Chapter Three

Finite Differences

In this chapter we will discuss types of finite differences with their tables, which are considered the basis on which the theory of numerical analysis was established.

The function used here is an equidistant discrete function and is as follows:

$$
\begin{aligned}
&x_0 && , y_0 = f(x_0) = f_0 \\
&x_1 = x_0 + h && , y_1 = f(x_1) = f(x_0 + h) = f_1 \\
&\vdots \\
&\vdots \\
&x_n = x_0 + nh && , y_n = f(x_n) = f(x_0 + nh) = f_n
\end{aligned}
$$

There are three types of finite differences:

1- Forward differences.

2- Backward differences.

3- Central differences.

Forward differences: It is symbolized by (Δ) the symbol and known as the equation

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta f(x) = f(x_1) - f(x_0)$$

$$\Delta_{y_0} = y_1 - y_0$$

It is called the first forward difference, and we get the second difference by taking the difference for the first difference.

$$\Delta^2_{y_0} = \Delta_{y_1} - \Delta_{y_0}$$

$$\Delta^2_{y_0} = y_2 - y_1 - y_1 + y_0$$

$$\Delta^2_{y_0} = y_2 - 2y_1 + y_0$$

$$\Delta^n_{y_0} = \Delta^{n-1}_{y_1} - \Delta^{n-1}_{y_0}$$

The differences in rank K can be expressed as follows:

$$\Delta^k_{y_i} = \Delta^{k-1}_{y_{i+1}} - \Delta^{k-1}_{y_i}$$

Forward Difference Table

The forward differences of any order can be obtained by subtracting the upper differences from the lower differences in the adjacent column to the left. Note that the location of the odd $(\Delta^1, \Delta^3, \Delta^5)$ differences is opposite the middle of the distance between the initial point values, while the even $(\Delta^2, \Delta^4, \Delta^6)$ differences are in a straight line opposite the initial values.

x_i	y_i	Δ_y	Δ^2_y	Δ^3_y	Δ^4_y
X_0	y_0				
		Δ_{y_0}			
X_1	y_1		$\Delta^2_{y_0}$		
		Δ_{y_1}		$\Delta^3_{y_0}$	
X_2	y_2		$\Delta^2_{y_1}$		$\Delta^4_{y_0}$
		Δ_{y_2}			
X_3	y_3				
X_{n-3}	y_{n-3}				
X_{n-2}	y_{n-2}				
X_{n-1}	y_{n-1}				
		$\Delta_{y_{n-1}}$			
X_n	y_n				

Example: Write a forward difference table for the following function:

$$f(x) = x^4 - 5x^2 + 3 \quad , x = -1, 0, 1, 2, 3$$

Solution: Substitute the value of x into the equation, extract the values of y, and then find the differences.

x_i	y_i	Δ_y	Δ_y^2	Δ_y^3	Δ_y^4
-1	-1				
		4			
0	3		-8		
		-4		12	
1	-1		4		24
		0		36	
2	-1		40		
		40			
3	39				

Homework (1): Make a forward difference table for the following data.

x_i	3	6	9	12	15
y_i	42	40	39	36	30

Homework (2): Calculate the first forward differences of the following function.

$$f(x) = x^2$$

2- Backward differences

Backward differences are denoted by the symbol $(\nabla)$and defined by the following equation.

$$\nabla f(x) = f(x) - f(x-h)$$

The first backward differences are calculated as follows:

$$\nabla y_0 = y_0 - y_{-1}$$
$$\nabla y_{-1} = y_{-1} - y_{-2}$$
$$\vdots$$
$$\nabla y_i = y_i - y_{i-1}$$

The second backward differences are calculated:

$$\nabla^2 y_0 = \nabla(\nabla y_0)$$
$$\nabla^2 y_0 = \nabla(y_0 - y_{-1})$$
$$\nabla^2 y_0 = y_0 - y_{-1} - y_{-1} - y_{-2}$$
$$\nabla^2 y_0 = y_0 - y_{-1} + y_{-2}$$
$$\vdots$$
$$\nabla^2 y_i = y_i - y_{i-1} + y_{i-2}$$

