$$h(z) = (f * g)(z)$$

$$= \int_{-\infty}^{\infty} f(z-x)\, g(x)\, dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{(z-x)^2}{2}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{(z-x)^2}{2} - \frac{x^2}{2}} dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4}} e^{-(x-\frac{z}{2})^2} dx$$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{4}} \int_{-\infty}^{\infty} e^{-(x-\frac{z}{2})^2} dx$$

Let $\frac{w}{\sqrt{2}} = x - \frac{z}{2} \Rightarrow \frac{1}{\sqrt{2}} dw = dx$, and $\frac{w^2}{2} = (x - \frac{z}{2})^2$

$$h(z) = \frac{1}{2\pi} e^{-\frac{z^2}{4}} \int_{-\infty}^{\infty} e^{-\frac{w^2}{2}} \cdot \frac{1}{\sqrt{2}} dw$$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{4}} \cdot \sqrt{\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}$$

$$= \frac{1}{2\sqrt{\pi}} e^{-\frac{z^2}{4}} = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2}\left(\frac{z-0}{\sqrt{2}}\right)^2}$$

Hence, sum of two standard Normal r.v's is again a normal r.v. with mean zero and variance 2.

Example (50)

what is the probability density of the sum of two independen r.v's, each of which is cauchy?

Solution: Let $Z = X + Y$, where $X \sim$ Cauchy (0, 1) and $Y \sim$ Cauchy . Hence the density function $f(x)$ of the r.v. X and Y is given

$$f(x) = \frac{1}{\pi(1+x^2)} \quad \text{and} \quad g(y) = \frac{1}{\pi(1+y^2)}$$

since X and Y are independent, the density function of Z can be obtained by the method of Convolution. The sum $Z = x + y$ is between $(-\infty)$ and $(+\infty)$. Hence, the density function of Z is:

$$h(z) = (f * g)(z)$$

$$= \int_{-\infty}^{\infty} f(z-x)\, g(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi[1+(z-x)^2]} \cdot \frac{1}{\pi(1+x^2)}\, dx$$

$$= \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{1}{1+(z-x)^2} \cdot \frac{1}{1+x^2}\, dx$$

we decompose the above integral using partial fraction decomposition. Hence

$$\frac{1}{1+(z-x)^2} \cdot \frac{1}{1+x^2} = \frac{2Ax+B}{1+x^2} + \frac{2C(z-x)+D}{1+(z-x)^2}$$

where; $A = \frac{1}{z(4+z^2)} = C$ and $B = \frac{1}{4+z^2} = D$

Then;

$$h(z) = \frac{1}{\pi^2 z^2(4+z^2)} \left[z \,\mathrm{Ln}\left(\frac{1+x^2}{1+(z-x)^2}\right) + z^2 \tan^{-1} x - z^2 \tan^{-1}(z-x) \right]_{-\infty}^{\infty}$$

$$= \frac{1}{\pi^2 z^2(4+z^2)} \left[0 + z^2\pi + z^2\pi \right]$$

$$= \frac{2}{\pi(4+z^2)}$$

Hence, the sum of two independent cauchy r.v's is not a cauchy r.v.

4.) Moment Generating Function Method:

we Know that if X and Y are independent r.v's, then

$$M_{x+y}(t) = M_x(t)\, M_y(t)$$

Example (51) Let $X \sim P(\lambda_1)$ and $Y \sim P(\lambda_2)$. what is the probability function of $X+Y$ if X and Y are independent.

Solution: since $X \sim P(\lambda_1)$ and $Y \sim P(\lambda_2)$, we get:

$$M_x(t) = e^{\lambda_1(e^t-1)} \text{ and } M_y(t) = e^{\lambda_2(e^t-1)}$$

$$M_{x+y}(t) = e^{\lambda_1(e^t-1)}\, e^{\lambda_2(e^t-1)}$$
$$= e^{(\lambda_1+\lambda_2)(e^t-1)}$$

that is $x+y \sim p(\lambda_1+\lambda_2)$. Hence the density Function of $Z = X+Y$ is given by:

$$h(z) = \begin{cases} \dfrac{e^{-(\lambda_1+\lambda_2)}}{z!}(\lambda_1+\lambda_2)^z &, z=0,1,2,\dots \\ 0 &, o.w \end{cases}$$

Example (52) what is the p.d.f. of the sum of two independent r.v's, each of which is gamma distribution with θ and α?

solutions

Let X and Y be two independent gamma r.v's with parameters θ and α, that is:

$X \sim Gam(\theta,\alpha)$ and $Y \sim Gam(\theta,\alpha)$.

Hence; $M_x(t) = (1-\theta)^{-\alpha}$
$M_y(t) = (1-\theta)^{-\alpha}$

$$M_{x+y}(t) = M_x(t)\, M_y(t)$$
$$= (1-\theta)^{-\alpha}(1-\theta)^{-\alpha}$$
$$= (1-\theta)^{-2\alpha}$$

Thus $X+Y$ has a m.g.f. of Gamma r.v. with parameters θ and 2α. Therefore

$$X+Y \sim Gam(\theta, 2\alpha).$$

[illegible]

M.Sc. Test. Aduansed Mathematical statistic

Prof. Dr. Jawad Al. Mosawi

1. Let $X \sim f(x) = \frac{2c}{3^x}$, $x = 1, 2, \ldots$ for some constant c. what is the probability that X is odd?

2. Let $X \sim f(x) = \frac{2}{x^2}$, $1 < x < 2$, find the p.d.f. of $Y = \sqrt{x}$.

3. Let $(X, Y) \sim p(x, y) = \frac{x-y}{20}$, $x = y, 1, 2, 3, 4$
$y = 0, 1, \ldots, x$ $\frac{1}{20}$

Find the marginal conditional probability of X given $Y = y$.

4. Let $X \sim f(x) = [\pi(1 + \{x - \theta\}^2]^{-1}$, $x \in R^1$.
Find the c.d.f. of X.

5. Let $(x, y) \sim f(x, y) = 4x$, $0 < x < \sqrt{y} < 1$, what is the density function of $\frac{X}{Y}$?

6. Let X and Y distributed as bivariate Bernoulli distⁿ. i.e. $(X, Y) \sim Ber(\theta, \lambda)$.
Find $E(X/Y)$.

– Good Luck –

أ.د. جواد كاظم خضير

٣٠/١٢/٢٠٢١

M.Sc. Course Advanced Mathematical Statistic
Prof. Dr. Jawad Al-Mosawi

1. A continuous r.v. X has the following density function:

$$f(x) = \begin{cases} \frac{3x^2}{\theta^3} & , \ 0 < x < \theta , \quad \theta > 0 \\ 0 & , \ o.w \end{cases}$$

what is the probability of X less than the ratio of the mode to the median, so that the mode of this distn. is θ?

$$P\left(X < \frac{\cancel{\theta}}{\frac{\cancel{\theta}}{\sqrt[3]{2}}}\right) = \int_0^{\sqrt[3]{2}} \frac{3x^2}{\theta^3}\, dx$$

2. Let $(X,Y) \sim Ber(\alpha, \beta)$. use the joint moment Generating function to find the $var(X/Y=y)$.

3. Let $f(x) = a e^{-ax}$, $x > 0$. Find $M_x(-3a)$.

4. Let $X \sim U(0,10)$. Compute $Pr\left[x + \frac{10}{x} > 7\right]$.

5. Let $f(x,y) = \begin{cases} 24xy & , \ x > 0, y > 0, \ 0 < x + y < 1. \\ 0 & , \end{cases}$

what is the conditional probability $P(X < \frac{1}{2} / Y = \frac{1}{4})$?

