional variance of Y given $X=x$, denoted by $var(Y/x)$ is denoted by :

$$var(Y/x) = E(Y^2/x) - [E(Y/x)]^2$$

Example (36) suppose the following j.p.d.f. :

$$f(x,y) = \begin{cases} e^{-y} & , \quad 0 < x < y < \infty \\ 0 & , \quad o.w \end{cases}$$

Compute $var(Y/x)$.

Solution:

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{x}^{\infty} e^{-y}\,dy = e^{-x}.$$

$$h(y/x) = \frac{f(x,y)}{f(x)} = \frac{e^{-y}}{e^{-x}} = e^{-(y-x)} \quad , y > x$$

$$E(Y/x) = \int_{-\infty}^{\infty} y\, h(y/x)\,dy = \int_{x}^{\infty} y\, e^{-(y-x)}\,dy$$

, Let $z = y - x$, then:

$z = 0 - 0 = 0$

$z = \infty - 0 = \infty$

$0 < z < \infty$

$$E(Y/x) = \int_0^{\infty} (z+x)\, e^{-z}\,dz$$

$$= \int_0^{\infty} z\, e^{-z}\,dz + x\int_0^{\infty} e^{-z}\,dz$$

$$= \Gamma(2) + x\,\Gamma(1) = x + 1$$

$$E(Y^2/x) = \int_0^{\infty} y^2\, h(y/x)\,dy = \int_x^{\infty} y^2 e^{-(y-x)}\,dy$$

$$= \int_0^{\infty} (z+x)^2 e^{-z}\,dz \quad , \text{where } z = y - x$$

$$= x^2\int_0^{\infty} e^{-z}\,dz + \int_0^{\infty} z^2 e^{-z}\,dz + 2x\int_0^{\infty} z\, e^{-z}\,dz$$

$$= x^2\,\Gamma(1) + \Gamma(3) + 2x\,\Gamma(2)$$

$$= x^2 + 2 + 2x = (1+x)^2 + 1$$

$$\therefore \; var(Y/x) = E(Y^2/x) - [E(Y/x)]^2$$
$$= (1+x)^2 + 1 - (1+x)^2$$
$$= \underline{1}$$

Theorem (22.) Let X and Y be two r.v's with mean μ_x and μ_y, and standard deviation σ_X and σ_Y respectively. If the conditional expectation of Y given $X=x$ is linear in x, then:

$$E_x[Var(Y/X)] = (1-\rho^2)\, var(Y)$$ حفظ

مثال

Example (37). Let $E(Y/X=x) = 2x$ and $Var(Y/X=x) = 4x^2$, and Let $X \sim U(0,1)$. what is the variance of Y?

$\sigma_Y^2 = Var(Y)$

Solution

If $E(Y/X=x)$ is linear function of x, then

$$E(Y/X=x) = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x-\mu_x)$$

and;

$$E_x[var(Y/X)] = \sigma_Y^2 (1-\rho^2)$$ حفظ

we are given that;

$$\mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x-\mu_x) = 2x$$

Hence, equating the coefficient of x terms, we get

$$\rho \frac{\sigma_Y}{\sigma_x} = 2$$

which is
$$\rho = 2\frac{\sigma_X}{\sigma_Y}$$

Further, we are given that $Var(Y/X=x) = 4x^2$.
Since $X \sim U(0,1)$, we get the density of X to be $f(x)=1$ on the interval $(0,1)$. Therefore,

$$E_x[Var(Y/X)] = \int_{-\infty}^{\infty} var(Y/X=x) f(x)\, dx$$

$$= \int_0^1 4x^2\, dx = \frac{4}{3}$$

$X \sim U(0,1)$ $f(x) = 1$

By Theorem :

$$\frac{4}{3} = E_x[var(Y/X)]$$
$$= \sigma_y^2 (1-\rho^2)$$
$$= \sigma_y^2 \left(1 - 4\frac{\sigma_x^2}{\sigma_y^2}\right)$$
$$= \sigma_y^2 - 4\sigma_x^2$$

Hence;
$$\sigma_y^2 = \frac{4}{3} + 4\sigma_x^2$$

since $X \sim U(0,1)$, the variance of X is given by $\sigma_x^2 = \frac{1}{12}$

Therefore, the variance of Y is given by:
$$\sigma_y^2 = \frac{4}{3} + \frac{4}{12}$$
$$= \frac{5}{3}$$

Example (38) Let $E(X/Y=y) = 3y$ and $var(X/Y=y) = 2$ and let Y have the p.d.f. $f(y) = e^{-y}$, $y>0$. Find $var(X)$.