In general

$$\nabla^n y_i = \sum_{j=0}^{n} (-1)^j \binom{n}{j} y_{i-j}$$

The table of backward differences is as follows:

x_i	y_i	∇_y	∇_y^2	∇_y^3	∇_y^4	∇_y^5	∇_y^6
X_{-6}	y_{-6}						
		$\nabla_{y_{-5}}$					
X_{-5}	y_{-5}		$\nabla^2_{y_{-4}}$				
		$\nabla_{y_{-4}}$		$\nabla^3_{y_{-3}}$			
X_{-4}	y_{-4}		$\nabla^2_{y_{-3}}$		$\nabla^4_{y_{-2}}$		
		$\nabla_{y_{-3}}$		$\nabla^3_{y_{-2}}$		$\nabla^5_{y_{-1}}$	
X_{-3}	y_{-3}		$\nabla^2_{y_{-2}}$		$\nabla^4_{y_{-1}}$		$\nabla^6_{y_0}$
		$\nabla_{y_{-2}}$		$\nabla^3_{y_{-1}}$		$\nabla^5_{y_0}$	
X_{-2}	y_{-2}		$\nabla^2_{y_{-1}}$		$\nabla^4_{y_0}$		
		$\nabla_{y_{-1}}$		$\nabla^3_{y_0}$			
X_{-1}	y_{-1}		$\nabla^2_{y_0}$				
		∇_{y_0}					
X_0	y_0						

Example: Construct backward differences table for the following data.

x_i	3	6	9	12	15
y_i	42	40	39	36	30

Solution:

x_i	y_i	∇_y	∇_y^2	∇_y^3	∇_y^4
3	42				
		-2			
6	40		1		
		-1		-3	
9	39		-2		2
		-3		-1	
12	36		-3		
		-6			
15	30				

H.w.1 Calculate the first backward differences of the following function $f(x) = x^2$

3- The central differences

The central differences are denoted by the symbol δ , and the equation for the differences is defined by the following formula.

$$\delta\ f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

The table of central differences is as follows:

x_i	y_i	δ_{y_i}	$\delta^2_{y_i}$	$\delta^3_{y_i}$	$\delta^4_{y_i}$	$\delta^5_{y_i}$	$\delta^6_{y_i}$
x_{-3}	y_{-3}						
		$\delta_{y_{\frac{-5}{2}}}$					
x_{-2}	y_{-2}		$\delta^2_{y_{-2}}$				
		$\delta_{y_{\frac{-3}{2}}}$		$\delta^3_{y_{\frac{-3}{2}}}$			
x_{-1}	y_{-1}		$\delta^2_{y_{-1}}$		$\delta^4_{y_{-1}}$		
		$\delta_{y_{\frac{-1}{2}}}$		$\delta^3_{y_{\frac{-1}{2}}}$		$\delta^5_{y_{\frac{-1}{2}}}$	
x_0	y_0		$\delta^2_{y_0}$		$\delta^4_{y_0}$		$\delta^6_{y_0}$
		$\delta_{y_{\frac{1}{2}}}$		$\delta^3_{y_{\frac{1}{2}}}$		$\delta^5_{y_{\frac{1}{2}}}$	
x_1	y_1		$\delta^2_{y_1}$		$\delta^4_{y_1}$		
		$\delta_{y_{\frac{3}{2}}}$		$\delta^3_{y_{\frac{3}{2}}}$			
x_2	y_2		$\delta^2_{y_2}$				
		$\delta_{y_{\frac{5}{2}}}$					
x_3	y_3						

The general equation for central differences is:

$$\delta^{2k}_{y_i} = \delta^{2k-1}_{y_{i+\frac{1}{2}}} - \delta^{2k-1}_{y_{i-\frac{1}{2}}}$$

$$when \quad n = 2k,\ i = 0,\mp1,\mp2,...$$

$$\delta^{2k+1}_{i+\frac{1}{2}} = \delta^{2k}_{i+1} - \delta^{2k}_{i}$$

$$\delta^{2k+1}_{i-\frac{1}{2}} = \delta^{2k}_{i} - \delta^{2k}_{i-1}$$

$$when\ n = 2k+1, \quad i = 0,\mp1,\mp2,...$$

For example:

$\delta_{y_{\frac{1}{2}}} = y_1 - y_0$

$\delta_{y_{-\frac{1}{2}}} = y_0 - y_1$

$\delta^2{}_{y_{\frac{1}{2}}} = \delta\, y_1 - \delta\, y_0$

H.W:

1- Find $\delta\ f(x)$ when $f(x) = x^2$

2- Calculate the forward differences of the function $y = x^3 - 5x^2 + 2, \quad x = -1,0,1,2,3,4$

3- Calculate the first and second forward differences of the following function

$y_s = s(s-1)(s-2)$