6. Let $f(x,y) = 4x$, $0 < x < \sqrt{y} < 1$.
w~~hat is the~~ density function of $X - Y$?

- Good Luck -

أ.د. جواد كاظم خضير

٢/١١/٢٠٢٢

Prof.Dr. **First attempt** **Department of Statistics**

Jawad kadhim khudhier **advanced mathematical statistics exam** **Master science degree**

2/4/2023

1. Let X and Y be two jointly random variables with j.p.d.f. :

$$f(x,y) = \begin{cases} 4x & ,0 < x < \sqrt{y} < 1 \\ 0 & ,o.w. \end{cases}$$

What is the distribution of $\frac{X}{Y}$?

2. If the random variable X is distributed as Normal with mean (1) and standard deviation (2), then what is $P(X^2 - 2X \leq 8)$.

3. Let

$$f(x) = \begin{cases} \frac{x}{4a} & ,0 < x \leq a \\ 0 & ,o.w. \end{cases}$$

Find the 25th percentile of the distribution.

4. Let X_1 and X_2 be two random variables with the following joint pmf:

$$p(x_1,x_2) = \begin{cases} \binom{x_2}{x_1}\left(\frac{1}{2}\right)^{x_1}\left(\frac{x_1}{15}\right) & ,x_1 = 1,\dots,5 \\ & ,x_2 = 0,\dots,x_1 \\ 0 & ,o.w. \end{cases}$$

Find $E(X_2)$.

5. Let X and Y be two jointly random variables with j.p.d.f. :

$$f(x,y) = \begin{cases} 2 & ,0 < y < x - 1\,; and\ 0 < x < 1 \\ 0 & ,o.w. \end{cases}$$

What is the conditional mean of Y given $X = x$?

Good luck

ترم ٤ ٢٠٢٢ - ٢٠٢٣

Non-Central Distributions

1- Non-central T-dist:

Let $T = \dfrac{Z+\delta}{\sqrt{\frac{W}{r}}}$ $\Rightarrow$ if $\delta = 0 \Rightarrow$ we get Central T-dist.

$Z \sim N(0,1)$ and $\therefore Z+\delta \sim N(\delta,1)$ indep. of

$W \sim \chi^2_{(r)}$

$$\therefore T = \frac{u}{\sqrt{w/r}}$$

$$g(Z,w) = g(Z) \cdot g(w)$$
$$= \frac{1}{\sqrt{2\pi}} e^{-Z^2/2} \cdot \frac{1}{\Gamma r/2 \; 2^{r/2}} w^{\frac{r}{2}-1} e^{-w/2}$$

Let $t = \dfrac{Z+\delta}{\sqrt{\frac{w}{r}}}$ and $u = w$

$$\therefore Z = \left(\frac{u}{r}\right)^{1/2} t - \delta$$

$$|J| = \begin{vmatrix} \frac{\partial Z}{\partial t} & \frac{\partial Z}{\partial u} \\ \frac{\partial w}{\partial t} & \frac{\partial w}{\partial u} \end{vmatrix} = \begin{vmatrix} \sqrt{\frac{u}{r}} & \frac{t}{2\sqrt{ur}} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{u}{r}}$$

$\therefore$ The j.p.d.f. of t and u is:-

$$h(t,u) = \frac{1}{\sqrt{2\pi}\,\Gamma\frac{r}{2}\, 2^{r/2}} u^{\frac{r}{2}-1} e^{-\frac{1}{2}\left[\left(\left(\frac{u}{r}\right)^{1/2} t - \delta\right)^2 + u\right]} \cdot \sqrt{\frac{u}{r}}$$

$$= \frac{u^{\frac{r-1}{2}}}{\sqrt{2\pi r}\,\Gamma\frac{r}{2}\, 2^{r/2}} e^{-\frac{1}{2}\left[\frac{u}{r}t^2 - 2t\delta\sqrt{\frac{u}{r}} + \delta^2 + u\right]}$$

but; $f(t,\delta) = \int_0^\infty h(t,u)\,du \quad \ldots(1)$

X

Let $u\left(\frac{t^2}{r}+1\right)=v^2 \Rightarrow u=\dfrac{v^2}{\left(\frac{t^2}{r}+1\right)}$

$$\Rightarrow \left|\frac{du}{dv}\right| = \frac{2vr}{t^2+r}$$

$$\therefore h(v,t) = \frac{\left(\frac{v^2}{t^2/r+1}\right)^{\frac{r-1}{2}}}{\sqrt{2\pi r}\; 2^{r/2}\,\Gamma r/2}\, e^{-\frac{1}{2}\left[v^2-2t\delta\frac{v}{\sqrt{t^2+r}}+\delta^2\right]} \cdot \frac{2vr}{t^2+r}$$

So, by adding $\mp \dfrac{t^2\delta^2}{t^2+r}$ to exponential term:

$$\therefore h(v,t) = \frac{v^{r-1}\left[\frac{r}{t^2+r}\right]^{\frac{r-1}{2}} \cdot \frac{2vr}{t^2+r}}{\sqrt{2\pi r}\; 2^{r/2}\,\Gamma\frac{r}{2}} \exp\left[-\frac{1}{2}\left\{v^2-2t\delta\frac{v}{\sqrt{t^2+r}} + \frac{t^2\delta^2}{t^2+r^2} - \frac{t^2\delta^2}{t^2+r^2} + \delta^2\right\}\right]$$

$$h(v,t) = \frac{v^r\left[\frac{r}{t^2+r}\right]^{\frac{r+1}{2}}}{\sqrt{\pi r}\; 2^{\frac{r-1}{2}}\,\Gamma\frac{r}{2}}\, e^{-\frac{1}{2}\left[\frac{-t^2\delta^2}{t^2+r}+\delta^2\right]-\frac{1}{2}\left[v-\frac{t\delta}{\sqrt{t^2+r^2}}\right]}$$

$$\therefore f(t,\delta) = \int_0^\infty \frac{v^r\left[\frac{r}{t^2+r}\right]^{\frac{r+1}{2}}}{\sqrt{\pi r}\; 2^{\frac{r-1}{2}}\,\Gamma\frac{r}{2}}\, e^{-\frac{1}{2}\left[\frac{-t^2\delta^2}{t^2+r}+\delta^2\right]-\frac{1}{2}\left[v-\frac{t\delta}{\sqrt{t^2+r^2}}\right]} dv$$

$$\boxed{\therefore f(t,\delta) = \frac{\left[\frac{r}{t^2+r}\right]^{\frac{r+1}{2}}}{2^{\frac{r-1}{2}}\sqrt{\pi r}\,\Gamma\frac{r}{2}}\, e^{-\frac{1}{2}\left(\frac{\delta^2 r}{t^2+r}\right)} \int_0^\infty v^r e^{-\frac{1}{2}\left[v-\frac{t\delta}{\sqrt{t^2+r}}\right]^2} dv}$$