Sol: By previous Theorem:
$$var(X/Y=y) = \sigma_x^2(1-\rho^2) = 2 \qquad \text{----}(2)$$

and
$$\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y) = 3y$$

$$\Rightarrow \quad \rho = 3\frac{\sigma_y}{\sigma_x}$$

Hence from (2), we get $E_y[var(X/Y)] = 2$ and thuse
$$\sigma_x^2\left(1 - 9\frac{\sigma_y^2}{\sigma_x^2}\right) = 2$$

which is:
$$\sigma_x^2 = 9\sigma_y^2 + 2$$

Now;
$$E(Y) = \int_0^\infty y f(y)\,dy = \int_0^\infty y e^{-y}\,dy = 1$$
$$E(Y^2) = \int_0^\infty y^2 f(y)\,dy = \int_0^\infty y^2 e^{-y}\,dy = 2$$

$$\Rightarrow \quad \sigma_y^2 = E(Y^2) - [E(Y)]^2 = 2-1 = 1$$

$$\Rightarrow \quad \sigma_x^2 = 9\sigma_y^2 + 2 = 9(1) + 2 = 11$$

Transformation of Random Variables and their distributions

There are several methods for finding the probability density function of a transformation r.v's. Some of these methods are:

(1) Distribution function method (c.d.f.)

(2) Transformation method

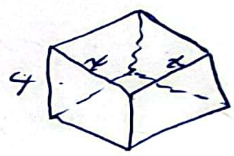
(3) Convolution method, and

(4) Moment generating function method.

(1) Distribution Function Method: c.d.f توضيح

مثال **Example (39)** A box is to be constructed so that the high is (4) inches بوصة and its base is X inches. If $X \sim N(0,1)$. what is the distribution of the volume of the box?

Sol: The volume is a r.v., since X is a r.v. The r.v. V is given by $V = 4X^2$.



$$G(V) = P(V \le v) = P(4X^2 \le v)$$

c.d.f

$$= P\left(-\tfrac{1}{2}\sqrt{v} < X < \tfrac{1}{2}\sqrt{v}\right)$$

$$= \int_{-\frac{1}{2}\sqrt{v}}^{+\frac{1}{2}\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = 2\int_0^{\frac{1}{2}\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$g(v) = \frac{d}{dv} G(v)$$

p.d.f

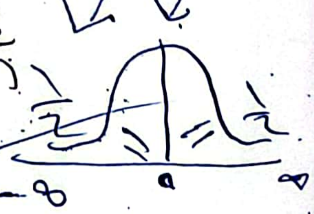
$$= \frac{d}{dv}\left(2\int_0^{\frac{1}{2}\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx\right)$$

$$= 2\,\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{1}{2}\sqrt{v}\right)^2} \cdot \left(\frac{1}{2}\right)\frac{d\sqrt{v}}{dv} + 0$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{8}v} \cdot \frac{1}{2\sqrt{v}}$$

$$= \frac{1}{\Gamma(\frac{1}{2})\, 8^{1/2}}\, v^{\frac{1}{2}-1}\, e^{-v/8}$$

$$\Rightarrow V \sim Gam\left(\tfrac{1}{2}, 8\right)$$



$V > 0 \neq V > \sqrt{x}$

Some useful Examples:

مهم

Example (40): Suppose $Y \sim U(\overset{a}{0}, \overset{b}{1})$ and $Y = \frac{1}{4}X^2$. what is the Probability function of X

Sol: By using cdf:

$$F(x) = P(X \le x) = P(X^2 \le x^2)$$
$$= P\left(\tfrac{1}{4}X^2 \le \tfrac{1}{4}x^2\right)$$
$$= P\left(Y \le \tfrac{1}{4}x^2\right)$$
$$= \int_0^{x^2/4} f(y)\,dy = \int_0^{x^2/4} 1\,dy$$
$$= \left[y\right]_0^{x^2/4} = \frac{x^2}{4}$$

$$f(x) = \frac{\partial}{\partial x}F(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le 2 \\ 0, & o.w \end{cases}$$

Hint: $0 \le Y \le 1 \Rightarrow 0 \le \frac{1}{4}x^2 \le 1 \Rightarrow 0 \le x^2 \le 4$
$\Rightarrow 0 \le x \le 2$

مهم

Example (4.1): Let $f(x) = \dfrac{e^{-x}}{(1+e^{-x})^2}$, $x \in \mathbb{R}$.

