2-2-2 **Newton-Raphson Method**

This method is considered one of the best ways to solve equations, and it is an iterative numerical method for solving, it can be used to find the approximate root for the equation when it easy to find the derivative.

Deriving Newton-Raphson's Formula:

Let $f(x)$ be a continuous and differentiable function on the [a, b], and let (x_0) be the initial value of the root. Then the value of the new root is equal to

$x_1 = x_0 + h$ Where h is the correction value in x_0:

$$f(x_1) = f(x_0 + h)$$

Using Taylor's formula at point x_0:

$$f(x_1) = f(x_0) + \frac{h\ f'(x_0)}{1!} + \frac{h^2 f''(x_0)}{2!} + \frac{h^3\ f'''(x_0)}{3!} + \cdots\cdots, but\ h = (x_1 - x_0)$$

$$f(x_1) = f(x_0) + (x_1 - x_0)\ f'(x_0) + \frac{(x_1 - x_0)^2 f''(x_0)}{2} + \frac{(x_1 - x_0)^3\ f'''(x_0)}{6} + \cdots\cdots$$

Let x_1 be close to the root of x_0, then $f(x) = 0$, and likewise, x_1 is close to x_2. Therefore, $(x_1 - x_0)$ has a very small value, and when this value is squared, it is even smaller and close to zero. The same applies to higher powers, so these powers of two or more can be neglected, resulting in the following equation.

$$f(x_0) + (x_1 - x_0)\ f'(x_0) = 0$$
$$(x_1 - x_0)\ f'(x_0) = -f(x_0)$$
$$(x_1 - x_0) = \frac{-f(x_0)}{f'(x_0)}$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We get the Newton-Raphson's equation

$$x_{n+1} = x_n - \frac{f(x_n)}{\acute{f}(x_n)} \quad n = 0, 1, 2, 3, \ldots..$$

Steps of Newton-Raphson method

1- Choose appropriate initial root $x_0 \in [a, b]$

2- Set $n = 0$

3- Calculate $x_{n+1} = x_n - \frac{f(x_n)}{\acute{f}(x_n)}$

4- Set $n = n + 1$ and continue in the iterative processes, until we get the stop condition is satisfied:

$$|x_{n+1} - x_n| < \epsilon$$

Example7: Use Newton- Raphson algorithm to find the approximate root of the following equation

$$f(x) = x^3 - x^2 + 2x - 1 \quad ,[0,1] \quad \varepsilon = 0$$

Solution:

$$x_0 = \frac{0+1}{2} = 0.5$$

$$f(x) = x^3 - x^2 + 2x - 1$$

$$f(x_0) = f(0.5) = (0.5)^3 - (0.5)^2 + 2(0.5) - 1$$

$$f(x_0) = f(0.5) = -0.125$$

$$f'(x_0) = 3x^2 - 2x + 2$$

$$f'(x_0) = f(0.5) = 3(0.5)^2 - 2(0.5) + 2$$

$$f'(x_0) = f(0.5) = 1.75$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.5 - \frac{-0.125}{1.75}$$

$$x_1 = 0.5714$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.5714 - \frac{-0.14}{1.836}$$

$$x_2 = 0.647$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 0.647 - \frac{0.1462}{1.9618}$$

$$x_3 = 0.572$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$x_4 = 0.572 - \frac{0.004}{1.8375}$$

$$x_4 = 0.57$$

$$x_5 = 0.57$$

$$\therefore |x_5 - x_4| \leq \varepsilon$$

$$\therefore \textit{the root is } (0.57)$$

2-3Recursive formulas

In numerical analysis, there is a reversal of the steps taken to reach a specific analytical solution

2-3-1 Square root

The square root of N can be obtained from the following equation

$$f(x) = x^2 - N$$

Then we apply the Newton-Raphson method as follows

$$x_{i+1} = x_i + \frac{f(x_i)}{f'(x_i)}$$

$$x_{i-1} = x_i - \frac{x_i^2 - N}{2x_i}$$

$$x_{i+1} = x_i - \frac{1}{2}x_i + \frac{N}{2x_i}$$

$$x_{i+1} = \frac{1}{2}x_i + \frac{N}{2x_i}$$

$$x_{i+1} = \frac{1}{2}\left[x_i + \frac{N}{x_i} \right]$$

Example10

Calculate the square root of $\sqrt{30}$ for six numbers?

Solution:

As $\sqrt{25}$ =5, $\sqrt{36}$ =6 then we assume $x_0 = 5.5$

Then

$$x_1 = \frac{1}{2}\left(5.5 + \frac{30}{5.5}\right) = 5.477273$$

$$x_2 = \frac{1}{2}\left(5.477273 + \frac{30}{5.477273}\right) = 5.477225$$

This result is equal for six numbers

H.W

1- Calculate the square root of $\sqrt{40}$ for five numbers?

2- Calculate the square root of $\sqrt{130}$ for four numbers?

2-3-2 The Reciprocal

The reciprocal of the number N can be obtained by applying Newton-Raphson's formula as follows

$$f(x) = \frac{1}{x_i} - N$$

$$\acute{f}(x) = -\frac{1}{x^2{}_i}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{\acute{f}(x_i)}$$

$$x_{i+1} = x_i + x_i - Nx_i^2$$

$$x_{i+1} = x_i(2 - Nx_i)$$

Example11:- Calculate the reciprocal of the number N = 3 by the iterative method using the initial value $x_0 = 0.3$

Solution:

$x_{i+1} = x_i(2 - 3x_i)$ then

$x_1 = (0.3)(1.1) = 0.33$

$x_2 = (0.33)(1.01) = 0.3333$

$x_3 = (0.3333)(1.0001) = 0.333333$

H.W

1- Calculate the reciprocal of the number N = 7 by the iterative method using the initial value $x_0 = 0.14$.

2- Calculate the reciprocal of the number N = 8 by the iterative method.

2-3-3 The Reciprocal of Square Root

i$f(x) = \frac{1}{x^2} - N$

So it is

$$\acute{f}(x) = -\frac{2}{x^3{}_i}$$

Substituting into the Newton-Raphson formula, we get the following iterative expression:

$$x_{i+1} = \frac{x_i}{2}(3 - Nx_i^2)$$

Proof

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_{i+1} = x_i - \frac{\frac{1}{x_i^2} - N}{\frac{-2}{x_i^3}}$$

$$x_{i+1} = x_i + \left(\frac{1}{x_i^2} - N\right)\frac{x_i^3}{2}$$

$$x_{i+1} = x_i + \frac{x_i^3}{2x_i^2} - \frac{Nx_i^3}{2}$$

$$x_{i+1} = x_i + \frac{x_i}{2} - \frac{Nx_i^3}{2}$$

$$x_{i+1} = \frac{3}{2}x_i - \frac{Nx_i^3}{2}$$

$$x_{i+1} = \frac{x_i}{2}\left(3 - Nx_i^2\right)$$

Example:-

Calculate the reciprocal of the square root of N = 20 for seven decimal places?

Solution

$$\frac{1}{\sqrt{25}} = \frac{1}{5} = 0.20$$

$$\frac{1}{\sqrt{16}} = \frac{1}{4} = 0.25$$

Then

$x_0 = 0.22$, *By compensation in* $(1-3)$ *we get*

$$x_1 = \frac{0.22}{2}(3 - 20(0.22)^2) = 0.22352$$

$$x_2 = \frac{0.22352}{2}(3 - 20(0.22352)^2) = 0.2236067$$

Etc ….

H.W

1- Calculate the reciprocal of the square root of N = 10 for seven decimal places?

2- Calculate the reciprocal of the square root of N = 30 for seven decimal places?