Home work: in equation (1-1), Let expand $e^{t\delta\sqrt{\frac{u}{v}}}$ in a Series Show that:

$$\boxed{f(t,\delta) = \frac{e^{-\delta^2/2}}{\sqrt{\pi}\,\Gamma\frac{r}{2}} \sum_{i=0}^{\infty} \frac{(\delta\sqrt{2})^i}{i!} \cdot \frac{\Gamma\frac{r+i+1}{2}}{r^{\frac{i+1}{2}}} \cdot \frac{t^i}{\left(1+\frac{t^2}{r}\right)\left(\frac{r+i+1}{2}\right)}}$$

Solution: the Eq. (1-1)

$$f(t,\delta)=\int_0^\infty h(u,t)\,du=\int_0^\infty \frac{u^{\frac{r-1}{2}}}{\sqrt{2\pi r}\;2^{r/2}\,\Gamma\frac{r}{2}}\,e^{-\frac{1}{2}\left[\frac{u}{r}t^2-2t\delta\sqrt{\frac{u}{r}}+\delta^2+u\right]}du$$

$$=\frac{e^{-\delta^2/2}}{\sqrt{2\pi r}\;2^{r/2}\,\Gamma r/2}\int_0^\infty u^{\frac{r-1}{2}}\,e^{-\frac{1}{2}u\left(\frac{t^2}{r}+1\right)+t\delta\sqrt{\frac{u}{r}}}\,du$$

$$=K\int_0^\infty u^{\frac{r-1}{2}}\,e^{-\frac{1}{2}u\left(\frac{t^2}{r}+1\right)}\cdot e^{t\delta\sqrt{\frac{u}{r}}}\,du$$

$$=K\int_0^\infty u^{\frac{r-1}{2}}\,e^{-\frac{1}{2}u\left(\frac{t^2}{r}+1\right)}\left[1+t\delta\sqrt{\frac{u}{r}}+t^2\delta^2\cdot\frac{u}{r}\cdot\frac{1}{2!}+\cdots\right]du$$

$$=K\left[\int_0^\infty u^{\frac{r-1}{2}}\,e^{-\frac{1}{2}u\left(\frac{t^2}{r}+1\right)}du+\frac{t\delta}{\sqrt{r}}\int_0^\infty u^{\frac{r}{2}}\,e^{-\frac{1}{2}u\left(\frac{t^2}{r}+1\right)}du\right.$$
$$\left.+\frac{t^2\delta^2}{r\,2!}\int_0^\infty u^{\frac{r}{2}+\frac{1}{2}}\,e^{-\frac{1}{2}u\left(\frac{t^2}{r}+1\right)}du+\cdots\right]$$

Let $x=\frac{u}{2}\left(\frac{t^2}{r}+1\right)\Rightarrow u=\frac{2x}{\left(\frac{t^2}{r}+1\right)}\Rightarrow du=\frac{2dx}{\left(\frac{t^2}{r}+1\right)}$

$$\therefore f(t,\delta)=K\left[\int_0^\infty\left[\frac{2x}{\left(\frac{t^2}{r}+1\right)}\right]^{\frac{r-1}{2}}e^{-x}\cdot\frac{2}{\left(\frac{t^2}{r}+1\right)}dx+\cdots\right]$$

$$=K\left[\Gamma\frac{r+1}{2}\left\{2\left(\frac{t^2}{r}+1\right)\right\}^{-\left(\frac{r+1}{2}\right)}+\frac{t\delta}{\sqrt{r}}\,\Gamma\frac{r+2}{2}\left\{2\left(\frac{t^2}{r}+1\right)\right\}^{-\left(\frac{r+2}{2}\right)}\right.$$
$$\left.+\cdots\right]$$

$$=K\left[\Gamma\frac{r+1}{2}\,2^{\frac{r+1}{2}}\left\{\left(\frac{t^2}{r}+1\right\}^{-\left(\frac{r+1}{2}\right)}+r^{-\frac{1}{2}}\delta t\,\Gamma\frac{r+2}{2}\cdot 2^{\frac{r+2}{2}}\left(\frac{t^2}{r}+1\right)^{-\left(\frac{r+2}{2}\right)}\right.$$
$$\left.+\cdots\right]$$

$$f(t,\delta) = \frac{e^{-\delta^2/2}}{\sqrt{\pi}\,\Gamma\frac{r}{2}}\Big[\Gamma\tfrac{r+1}{2}\left(\tfrac{t^2}{r}+1\right)^{-\frac{(r+1)}{2}} r^{-\frac{1}{2}} + t\delta r^{-\frac{2}{2}}\,\Gamma\tfrac{r+2}{2}\,2^{\frac{1}{2}}\left(\tfrac{t^2}{r}+1\right)^{-\frac{(r+2)}{2}} + r^{-\frac{3}{2}} t^2\delta^2\,\Gamma\tfrac{r+3}{2}\,2^{1}\left(\tfrac{t^2}{r}+1\right)^{-\frac{(r+3)}{2}} + \cdots\Big]$$

$$= \frac{e^{-\delta^2/2}}{\sqrt{\pi}\,\Gamma\frac{r}{2}}\Big[\frac{\Gamma\frac{r+1}{2}}{r^{\frac{1}{2}}\left(\frac{t^2}{r}+1\right)^{\frac{r+1}{2}}} + \frac{(\delta t\sqrt{2})\,\Gamma\frac{r+2}{2}}{r^{\frac{3}{2}}\left(\frac{t^2}{r}+1\right)^{\frac{r+2}{2}}} + \frac{(t\delta\sqrt{2})^2}{r^{3/2}\left(\frac{t^2}{r}+1\right)^{\frac{r+3}{2}}} + \cdots\Big]$$

$$\therefore f(t,\delta) = \frac{e^{-\delta^2/2}}{\sqrt{\pi}\,\Gamma\frac{r}{2}}\sum_{i=0}^{\infty}\frac{(t\delta\sqrt{2})^{i}\,\Gamma\frac{r+i+1}{2}}{r^{\frac{i+1}{2}}\left(1+\frac{t^2}{r}\right)^{\frac{r+i+1}{2}}}$$

Home work:

Show that

$$E[w^{-K/2}] = \frac{2^{-\frac{K}{2}}\,\Gamma\frac{r-K}{2}}{\Gamma\frac{r}{2}}, \quad r>K.$$

Solution

$$T = \frac{Z+\delta}{\sqrt{\frac{w}{r}}} = (Z+\delta)\left(\frac{w}{r}\right)^{-1/2}$$

$$T = (Z+\delta)\,r^{\frac{1}{2}}w^{-\frac{1}{2}}$$

$$= Z r^{1/2} w^{-1/2} + \delta v^{1/2} w^{-1/2}$$

$$E[T] = E[Z r^{1/2} w^{-1/2}] + E[\delta \cdot v^{1/2} w^{-1/2}]$$

$Z \sim N(0,1)$

$w \sim \chi^2_{(r)} \quad \Rightarrow \quad E(Z) = 0$

$$\therefore E[T] = 0 + \delta r^{1/2} E(w^{-1/2}) = \delta r^{1/2} E(w^{-1/2}) \ldots\ldots ①$$

$$T^3 = (Z+S)^3 = (Z+S)(Z^2+S^2+2SZ)$$

$$ET^3 = (0+S)(1+S^2+0) = S(1+S^2)$$

$$= S\,S^2 + S^3 E W^{-\frac{3}{2}} r^{\frac{3}{2}}$$

$$E(W^{-k/2}) = \int_0^\infty W^{-k/2} f(w)\,dw$$

$$= \int_0^\infty w^{-k/2} \frac{w^{r/2-1}}{\Gamma\frac{r}{2}\,2^{r/2}} e^{-w/2}\,dw$$

$$= \int_0^\infty \frac{w^{\frac{r-k}{2}-1}}{\Gamma\frac{r}{2}\,2^{r/2}} e^{-w/2}\,dw$$

$$= \frac{\Gamma\frac{r-k}{2}\,2^{\frac{r-k}{2}}}{\Gamma\frac{r}{2}\,2^{r/2}} \underbrace{\int_0^\infty \frac{w^{\frac{r-k}{2}-1}}{\Gamma\frac{r-k}{2}\,2^{\frac{r-k}{2}}} e^{-w/2}\,dw}_{=1} \Big\} \cancel{\times} \frac{\sqrt{\frac{r-k}{2}}\,2^{\frac{r-k}{2}}}{\sqrt{\frac{r-k}{2}}\,2^{\frac{r-k}{2}}}$$

$$1$$

$$= \frac{\Gamma\frac{r-k}{2}\,2^{\frac{r-k}{2}}}{\Gamma\frac{r}{2}\,2^{r/2}} \times 1 = \frac{2^{-k/2}\,\Gamma\frac{r-k}{2}}{\Gamma\frac{r}{2}}$$

$$\therefore E(T) = \delta\, r^{1/2} E(W^{-1/2})$$

عوض k

$$= \delta\, \frac{r^{1/2}\, 2^{-1/2}\,\Gamma\frac{r-1}{2}}{\Gamma\frac{r}{2}} = \frac{\sqrt{r/2}\,\Gamma\frac{r-1}{2}}{\Gamma\frac{r}{2}}\,\delta$$

To Find ET^2 ?

$$T = \frac{Z+\delta}{\sqrt{W/r}} \Rightarrow T^2 = \frac{(Z+\delta)^2}{W/r}$$

$$T^2 = (Z+\delta)^2 \left(\frac{W}{r}\right)^{-1}$$

$$T^2 = (Z^2+2Z\delta+\delta^2)\left(\frac{W}{r}\right)^{-1}$$

$$ET^2 = [EZ^2 + 2\delta EZ + E\delta^2] \cdot r\, EW^{-1} = \frac{-2}{2} = -1 \quad \therefore k = 2$$

we have $EZ^2 = 1$, $EZ = 0$, $E\delta^2 = \delta^2$ and

$$E(W^{-1}) = \frac{2^{-2/2}\,\Gamma\frac{r-2}{2}}{\Gamma\frac{r}{2}} \rightarrow$$ عوض $k = 2$

$$= E\,W^{-\frac{2}{2}}$$

$$(Z+S)^3 W^{-1} r = W^{-\frac{1}{2}} r^{\frac{1}{2}}$$

$$= W^{-\frac{3}{2}} r^{\frac{3}{2}}$$

نعوضها

$$T^3 = \frac{(Z+S)^3}{\sqrt{\frac{W}{r}}^3} \longrightarrow \sqrt{\frac{W}{r}}\sqrt{\frac{W}{r}}\sqrt{\frac{W}{r}} = \frac{W}{r}\sqrt{\frac{W}{r}}$$

$$\frac{(Z+S)^3}{} =$$

$$\therefore ET^2 = (1+0+\delta^2)\, r \left[\frac{\Gamma\frac{r-2}{2}}{2\,\Gamma\frac{r}{2}} \right]$$

$$= (1+\delta^2)\, r \; \frac{[\frac{r-2}{2}-1]!}{2[\frac{r}{2}-1]!}$$

$$= (1+\delta^2)\, r \; \frac{[\cancel{\frac{r-4}{2}}]!\,[\frac{r-2}{2}-1]}{2[\frac{r}{2}-1][\cancel{\frac{r}{2}-2}]!} = (1+\delta^2)\, r \cdot \frac{1}{\cancel{2}(\frac{r-2}{\cancel{2}})}$$

$$\boxed{\therefore ET^2 = \frac{(1+\delta^2)\, r}{r-2}}$$

2. Non-Central χ^2 dist:

Let $$g(u) = \sum_{m=0}^{\infty} \frac{e^{-\theta/2} (\frac{\theta}{2})^m}{m!} \cdot \frac{u^{m+\frac{n}{2}-1}\, e^{-u/2}}{2^{m+\frac{n}{2}}\, \Gamma(m+\frac{n}{2})}, \quad u>0$$

(i) Show that $g(u)$ is a density function

(ii) Find the m.g.f. of u.

Sol

(i) $$\int_{\forall u} g(u)\, du \overset{?}{=} 1$$

$$\therefore \int_0^\infty \sum_{m=0}^{\infty} \frac{e^{-\theta/2} (\frac{\theta}{2})^m}{m!} \cdot \frac{u^{m+\frac{n}{2}-1}\, e^{-u/2}}{2^{m+\frac{n}{2}}\, \Gamma(m+\frac{n}{2})}\, du$$

$$\underbrace{\sum_{m=0}^{\infty} \frac{e^{-\theta/2} (\frac{\theta}{2})^m}{m!}}_{\substack{\Sigma \text{ Poisson dist.} \\ =1}} \; \underbrace{\int_0^\infty \frac{u^{m+\frac{n}{2}-1}\, e^{-u/2}}{2^{m+\frac{n}{2}}\, \Gamma(m+\frac{n}{2})}\, du}_{\substack{\int \text{ gamma dis} \\ =1}}$$

$\therefore$ $g(u)$ is a density function

(iii) $M_x(t) = E e^{xt}$

$$\therefore M_u(t) = E e^{ut} = \int_0^\infty e^{ut} g(u)\, du$$

$$= \sum_{m=0}^{\infty} \frac{e^{-\theta/2} (\frac{\theta}{2})^m}{m!} \int_0^\infty \frac{u^{m+\frac{n}{2}-1}}{2^{m+\frac{n}{2}} \Gamma m+\frac{n}{2}} e^{-u(\frac{1}{2}-t)} du$$

تحويل

Let $\underline{y = u(\frac{1}{2}-t)} \Rightarrow u = \frac{y}{1/2 - t} \Rightarrow du = \frac{dy}{1/2 - t}$

$$M_u(t) = \sum_{m=0}^{\infty} \frac{e^{-\theta/2} (\frac{\theta}{2})^m}{m!} \int_0^\infty (\frac{y}{1/2 - t})^{m+\frac{n}{2}-1} e^{-y} \cdot \frac{dy}{\frac{1}{2}-t}$$

$$= \sum_{m=0}^{\infty} \frac{e^{-\theta/2} (\frac{\theta}{2})^m}{m!} \cdot \frac{1}{2^{m+\frac{n}{2}} \Gamma m+\frac{n}{2}} \cdot \frac{1}{(\frac{1}{2}-t)^{m+\frac{n}{2}}} \int_0^\infty y^{m+\frac{n}{2}-1} e^{-y} dy$$

$$= \sum_{m=0}^{\infty} \frac{e^{-\theta/2} (\frac{\theta}{2})^m}{m!} \cdot \frac{1}{2^{m+\frac{n}{2}} \cancel{\Gamma m+\frac{n}{2}}} \cdot \frac{1}{(1/2-t)^{m+\frac{n}{2}}} \cdot \cancel{\Gamma m+\frac{n}{2}}$$

$$= \sum_{m=0}^{\infty} \frac{e^{-\theta/2} (\frac{\theta}{2})^m}{m!} \cdot \frac{1}{\cancel{2^{m+\frac{n}{2}}}} \cdot \frac{\cancel{2^{m+\frac{n}{2}}}}{(1-2t)^{m+n/2}}$$

$$= \sum_{m=0}^{\infty} \frac{e^{-\theta/2} (\frac{\theta}{2})^m}{m!} \cdot \frac{1}{(1-2t)^{m+n/2}}$$

$$= \frac{e^{-\theta/2}}{(1-2t)^{n/2}} \sum_{m=0}^{\infty} \frac{(\frac{\theta}{2})^m}{m!} (\frac{1}{1-2t})^m$$

$$= \frac{e^{-\theta/2}}{(1-2t)^{n/2}} \sum_{m=0}^{\infty} \frac{(\frac{\theta}{2-4t})^m}{m!} \frac{e^{-\frac{\theta}{2-4t}}}{e^{-\frac{\theta}{2-4t}}}$$

$$= \frac{e^{-\theta/2}}{(1-2t)^{n/2}} \cdot \frac{1}{e^{-\theta/(2-4t)}} \cdot \sum_{m=0}^{\infty} \frac{e^{-\theta/(2-4t)} [\theta/(2-4t)]^m}{m!}$$

$$= \frac{e^{-\theta/2 + \theta/(2-4t)}}{(1-2t)^{n/2}} =$$

$$M_u(t) = \frac{e^{\frac{\theta t}{(1-2t)}}}{(1-2t)^{n/2}} .$$

$$\frac{e^{-\theta/2}}{(1-2t)^{\frac{n}{2}}} \ast$$

$$\left\{ \frac{(\frac{\theta}{2-4t})^n}{m!} \right.$$

$$= \frac{e^{-\theta/2}}{(1-2t)^{\frac{n}{2}}} \cdot e^{\frac{\theta}{2-4t}}$$

$$= \frac{1}{(1-2t)^{\frac{n}{2}}} \cdot e^{-\frac{\theta}{2}+\frac{\theta}{2-4t}} \searrow e^{\frac{+4\theta t}{2(2-4t)}}$$

$$= \frac{1}{(1-2t)^{\frac{n}{2}}} \cdot e^{\theta t / 1-2t}$$

2 [illegible]

$$(\frac{1}{2}-t)^{m+\frac{n}{2}}$$

$$= 2^{m+\frac{n}{2}} (1-2t)^{m+\frac{n}{2}}$$

$$= \frac{1}{(\frac{1-2t}{2})^{m+\frac{n}{2}}}$$

$$= \frac{1}{\frac{(1-2t)^{m+\frac{n}{2}}}{2^{m+\frac{n}{2}}}}$$

$$= \frac{2^{m+\frac{n}{2}}}{(1-2t)^{m+\frac{n}{2}}}$$

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3- Non-Central F-dist:

If u and v are indep. r.v's

$$u \sim \chi'^{2}_{(r_1)}(\theta)$$

$$v \sim \chi'^{2}_{(r_2)}(\theta) \Rightarrow F = \frac{u/r_1}{v/r_2} \sim F'_{(r_1,r_2)}(\theta)$$

we seek prob. density function of j.p.d.f. of u and v.

$$h_1(u,v) = \frac{v^{\frac{r_2}{2}-1}\, e^{-v/2}}{2^{r_2/2}\,\Gamma\left(\frac{r_2}{2}\right)} \sum_{m=0}^{\infty} \frac{e^{-\theta/2}\left(\frac{\theta}{2}\right)^m}{m!} \cdot \frac{u^{m+\frac{r_1}{2}-1}\, e^{-u/2}}{2^{m+\frac{r_1}{2}}\,\Gamma\left(m+\frac{r_1}{2}\right)}$$

$$f = \frac{u/r_1}{v/r_2} = \frac{r_2 u}{r_1 v} \Rightarrow \boxed{u = \frac{f r_1 v}{r_2}} \text{ and } \boxed{w = v}$$

$$\Rightarrow u = \frac{f r_1 w}{r_2} \Rightarrow |J| = \frac{r_1 w}{r_2}, \text{ then:}$$

$$h_2(w,f) = \frac{w^{\frac{r_2}{2}-1}\, e^{-w/2}}{2^{r_2/2}\,\Gamma\left(\frac{r_2}{2}\right)} \sum_{m=0}^{\infty} \frac{e^{-\theta/2}\left(\frac{\theta}{2}\right)^m}{m!} \frac{\left[\frac{f r_1 w}{r_2}\right]^{m+\frac{r_1}{2}-1} e^{-\frac{1}{2}\left[\frac{f r_1 w}{r_2}\right]}}{2^{m+\frac{r_1}{2}}\,\Gamma\left(m+\frac{r_1}{2}\right)}$$

$$= \frac{w^{\frac{r_2}{2}-1}\, e^{-w/2}}{2^{r_2/2}\,\Gamma\left(\frac{r_2}{2}\right)} \sum_{m=0}^{\infty} \frac{e^{-\theta/2}\left(\frac{\theta}{2}\right)^m}{m!} \frac{\left(\frac{r_1}{r_2}\right)^{m+\frac{r_1}{2}-1} e^{-\frac{r_1 w f}{2}}}{2^{m+\frac{r_1}{2}}\,\Gamma\left(m+\frac{r_1}{2}\right)}$$

$$* f^{m+\frac{r_1}{2}-1}\, w^{m+\frac{r_1}{2}-1}$$

$$h_2(w,f) = \sum_{m=0}^{\infty} \frac{e^{-\theta/2}\left(\frac{\theta}{2}\right)^m}{m!} \cdot \frac{1}{2^{m+\frac{r_1}{2}}\,\Gamma\left(\frac{r_2}{2}\right)} \cdot \frac{\left(\frac{r_1}{r_2}\right)^{m+\frac{r_1}{2}-1}}{2^{m+\frac{r_1}{2}}\,\Gamma\left(m+\frac{r_1}{2}\right)}$$

$$* f^{m+\frac{r_1}{2}-1}\, w^{m+\frac{r_1}{2}-1+\frac{r_2}{2}}\, e^{-\frac{w}{2}-\frac{r_1 f w}{2 r_2}}$$

Let $(\frac{w}{2} + \frac{r_1 f w}{2 r_2}) = t$

$$\Rightarrow \quad \frac{w}{2}[1 + \frac{r_1}{r_2} f] = t$$

$$\Rightarrow \quad w = \frac{2t}{[1 + \frac{r_1}{r_2} f]} \quad , \; dw = \frac{2dt}{[1 + \frac{r_1}{r_2} f]}$$

$$h_3(t,f) = \sum_{m=0}^{\infty} \frac{e^{-\theta/2} (\frac{\theta}{2})^m}{m!} \cdot \frac{(\frac{r_1}{r_2})^{m+\frac{r_1}{2}-1}}{\Gamma\frac{r}{2}\,\Gamma m+\frac{r_1}{2}} \; f^{m+\frac{r_1}{2}-1}$$

$$* \frac{1}{2^{\frac{r_1+r_2}{2}+m}} \cdot e^{-t} \cdot t^{\frac{r_1+r_2}{2}+m-1} \cdot \frac{1}{[1+\frac{r_1}{r_2}f]^{\frac{r_1+r_2}{2}+m}}$$

Finaly ; to find $g(f)$

$$g(f) = \int_0^{\infty} h(t,f)\, dt$$

$$= \sum_{m=0}^{\infty} \frac{e^{-\frac{\theta}{2}} (\frac{\theta}{2})^m}{m!} \cdot \frac{(\frac{r_1}{r_2})^{m+\frac{r_1}{2}}}{\beta(m+\frac{r_1}{2}, \frac{r_2}{2})} \cdot \frac{f^{m+\frac{r_1}{2}-1}}{(1+\frac{r_1}{r_2}f)^{\frac{r_1+r_2}{2}+m}}$$

$$, \; f > 0$$

where $\beta(a,b)$ is a Beta function with $a, b > 0$

Some Special Discrete Bivariate Distributions

1. Bivariate Bernoulli Distribution:

توزيع برنولي الثنائي

Definition (42)

A discrete bivariate random variable (X, Y) is said to have the bivariate Bernoulli distribution if its joint probability density is of the form:

حفظ

$$P(x, y) = \begin{cases} \frac{1}{x!\,y!\,(1-x-y)!} P_1^x P_2^y (1-P_1-P_2)^{1-x-y} & , x, y = 0, 1 \\ 0 & , o.w \end{cases}$$

where $0 < P_1, P_2, P_1+P_2 < 1$ and $x+y \leq 1$. we denote a bivariate Bernoulli random variable by writing:

$$(X, Y) \sim Ber(P_1, P_2)$$

حفظ

Recall that; the joint moment generating function of X and Y is defined as

$$M(s, t) = E(e^{sX+tY})$$

Theorem (26)

Let $(X, Y) \sim Ber(P_1, P_2)$, where P_1 and P_2 are parameters. Then:

$$E(X) = P_1$$
$$E(Y) = P_2$$
$$var(X) = P_1(1-P_1)$$
$$var(Y) = P_2(1-P_2)$$
$$cov(X, Y) = -P_1P_2$$
$$M(s, t) = 1-P_1-P_2+P_1e^s+P_2e^t.$$

حفظ

proof: The joint m.g.f. of X and Y is given by:

$$M(s,t) = E(e^{sX+tY})$$
$$= \sum_{x=0}^{1}\sum_{y=0}^{1} p(x,y)\, e^{sx+ty}$$
$$= p(0,0) + p(1,0)\,e^{s} + p(0,1)\,e^{t} + p(1,1)\,e^{s+t}$$
$$= 1 - P_1 - P_2 + P_1 e^{s} + P_2 e^{t} + \boxed{0}\, e^{t+s}$$
$$= 1 - P_1 - P_2 + P_1 e^{s} + P_2 e^{t}.$$

$1-P_1-P_2$ ليه ٥

$\frac{1 \qquad P_1P_2}{!!\;!!\;\boxed{(-1)}}$ $(1-P_1-P_2)$ $=1$

يا صفر مشلون

مشلون ٥

The expected value of X is given by:

$$E(X) = \frac{\partial M(s,t)}{\partial s}\Big|_{(0,0)}$$
$$= \frac{\partial}{\partial s}\left(1 - P_1 - P_2 + P_1 e^{s} + P_2 e^{t}\right)\Big|_{(0,0)}$$
$$= P_1 e^{s}\Big|_{(0,0)}$$

$$\therefore \boxed{E(X) = P_1}$$

Similarly, the expected value of Y is given by:

$$E(Y) = \frac{\partial M(s,t)}{\partial t}\Big|_{(0,0)}$$
$$= \frac{\partial}{\partial t}\left(1 - P_1 - P_2 + P_1 e^{s} + P_2 e^{t}\right)\Big|_{(0,0)}$$
$$= P_2 e^{t}\Big|_{(0,0)}$$

$$\therefore E(Y) = P_2$$

Therefore the covariance of X and Y is

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y)$$

The product moment of X and Y is:

$$E(XY) = \frac{\partial^2 M(s,t)}{\partial t \partial s}\Big|_{(0,0)}$$

$$= \frac{\partial^2}{\partial t \partial s}\left(1 - P_1 - P_2 + P_1 e^{s} + P_2 e^{t}\right)\Big|_{(0,0)}$$

$$= \frac{\partial}{\partial t}\left(P_1 e^{s}\right)\Big|_{(0,0)}$$

$$= 0$$

Then; $Cov(X,Y) = 0 - P_1 P_2 = \boxed{-P_1 P_2}$

Similarly, it can be shown that:

$\boxed{E(X^2) = P_1}$ and $\boxed{E(Y^2)} = P_2$

Thus, we have:

$$var(X) = E(X^2) - [E(X)]^2 = P_1 - P_1^2 = P_1(1-P_1)$$

$$var(Y) = E(Y^2) - [E(Y)]^2 = P_2 - P_2^2 = P_2(1-P_2).$$

Theorem (27) Let $(X,Y) \sim B(P_1, P_2)$, where P_1 and P_2 are parameters. Then the Conditional distribution $p(y/x)$ and $p(x/y)$ are also Bernoulli and

$$E(Y/x) = \frac{P_2(1-x)}{1-P_1} = E(Y^2/x)$$

$$E(X/y) = \frac{P_1(1-y)}{1-P_2} = E(x^2/y)$$

$$var(Y/x) = \frac{P_2(1-P_1-P_2)(1-x)}{(1-P_1)^2}$$

$$var(X/y) = \frac{P_1(1-P_1-P_2)(1-y)}{(1-P_2)^2}$$

Proof:

$$P(y/x) = \frac{p(x,y)}{P(x)}$$

$$= \frac{p(x,y)}{\sum_{y=0}^{1} p(x,y)}$$

2 ✓

$$P(y/X=x) = \frac{P(x,y)}{P(x)} = \frac{P(x,y)}{\sum_{y=0} P(x,y)}$$

$$= \frac{P(x,y)}{P(x,0)+P(x,1)} \quad , x=0,1;\ y=0,1;\ 0 \le x+y \le 1. = \frac{P(x,y)}{P(x,0)+P(x,1)}$$

Hence; $P(1/0) = \dfrac{P(0,1)}{P(0,0)+P(0,1)} = \dfrac{P_2}{1-P_1-\cancel{P_2}+\cancel{P_2}}$

$$= \frac{P_2}{1-P_1}$$

and; $P(1/1) = \dfrac{P(1,1)}{P(1,0)+P(1,1)} = \dfrac{0}{P_1+0} = 0$

$$\Rightarrow E(Y/x=0) = \sum_{y=0}^{1} y\, P(y/0) = P(1/0) = \frac{P_2}{1-P_1}$$

$$\text{and } E(Y/x=1) = \sum_{y=0}^{1} y\, P(y/1) = P(1/1) = 0$$

Merging these together, we have

$$E(Y/x) = \frac{P_2}{1-P_1} \quad x0$$

Similarly, we compute:

$$E(Y^2/x=0) = \sum_{y=0}^{1} y^2\, P(y/0) = P(1/0)$$

$$= \frac{P_2}{1-P_1}$$

and: $E(Y^2/x=1) = P(1/1) = 0$

Therefore;

$$\text{var}(Y/x=0) = E(Y^2/x=0) - [E(Y/x=0)]^2$$

$$= \frac{P_2}{1-P_1} - \left(\frac{P_2}{1-P_1}\right)^2$$

$$= \frac{P_2(1-P_1-P_2)}{(1-P_1)^2}$$

and $\text{var}(Y/x=1) = 0$

Merging these together, we have جاء 2015 التالي

$$\boxed{\text{var}(Y/x) = \frac{P_2(1-P_1-P_2)(1-x)}{(1-P_1)^2}} \quad , x=0,1$$

The $E(X/Y)$ and $\text{var}(X/Y)$ can be obtained in a similar manner.

2. Bivariate Binomial Distribution:

Definition (43)

A discrete bivariate random variable (X, Y) is said to have the bivariate Binomial distribution with parameters n, P_1, P_2 if its joint probability density is of the form:

$$p(x,y) = \begin{cases} \dfrac{n!}{x!\,y!\,(n-x-y)!} P_1^{x} P_2^{y} (1-P_1-P_2)^{n-x-y} & , x,y = 0,1,\dots,n \\ 0 & , \text{o.w} \end{cases}$$

where $0 < P_1, P_2$, $P_1 + P_2 < 1$, $x+y \leq n$ and n is a positive integer. We denote a bivarite binomial random variable by writing:

$$(X,Y) \sim \operatorname{Bin}(n, P_1, P_2).$$

Example (53) A certain game involves rolling a fair die and watching the numbers of rolls of 4 and 5. What is the probability that in 10 rolls of the die one 4 and three 5 will be observed?

Solution:

Let X denote the number of 4 and Y denote the number of 5. Then the joint distribution of X and Y is bivariate binomial with $n = 10$, $P_1 = \frac{1}{6}$, $P_2 = \frac{1}{6}$ and $1 - P_1 - P_2 = \frac{4}{6}$. Hence the probabitiy that in 10 rolls of the die one 4 and three 5 will be observed is:

$$P(X = 1, Y = 3) = p(1,3)$$

$$= \frac{n!}{x!\,y!\,(1-x-y)!} P_1^{x} P_2^{y} (1-P_1-P_2)^{n-x-y}$$

$$= \frac{10!}{1!\,3!\,(10-1-3)!} \left(\frac{1}{6}\right)^{1} \left(\frac{1}{6}\right)^{3} \left(\frac{4}{6}\right)^{10-1-3}$$

$$= 0.0569$$

Hint: The following result generalizes the bivariate theorem and can be called trinomial theorem. Similar to the proof of binomial theorem, one can establish:

$$(a+b+c)^n = \sum_{x=0}^{n} \sum_{y=0}^{n} C_{x,y}^{n} \, a^x \, b^y \, c^{n-x-y}$$

where $0 \le x+y \le n$ and

$$C_{x,y}^{n} = \frac{n!}{x!\, y!\, (n-x-y)!}$$

Theorem (28)

Let $(X,Y) \sim Bin(n, P_1, P_2)$, where n, P_1 and P_2 are parameters. Then

$$E(X) = nP_1$$
$$E(Y) = nP_2$$
$$var(X) = nP_1(1-P_1)$$
$$var(Y) = nP_2(1-P_2)$$
$$cov(X,Y) = -nP_1P_2$$
$$M(s,t) = (1-P_1-P_2+P_1e^s+P_2e^t)^n.$$

Proof: First, we find the joint moment generating function of X and Y. The M(s,t) is given by:

$$M(s,t) = E(e^{sX+tY})$$
$$= \sum_{x=0}^{n}\sum_{y=0}^{n} e^{sx+ty} \, p(x,y)$$
$$= \sum_{x=0}^{n}\sum_{y=0}^{n} e^{sx+ty} \frac{n!}{x!\, y!\,(n-x-y)!} P_1^x P_2^y (1-P_1-P_2)^{n-x-y}$$
$$= \sum_{x=0}^{n}\sum_{y=0}^{n} \frac{n!}{x!\,y!\,(n-x-y)!} (e^sP_1)^x (e^tP_2)^y (1-P_1-P_2)^{n-x-y}$$
$$= (1-P_1-P_2+P_1e^s+P_2e^t)^n \text{, by theorem.}$$

$$\underline{E(X)} = \left.\frac{\partial M(s,t)}{\partial s}\right|_{(0,0)}$$

$$= \left.\frac{\partial}{\partial s}(1-P_1-P_2+P_1e^{s}+P_2e^{t})^{n}\right|_{(0,0)}$$

$$= \left.n(1-P_1-P_2+P_1e^{s}+P_2e^{t})^{n-1}P_1e^{s}\right|_{(0,0)}$$

$$= nP_1$$

$$\underline{E(Y)} = \left.\frac{\partial M(s,t)}{\partial t}\right|_{(0,0)}$$

$$= \left.\frac{\partial}{\partial t}(1-P_1-P_2+P_1e^{s}+P_2e^{t})^{n}\right|_{(0,0)}$$

$$= \left.n(1-P_1-P_2+P_1e^{s}+P_2e^{t})^{n-1}P_2e^{t}\right|_{(0,0)}$$

$$= nP_2$$

The product moment of X and Y is:

$$\underline{E(XY)} = \left.\frac{\partial^2 M(s,t)}{\partial t\partial s}\right|_{(0,0)}$$

$$= \left.\frac{\partial^2}{\partial t\partial s}(1-P_1-P_2+P_1e^{s}+P_2e^{t})^{n}\right|_{(0,0)}$$

$$= \left.\frac{\partial}{\partial t}\left[n(1-P_1-P_2+P_1e^{s}+P_2e^{t})^{n-1}P_1e^{s}\right|_{(0,0)}$$

$$= n(n-1)P_1P_2$$

$$\frac{\delta^2 M(s,t)}{\delta t^2} = nP_2\big[(1-P \dots P_2e^{t})^{n-1}\cdot e^{t} + e^{t}\{(n-1)(\dots P_2e^{t})^{n-2}P_2e^{t}$$

$$= nP_2 + n(n-1)P_2^2$$

Therefore the covariance of X and Y is

$$\underline{\mathrm{Cov}(X,Y)} = E(XY) - E(X)E(Y)$$

$$= n(n-1)P_1P_2 - (nP_1)(nP_2)$$

$$= n(n-1)P_1P_2 - n^2P_1P_2$$

$$= -nP_1P_2$$

$$var(x) = E x^2 - [E(x)]^2$$

$$E(x^2) = n(n-1)p_1^2 + np_1$$

$$\Rightarrow var(x) = n(n-1)p_1^2 + np_1 - n^2p_1^2$$

$$= n^2p_1^2 - np_1^2 + np_1 - n^2p_1^2$$

$$= np_1(1-p_1)$$

Similarly;

$$E(y^2) = n(n-1)p_2^2 + np_2$$

$$var(y) = n(n-1)p_2^2 + np_2 - n^2p_2^2$$

$$= np_2(1-p_2).$$

Remark: For any real numbers a and b, we have

$$\sum_{y=0}^{m} y \binom{m}{y} a^y b^{m-y} = ma(a+b)^{m-1}$$

[illegible]

and;

$$\sum_{y=0}^{m} y^2 \binom{m}{y} a^y b^{m-y} = ma(ma+b)(a+b)^{m-2}$$

where m is a positive integer.

[illegible]

Example (54)

If X equals the number of ones and Y equals the number of twos and threes when a pair of fair dice are rolled, then what the correlation coefficient of X and Y.

Solution:

The joint density of X and Y is bivariate Binomial and is given by:

$$P(x,y) = \frac{2!}{x!\,y!\,(2-x-y)!} \left(\frac{1}{6}\right)^x \left(\frac{2}{6}\right)^y \left(\frac{3}{6}\right)^{2-x-y}, \quad 0 \leq x+y \leq 2$$

By theorem

$$var(x) = np_1(1-p_1) = 2\left(\frac{1}{6}\right)\left(1-\frac{1}{6}\right) = \frac{10}{36}$$

$$var(Y) = nP_2(1-P_2) = 2\left(\frac{2}{6}\right)\left(1-\frac{2}{6}\right) = \frac{16}{36}$$

$$cov(X,Y) = -nP_1P_2$$
$$= -2\left(\frac{1}{6}\right)\left(\frac{2}{6}\right) = -\frac{4}{36}$$

Therefore;

$$corr(X,Y) = \frac{cov(X,Y)}{\sqrt{var(X)\,var(Y)}}$$
$$= \frac{-4/36}{\sqrt{\frac{10}{36}\cdot\frac{16}{36}}} = \frac{-4}{4\sqrt{10}}$$
$$= -0.3162$$

Theorem (29)

Let $(X,Y) \sim Bin(n, P_1, P_2)$, where n, P_1 and P_2 are parameters. Then the conditional distributions $p(y/x)$ and $p(x/y)$ are also binomial and:

$$E(Y/x) = \frac{P_2(n-x)}{1-P_1}$$

$$E(X/y) = \frac{P_1(n-y)}{1-P_2}$$

$$Var(Y/x) = \frac{P_2(1-P_1-P_2)(n-x)}{(1-P_1)^2}$$

$$Var(X/y) = \frac{P_1(1-P_1-P_2)(n-y)}{(1-P_2)^2}$$

Proof:

2. The marginal density $P_1(x)$ of X is given by:

$$P_1(x) = \sum_{y=0}^{n-x} \frac{n!}{x!\,y!\,(n-x-y)!}\cdot P_1^{x} P_2^{y}(1-P_1-P_2)^{n-x-y} \Big\} \times \frac{(n-x)!}{(n-x)!}$$

$$= \frac{n!\,P_1^{x}}{x!(n-x)!} \sum_{y=0}^{n-x} \frac{(n-x)!}{y!(n-x-y)!} P_2^{y}(1-P_1-P_2)^{n-x-y}$$

$$= C_x^n P_1^{x}(1-P_1\cancel{-P_2}\cancel{+P_2})^{n-x}$$

$$= C_x^n P_1^{x}(1-P_1)^{n-x} \quad , x = 0, 1, \dots, n$$

by binomial Theorem.

The conditional density of Y given X = x is

$$P(y/x) = \frac{P(x,y)}{P_1(x)} = \frac{P(x,y)}{C_x^n P_1^x (1-P_1)^{n-x}}$$

$$\frac{\cancel{x!}}{\cancel{x!}\,y!\,(n-x-y)!} \Big/ \frac{\cancel{x!}}{\cancel{x!}\,(n-x)!}$$

$$= \frac{(n-x)!}{(n-x-y)!\,y!} P_2^y (1-P_1-P_2)^{n-x-y} \quad (1-P_1)^{x-n}$$

$$= (1-P_1)^{x-n} C_y^{n-x} P_2^y (1-P_1-P_2)^{n-x-y}$$

Hence;

$$E(Y/x) = \sum_{y=0}^{n-x} y\, P(y/x)$$

$$= \sum_{y=0}^{n-x} y (1-P_1)^{x-n} C_y^{n-x} P_2^y (1-P_1-P_2)^{n-x-y}$$

$$= (1-P_1)^{x-n} \sum_{y=0}^{n-x} y\, C_y^{n-x} P_2^y (1-P_1-P_2)^{n-x-y}$$

$$= \sum y\, C_y^n a^y b^{n-1} = a n (a+b)^{n-1}$$

$$= (1-P_1)^{x-n} P_2 (n-x)(1-P_1)^{n-x-1} \qquad \cancel{-P_1-P_2}$$

تكون صحيح في التعويض موجود

$$= \frac{P_2(n-x)}{1-P_1}$$

$$= \frac{P_2(n-x)}{(1-P_1-P_2+P_2)} = \frac{P_2(n-x)}{(1-P_1)^{n-x}}$$

Next, we find the conditional variance of Y ginve X = x.

$$E(Y^2/x) = \sum_{y=0}^{n-x} y^2\, P(Y/x)$$

$$= \sum_{y=0}^{n-x} y^2 (1-P_1)^{x-n} C_y^{n-x} P_2^y (1-P_1-P_2)^{n-x-y}$$

$$= (1-P_1)^{\boxed{x-n} \leftarrow -(n-x)} \sum_{y=0}^{n-x} y^2 C_y^{n-x} P_2^y (1-P_1-P_2)^{n-x-y}$$

$$= (1-P_1)^{x-n} P_2 (n-x)(1-P_1)^{n-x-2} [(n-x)P_2+1-P_1-P_2]$$

$$= \frac{P_2(n-x)[(n-x)P_2+1-P_1-P_2]}{(1-P_1)^2}$$

$$\sum y^2 C_y^m a^y b^{m-1} = a m (am+b)(a+b)^{m-2}$$

Hence, the Conditional variance of Y given $X = x$ is:

$$var(Y/x) = E(Y^2/x) - [E(Y/x)]^2$$

$$= \frac{P_2(n-x)[(n-x)P_2 + 1 - P_1 - P_2]}{(1-P_1)^2} - \left(\frac{P_2(n-x)}{1-P_1}\right)^2$$

$$= \frac{P_2(1-P_1-P_2)(n-x)}{(1-P_1)^2}$$

similarly, one can establish:

$$E(X/y) = \frac{P_1(n-y)}{1-P_2} \quad \text{and}$$

$$var(X/y) = \frac{P_1(1-P_1-P_2)(n-y)}{(1-P_2)^2} .$$

Note: $P(y/x)$ in the previous theorem is a univariate binomial probability density function. To see this observe that:

$$(1-P_1)^{x-n} C_y^{n-x} P_2^y (1-P_1-P_2)^{n-x-y} = C_y^{n-x}\left(\frac{P_2}{1-P_1}\right)^y\left(1 - \frac{P_2}{1-P_1}\right)^{n-x-y}$$

Hence;

$$Y/x \sim Bin\left((n-x), \frac{P_2}{1-P_1}\right).$$

Example(55)

Let W equal the weight of soap in a 1-Kilogram box. suppose $P(W<1) = 0.02$ and $P(W>1.072) = 0.08$. Call a box of soap light, good, or heavy depending on weether on $W<1$, $1 \leq W \leq 1.072$ or $W>1.072$, respectively. In a random Sample of 50 boxes, let X equal the number of light boxes and Y the number of good boxes.

Compute:

1. the regression curve $[E(Y/x)]$ of Y on X.
2. the scedastic curve $[Var(Y/x)]$ of Y on X.

C_0^{n-y}