Find the pdf of $Y = \dfrac{1}{1+e^{-x}}$.

Sol By using cdf:

$$F(y) = P(Y \le y) = P\left(\frac{1}{1+e^{-x}} \le y\right)$$
$$= P\left(1+e^{-x} \ge \frac{1}{y}\right)$$
$$= P\left(e^{-x} \ge \frac{1}{y} - 1\right) = P\left(-x \ge \ln\left(\frac{1}{y}-1\right)\right)$$

نضرب ال Ln بجد -

$$= P\left(X \le -\ln\left[\frac{1}{y}-1\right]\right) = P\left(X \le -\ln\left(\frac{1-y}{y}\right)\right)$$
$$= P\left(X \le \ln\left[\frac{1-y}{y}\right]^{-1}\right) = P\left(X \le \ln\frac{y}{1-y}\right)$$
$$= \int_{-\infty}^{\ln\frac{y}{1-y}} f(x)\,dx = \int_{-\infty}^{\ln\frac{y}{1-y}} \frac{e^{-x}}{(1+e^{-x})^2}\,dx$$
$$= \int_{-\infty}^{\ln\frac{y}{1-y}} (1+e^{-x})^{-2}(e^{-x})\,dx$$

$$\therefore F(y) = -\frac{[1+e^{-x}]^{-2+1}}{-2+1}\Bigg]_{-\infty}^{\ln\frac{y}{1-y}} = [1+e^{-x}]^{-1}\Bigg]_{-\infty}^{\ln\frac{y}{1-y}}$$

$$= \left[(1+e^{-\ln\frac{y}{1-y}})^{-1} - (1+e^{\infty})^{-1}\right]$$

$$= \left[(1+e^{\ln(\frac{y}{1-y})^{-1}})^{-1} - \frac{1}{1+e^{\infty}}\right] \qquad e^{\infty}=\infty \quad e^{-\infty}=0$$

$$= \left[(1+(\frac{y}{1-y})^{-1})^{-1} - 0\right] = \left[1+\frac{1-y}{y}\right]^{-1}$$

$$= \left[\frac{y+1-y}{y}\right]^{-1} = \left[\frac{1}{y}\right]^{-1} = y$$

$\therefore F(y) = y$

Hint: $-\infty < x < \infty \Rightarrow$ if $x \to -\infty \Rightarrow y = \frac{1}{1+e^{\infty}} = 0$

if $x \to \infty \Rightarrow y = \frac{1}{1+e^{-\infty}} = 1$

$\Rightarrow 0 < y < 1$

$$\therefore f(y) = \frac{\partial}{\partial y} F(y) = \begin{cases} 1 & , 0<y<1 \\ 0 & , o.w \end{cases}$$

$\Rightarrow Y \sim U(0,1).$

Example (42) If $X \sim Gam(\alpha=1, \beta=2)$. Find the pdf. of $Y = e^{X}$.

Sol: By using cdf:

$$F(y) = P(Y \le y) = P(e^{X} \le y) = P(X \le \ln y)$$

$$= \int_0^{\ln y} f(x)\,dx = \int_0^{\ln y} \frac{1}{2} e^{-\frac{x}{2}}\,dx$$

$$= -e^{-x/2}\Big]_0^{\ln y} = -e^{-\frac{1}{2}\ln y} + e^{0}$$

$$= -\cancel{e}^{\cancel{\ln} y^{-1/2}} + 1 = \left(1 - \frac{1}{\sqrt{y}}\right)$$

$$g(y) = \frac{\partial}{\partial y} F(y) = \frac{d}{dy}\left(1 - \frac{1}{\sqrt{y}}\right) = \frac{1}{2y\sqrt{y}}$$

$\because$ Hint: $0 < X < \infty \Rightarrow$ if $x \to 0 \Rightarrow y = e^{0} = 1$

if $y \to \infty \Rightarrow y = e^{\infty} = \infty$

$$\therefore g(y) = \begin{cases} \frac{1}{2y\sqrt{y}} & , y > 1 \\ 0 & , o.w \end{cases}$$

!! $y = e^{x}$

مهم

حلو.

Example (43): Let $X \sim N(\mu, \sigma^2)$

show that $\left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi^2_{(1)}$

Q/ **Sol:** by using cdf: let $\left(\frac{X-\mu}{\sigma}\right)^2 = W$

$$F(\omega) = P(W \leq \omega)$$

$$= P\left(\left(\frac{X-\mu}{\sigma}\right)^2 \leq \omega\right)$$

$$= P\left(-\sqrt{\omega} \leq \frac{X-\mu}{\sigma} \leq \omega\right)$$

$$= P\left(-\sqrt{\omega} \leq Z \leq \omega\right) = \int_{-\sqrt{\omega}}^{\sqrt{\omega}} f(z)\,dz$$

$$f(\omega) = \frac{d}{d\omega} F(\omega)$$

$$= \frac{d}{d\omega} \int_{-\sqrt{\omega}}^{\sqrt{\omega}} f(z)\,dz$$

$$= f(\sqrt{\omega}) \frac{d}{d\omega}(\sqrt{\omega}) - f(-\sqrt{\omega}) \cdot \frac{d}{d\omega}(-\sqrt{\omega})$$

$$= f(\sqrt{\omega}) \cdot \frac{1}{2\sqrt{\omega}} - f(-\sqrt{\omega}) \cdot \frac{-1}{2\sqrt{\omega}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\omega/2} \cdot \frac{1}{2\sqrt{\omega}} + \frac{1}{\sqrt{2\pi}} e^{-\omega/2} \cdot \frac{1}{2\sqrt{\omega}}$$

$$= \cancel{2} * \frac{1}{\sqrt{2\pi}} e^{-\omega/2} \cdot \frac{1}{\cancel{2}\sqrt{\omega}} = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\omega}} e^{-\omega/2}$$

$$\therefore f(\omega) = \frac{1}{\sqrt{2}\sqrt{\pi}} \cdot \frac{1}{\sqrt{\omega}} e^{-\omega/2} = \begin{cases} \frac{1}{2^{1/2}\Gamma\frac{1}{2}} \omega^{1/2-1} e^{-\omega/2} & \omega > 0 \\ 0 & \text{o.w} \end{cases}$$

$$\Rightarrow W \sim \chi^2_{(1)}.$$

(2) Transformation Method   نظريات التحويل

① (a) Univariate case

Theorem (23) Let X be a continuous r.v. with p.d.f. $f(x)$. Let $y = T(x)$ be an increasing (or decreasing) function. Then the density function of the r.v. $Y = T(X)$ is given by:

$$g(y) = \left|\frac{dx}{dy}\right| f(w(y))$$

where $x = w(y)$ is the inverse function of $T(x)$.

مثال **Example (44).** Let $Y = -\ln X$. If $X \sim U(0,1)$, then what is the density function of Y where nonzero?

Solution:

$$y = T(x) = -\ln x$$

$$x = e^{-y}$$

$$\frac{dx}{dy} = -e^{-y}$$

$$X = e^{-y}$$

$$g(y) = \left|\frac{dx}{dy}\right| f(w(y))$$

$$= e^{-y} \cdot 1 \qquad = \begin{cases} e^{-y} & , y > 0 \\ 0 & , o.w \end{cases}$$

$\Rightarrow Y \sim exp(1)$.

② (b) Bivariate case

Theorem (24). Le X and Y be two continuous r.v's with j.p.d.f. $f(x,y)$. Let $u = P(X,Y)$ and $V = Q(X,Y)$ be functions of X and Y. If the functions $P(x,y)$ and $Q(x,y)$ have single valued inverses, say $X = R(u,v)$ and $Y = S(u,v)$, then the joint density $g(u,v)$ of u and V is given by:

$$g(u,v) = |J|\, f(R(u,v), S(u,v))$$

where
$$J = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

مهم

Example (45) Let X and Y have the j.p.d.f.,

$$f(x,y) = \begin{cases} 8xy & , \quad 0<x<y<1 \\ 0 & , \quad o.w \end{cases}$$

what is the joint density of $U = \frac{X}{Y}$ and $V = Y$?

Solution:

$$u = \frac{x}{y}, \quad v = y \quad \Rightarrow \quad x = uv, \quad y = v$$

$\frac{\partial x}{\partial v} = u$, $\quad \frac{\partial x}{\partial u} = v$, $\quad \frac{\partial y}{\partial v} = 1$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

The joint density function of U and V is:

$$g(u,v) = |J|\, f(R(u,v), S(u,v)) = |J|\, f(x,y)$$

$x = uv$, $y = v$

$$= v * 8(uv)v = |J| \cdot 8xy = v * 8(uv)v$$

$$= 8uv^3$$

$f(uv, v) = 8(uv * v)$

$f(R(u,v), S(u,v)) = f(x,y) = f(x = uv,\ y = v)$

Note that, since

$$0 < x < y < 1 \quad \Rightarrow \quad 0 < uv < v < 1$$

The last inequalities yield:

$$0 < uv < v \;\; \div v \Rightarrow 0 < u < 1$$

$$0 < v < 1 \Rightarrow 0 < v < 1$$

Thus,

$$g(u,v) = \begin{cases} 8uv^3 & , \quad 0 \le u \le 1 \,;\; 0 \le v \le 1 \\ 0 & , \quad o.w \end{cases} = 8\, uv * v$$

مهم

Example (46) Let each of the independent r.v's X and Y have the density function:

$$f(x) = \begin{cases} e^{-x} & , \quad 0<x<\infty \\ 0 & , \quad o.w \end{cases}$$

what is the j.p.d.f. of $U = X$ and $V = 2X + 3Y$ and the domain on which this density is positive?

Solution:

$u = x$, $v = 2x + 3y$ $\Rightarrow$ $x = u$, $y = \frac{1}{3}v - \frac{2}{3}u$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}$$

The joint function of U and V is

$$g(u,v) = |J|\, f(R(u,v), S(u,v))$$

$$= \frac{1}{3}\, e^{-u}\, e^{-\frac{1}{3}v + \frac{2}{3}u}$$

$$= \begin{cases} \frac{1}{3} e^{-\left(\frac{u+v}{3}\right)} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \end{cases}$$

Further, since $v = 2u + 3y$ and $3y > 0$

$\Rightarrow v > 2u \Rightarrow 0 < 2u < v < \infty$

$$\therefore\ g(u,v) = \begin{cases} \frac{1}{3} e^{-\left(\frac{u+v}{3}\right)} & , 0 < 2u < v < \infty \\ 0 & , \text{o.w} \end{cases}$$

Example (47) Let X and Y be independent r.v's, each with density function:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad -\infty < x < \infty$$

Let $u = \frac{X}{Y}$, and $v = y$. What is the joint density of u and v? What is the density of u?

Solution:

$u = \frac{x}{y}$, $v = y$ $\Rightarrow$ $x = uv$, $y = v$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

[illegible]

$$f(x,y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \qquad R(u,v) = x = uv$$

$$S(u,v) = y = v$$

The joint density function of u and v is

$$\underline{g(u,v)} = |J| \; f(R(u,v), S(u,v))$$

من تحويل السؤال

$$= |v| \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u^2v^2)} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(v^2)}$$

$$= \begin{cases} \dfrac{|v|}{2\pi} e^{-\frac{1}{2}v^2(u^2+1)} & , -\infty < u < \infty, \; -\infty < v < \infty \\ 0 & , \text{o.w} \end{cases}$$

The marginal $g_1(u)$ of u is given by:

$$g_1(u) = \int_{-\infty}^{\infty} g(u,v)\, dv = \int_{-\infty}^{\infty} \frac{|v|}{2\pi} e^{-\frac{1}{2}v^2(u^2+1)} dv$$

$$= \int_{-\infty}^{0} (-v) \frac{1}{2\pi} e^{-\frac{1}{2}v^2(u^2+1)} dv + \int_{0}^{\infty} (v) \cdot \frac{1}{2\pi} e^{-\frac{1}{2}v^2(u^2+1)} dv$$

$$= \frac{1}{2\pi}\left(\frac{1}{\cancel{2}}\right)\left[\frac{\cancel{2}}{u^2+1} e^{-\frac{1}{2}v^2(u^2+1)}\right]_{-\infty}^{0} + \frac{1}{2\pi}\left(\frac{1}{\cancel{2}}\right)\left[\frac{-\cancel{2}}{u^2+1} e^{-\frac{1}{2}v^2(u^2+1)}\right]_{0}^{\infty}$$

مشتقة تعويض

$2v(u^2+1)$

$(-v)$

$$= \frac{1}{2\pi} \cdot \frac{1}{u^2+1} + \frac{1}{2\pi} \frac{1}{u^2+1}$$

$$= \begin{cases} \dfrac{1}{\pi(u^2+1)} & , -\infty < x < \infty \\ 0 & , \text{o.w} \end{cases} \quad \Rightarrow \underline{u \sim \text{Cauchy}(1,0)}$$

<u>Theorem</u> (25)

Let the joint density of the r.v's X and Y be $f(x,y)$. Then the probability density functions $X+Y$, XY, and $\frac{X}{Y}$ are given by:

$$h_{X+Y}(v) = \int_{-\infty}^{\infty} f(u, v-u)\, du$$

$$h_{XY}(v) = \int_{-\infty}^{\infty} \frac{1}{|u|} f\left(u, \frac{v}{u}\right) du$$

$$h_{\frac{X}{Y}}(v) = \int_{-\infty}^{\infty} |u| \, f(u, vu)\, du$$

In addition, if the r.v's X and Y are independent and have the m.p.d.f's $f(x)$ and $g(y)$ respectively, then we have

$$h_{X+Y}(z) = \int_{-\infty}^{\infty} g(y)\, f(z-y)\, dy$$

$$h_{XY}(z) = \int_{-\infty}^{\infty} \frac{1}{|y|}\, g(y)\, f\left(\frac{z}{y}\right) dy$$

$$h_{\frac{X}{Y}}(z) = \int_{-\infty}^{\infty} |y|\, g(y)\, f(zy)\, dy$$

3. Convolution Method for Sums of Random Variables

Definition (41) Let f and g be two real-valued functions. The convolution of f and g denoted by $f*g$, is defined as

$$(f*g)(z) = \int_{-\infty}^{\infty} f(z-y)\, g(y)\, dy$$

$$= \int_{-\infty}^{\infty} g(z-x)\, f(x)\, dx$$

Hint: (1) $f*g = g*f$

(2) If X and Y be two independent r.v's with p.m.f $f(x)$ and $g(y)$. Then, we get:

$$h(z) = \int_{-\infty}^{\infty} f(z-y)\, g(y)\, dy$$

Example (48) what is the probability density of the sum of two independent r.v's, each of which is gamma with $\alpha = 1$ and $\beta = 1$?

Solution: Let $Z = X+Y$, where $X \sim Gam(1,1)$ and $Y \sim Gam(1,1)$ with the following densities:

$$f(x) = \begin{cases} e^{-x} & , x>0 \\ 0 & \end{cases}$$

$$g(y) = \begin{cases} e^{-y} & , y>0 \\ 0 & \end{cases}$$

since X and Y are independent, the $f(z)$ can be obtained by the convolution method. Note that the sum $Z = x+y$ is between

(0) and (∞), and $0<x<z$. Hence $h(z)$ is given by:

$$h(z) = (f*g)(z)$$

$$= \int_{-\infty}^{\infty} f(z-x)\, g(x)\, dx$$

$Z = X+Y$

$\therefore Y = Z-X$

$$= \int_{0}^{\infty} f(z-x)\, g(x)\, dx$$

$g(y) = g(z-x)$

$= f(z-x)$

$$= \int_{0}^{z} e^{-(z-x)}\, e^{-x}\, dx$$

$$= \int_{0}^{z} e^{-z}\, dx = x e^{-z}\Big|_{0}^{z} = z e^{-z}$$

$$= \frac{1}{\sqrt{2}\, 1^{2}}\, z^{2-1}\, e^{-z} \quad , z>0$$

Hence $Z \sim Gam(1,2)$. Thus, if $X \sim Gam(1,1)$ and $Y \sim Gam(1,1)$ then $X+Y \sim Gam(1,2)$.

Example (49) what is the Probability density of the sum of two independent r.v's, each of which is Standard Normal?

Solution: Let $Z = X+Y$, where $X \sim N(0,1)$ and $Y \sim N(0,1)$

- Hence, the density of X is given by:

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$$

Similarly, the density of Y is given by:

$$g(y) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{y^2}{2}}$$

Since X and Y are independent, the density of Z can be obtained by the method of convolution. The sum $Z = X+Y$ is between ($-\infty$) and (∞).

Hence, the density function of Z is given by: