المادة الدراسية :- هياكل متقطعة (Discrete structures)

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Introduction

The study of the discrete structures used to represent discrete objects

• Many discrete structures are built using sets

Sets = collection of objects

Examples of discrete structures built with the help of sets:

• Combinations

• Relations

• Graphs

Set

A **set** is a (unordered) collection of objects. Theseobjects are sometimes called **elements** or **members** of the set. The ordering of the elements is not important and repetition of elements is ignored, for example {1, 3, 1, 2, 2, 1} = {1, 2, 3}.

One usually uses capital letters, A,B,X, Y, . . . , to denote sets, and lowercase letters, a, b, x, y, . . ., to denote elements of sets.

Below you'll see just a sampling of items that could be considered as sets:

- The items in a store
- The English alphabet
- Even numbers

A set could have as many entries as you would like. It could have one entry, 10 entries, 15 entries, infinite number of entries, or even have no entries at all! For example, in the above list the English alphabet would have 26 entries, while the set of even numbers would have an infinite number of entries.

Each entry in a set is known as an **element or member**

Sets are written using curly brackets "{" and "}", with their elements listed in between. For example the English alphabet could be written as {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z}, and even numbers could be {0,2,4,6,8,10,...}

Note: the dots at the end indicating that the set goes on infinitely .

Principles:

∈belong to

∉not belong to

⊆ subset

⊂proper subset, For example, {a, b} is a proper subset of {a, b, c}, but {a, b, c} is not a proper subset of {a, b, c}.

⊄not subset

So we could replace the statement "a is belong to the alphabet" with a {alphabet} and replace the statement "3 is not belong to the set of even numbers" with 3 {Even numbers}

Now if we named our sets we could go even further. Give the set consisting of the **alphabet** the name A, and give the set consisting of **even numbers** the name E.

We could now write

a∈ A

and 3∉ E.

Problem

Let A = {2, 3, 4, 5} and C = {1, 2, 3, . . ., 8, 9}, Show that A is a proper subset of C.

Answer

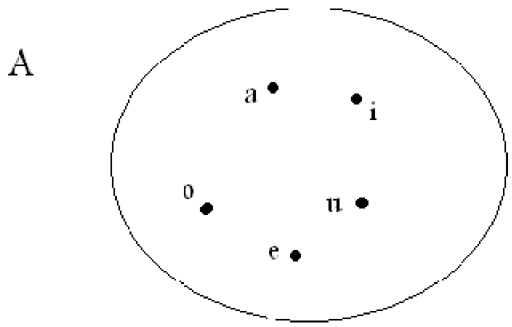
Each element of A belongs to C so A⊆ C. On the other hand, 1 ∈ C but 1 ∉ A. Hence A ≠ C. Therefore A is a proper subset of C.

There are three ways to specify a particular set:

1) By list its members separated by commas and contained in braces { }, (if it is possible), for example, A= {a,e,i,o,u}

2) By state those properties which characterize the elements in the set, for example, A={x:x is a letter in the English alphabet, x is a vowel}

3) Venn diagram: (A graphical representation of sets).



Example (1)

A={x:x is a letter in the English alphabet, x is a vowel}

e∈ A (e is belong to A)

f∉ A (f is not belong to A)

Example (2)

X is the set {1,3,5,7,9}

3 ∈ X

and 4 ∉ X

Example (3)

Let E = {x | x2 − 3x + 2 = 0} → (x-2)(x-1)=0 → x=2 & x=1

E = {2, 1}, and 2∈ E

Universal set, empty set:

In any application of the theory of sets, the members of all sets under investigation usually belong to some fixed large set called the universal set. For example, in human population studies the universal set consists of all the people in the world. We will let the symbol U denotes the universal set.

The set with no elements is called the empty set or null set and is denoted by ∅ or {}

Subsets:

Every element in a set A is also an element of a set B, then A is called a subset of B. We also say that B contains A. This relationship is written:

$A \subset B$ or $B \supset A$

If A is not a subset of B, i.e. if at least one element of Adose not belong to B,

we write $A \not\subset B$.

Example 4:

Consider the sets. A = {1,3,4,5,8,9} B = {1,2,3,5,7} and C ={1,5}

Then $C \subset A$ and $C \subset B$ since 1 and 5, the element of C, are also members of A and B.

But $B \not\subset A$ since some of its elements, e.g. 2 and 7, do not belong to A. Furthermore, since the elements of A,B and C must also belong to the universal set U, we have that U must at least the set {1,2,3,4,5,7,8,9}.

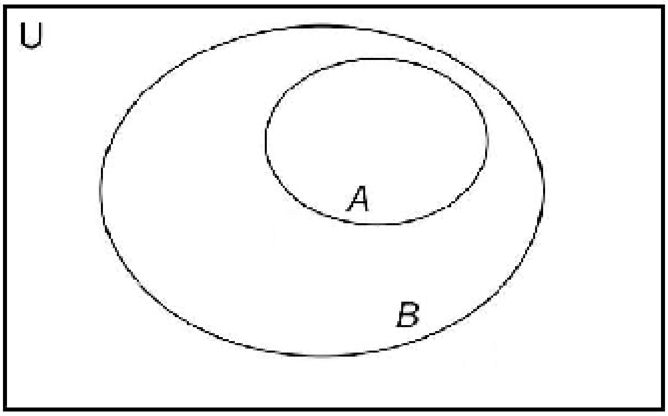
$A\subset B : \{\forall\ x\in A \Rightarrow x\in B$

$A\not\subset B :\{\exists\ x\in A$ but $x\notin B$

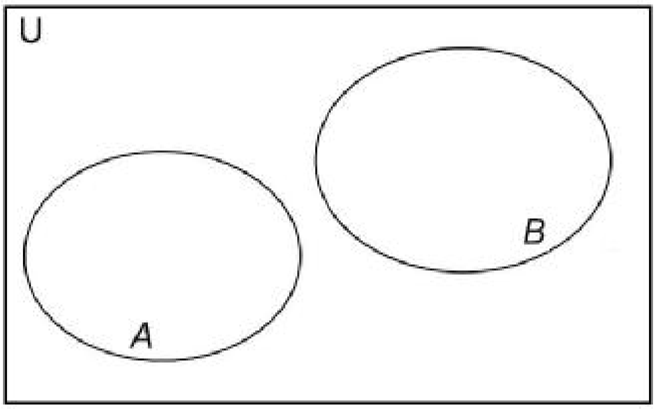
∀: For all لكل

∃: There exists يوجدعلىالاقل

The notion of subsets is graphically illustrated below:



A is entirely within B so A⊂ B.



A and B are disjoint or (A ∩ B = ∅) so we could write A ⊄ B and B ⊄ A.

Set of numbers:

Several sets are used so often, they are given special symbols.

N = the set of *natural numbers* or positive integers

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

Z = the set of all integers: . . . ,−2,−1, 0, 1, 2, . . .

$$\mathbb{Z} = \mathbb{N} \cup \{\ldots, -2, -1\}$$

Q = the set of rational numbers

$$\mathbb{Q} = \mathbb{Z} \cup \{\ldots, -1/3, -1/2, 1/2, 1/3, \ldots, 2/3, 2/5, \ldots\}$$

Where Q ={ a/b : a , b ∈ Z, b ≠ 0}

R = the set of real numbers

$$\mathbb{R} = \mathbb{Q} \cup \{\ldots, -\pi, -\sqrt{2}, \sqrt{2}, \pi, \ldots\}$$

C = the set of complex numbers

$$\mathbb{C} = \mathbb{R} \cup \{i, 1+i, 1-i, \sqrt{2}+\pi i, \ldots\}$$

Where C={ x + iy ; x , y ∈R; i = √-1}

Observe that **N** ⊂ **Z** ⊂ **Q** ⊂ **R** ⊂ **C**.

Theorem 1:

For any set A, B, C:

1- ∅ ⊂ A ⊂ U.

2- A ⊂ A.

3- If A ⊂ B and B ⊂ C, then A⊂ C.

4- A = B if and only if A ⊂ B and B ⊂ A.

Set operations:

A is said to be a subset of B if and only if every element of A is also an element of B,in which case we write A ⊆ B. A is a strict subset of B if A is a subset of B and A isnot equal to B, which is denoted by A⊂B. For example, {4, 23} ⊂ {2, 4, 17, 23} ⊆{2, 4, 17, 23}.
Two sets A and B are considered equal if and only if they have the same elements.This is denoted by A = B. More formally, A = B if and only if A ⊆B and B ⊆A.For two sets A and B, the operations of union, intersection, difference ,complement andSymmetric differencearedefined as follows:

1) UNION:
The *union* of two sets *A* and *B*, denoted by *A*∪ *B*, is the set of all elements whichbelong to *A* or to *B*;

A ∪ B = { x : x ∈ A or x ∈ B}

ExampleA={1,2,3,4,5} B={5,7,9,11,13}
$A \cup B$ = {1,2,3,4,5,7,9,11,13}

2) INTERSECTION

The *intersection* of two sets *A* and *B*, denoted by $A \cap B$, is the set of elements which belong to both *A*and *B*;

$A \cap B = \{ x : x \in A \text{ and } x \in B\}$.

Example 1A={1,3,5,7,9} B={2,3,4,5,6}

The elements they have in common are 3 and 5
$A \cap B = \{3,5\}$

Example 2A={The English alphabet} B={vowels}

So $A \cap B$ = {vowels}

Example 3A={1,2,3,4,5} B={6,7,8,9,10}

In this case A and B have nothing in common.
$A \cap B = \varnothing$

3) THE DIFFERENCE:
The difference of two sets A\B or A-B is those elements which belong to A butwhich do not belong to B.

$A \setminus B = \{x : x \in A, x \notin B\}$

4) COMPLEMENT OF SET:
Complement of set A^cor A', is the set of elements which belong to U but which do not belong to A .
$A^c = \{x : x \in U, x \notin A\}$

Example:
let A={1,2,3} B={3,4} U={1,2,3,4,5,6}
Find:
$A \cup B = \{1, 2, 3, 4\}$

$A \cap B = \{3\}$
$A - B = \{1, 2\}$
$A^c = \{4, 5, 6\}$

5) Symmetric difference of sets

The symmetric difference of sets A and B, denoted by $A \oplus B$, consists of those elements which belong to A or B but not to both. That is,

$$A \oplus B = (A \cup B)\backslash(A \cap B) \text{ or } A \oplus B = (A\backslash B) \cup (B\backslash A)$$

Example:

Suppose $U = N = \{1, 2, 3, \ldots\}$ is the universal set.
Let $A = \{1, 2, 3, 4\}$, $B= \{3, 4, 5, 6, 7\}$, $C= \{2, 3, 8, 9\}$, $E= \{2, 4, 6, 8, \ldots\}$
Then:
$A^c = \{5, 6, 7, \ldots\}$, $B^c = \{1, 2, 8, 9, 10, \ldots\}$, $C^c =\{1,4,5,6,7,10,\ldots\}$
$E^c=\{1, 3, 5, 7, \ldots\}$

$A\backslash B = \{1, 2\}$, $A\backslash C = \{1, 4\}$, $B\backslash C = \{4, 5, 6, 7\}$, $A\backslash E = \{1, 3\}$,
$B\backslash A = \{5, 6, 7\}$, $C\backslash A = \{8, 9\}$, $C\backslash B = \{2, 8, 9\}$, $E\backslash A = \{6, 8, 10, 12, \ldots\}$.

Furthermore:
$A\oplus B = (A\backslash B) \cup (B\backslash A) = \{1, 2, 5, 6, 7\}$, $B\oplus C = \{2, 4, 5, 6, 7, 8, 9\}$,
$A\oplus C = (A\backslash C) \cup (B\backslash C) = \{1, 4, 8, 9\}$, $A\oplus E = \{1, 3, 6, 8, 10, \ldots\}$.

Theorem 2 :

$A \subset B$, $A \cap B = A$, $A \cup B = B$ are equivalent

Theorem 3: (Algebra of sets)

Sets under the above operations satisfy various laws or identities which are listed below:

1- $A \cup A = A$
$A \cap A = A$
2- $(A \cup B) \cup C = A \cup (B \cup C)$ Associative laws
$(A \cap B) \cap C = A \cap (B \cap C)$
3- $A \cup B = B \cup A$ Commutativity
$A \cap B = B \cap A$

4- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ Distributive laws

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

5- $A \cup \varnothing = A$ Identity laws

$A \cap U = A$

6- $A \cup U = U$ Identity laws

$A \cap \varnothing = \varnothing$

7- $(A^c)^c = A$ Double complements

8- $A \cup A^c = U$ Complement intersections and unions

$A \cap A^c = \varnothing$

9- $U^c = \varnothing$

$\varnothing^c = U$

10- $(A \cup B)^c = A^c \cap B^c$ De Morgan's laws

$(A \cap B)^c = A^c \cup B^c$

We discuss two methods of proving equations involving set operations. The first is to break down what it means for an object x to be an element of each side, and the second is to use Venn diagrams.

For example, consider the first of De Morgan's laws:

$(A \cup B)^c = A^c \cap B^c$

We must prove:

1) $(A \cup B)^c \subset A^c \cap B^c$

2) $A^c \cap B^c \subset (A \cup B)^c$

We first show that $(A \cup B)^c \subset A^c \cap B^c$

Let's pick an element at random $x \in (A \cup B)^c$. We don't know anything about x, it could be a number, a function. All we do know about x, is that:

$x \in (A \cup B)^c$, so

$x \notin A \cup B$

Because that's what complement means. Therefore

$x \notin A$ and $x \notin B$,

by pulling apart the union. Applying complements again we get

$x \in A^c$ and $x \in B^c$

Finally, if something is in 2 sets, it must be in their intersection, so

$x \in A^c \cap B^c$

So, any element we pick at random from: $(A \cup B)^c$ is definitely in, $A^c \cap B^c$, so by definition

$(A \cup B)^c \subset A^c \cap B^c$

Next we show that $(A^c \cap B^c) \subset (A \cup B)^c$.

This follows a very similar way. Firstly, we pick an element at random from the first set, $x \in (A^c \cap B^c)$

Using what we know about intersections, that means

$x \in A^c$ and $x \in B^c$

Now, using what we know about complements,

$x \notin A$ and $x \notin B$.

If something is in neither A nor B, it can't be in their union, so

$x \notin A \cup B$,And finally

$\therefore x \in (A \cup B)^c$

We have prove that every element of $(A \cup B)^c$ belongs to $A^c \cap B^c$ and that every element of $A^c \cap B^c$ belongs to $(A \cup B)^c$. Together,

These inclusions prove that the sets have the same elements, i.e. that

$(A \cup B)^c = A^c \cap B^c$

Power set

The power set of some set S, denoted P(S), is the set of all subsets of S (including S itself and the empty set)

Example 1: Let A = { 1,2,3}

Power set of set A = P(A) = [{1},{2},{3},{1,2},{1,3},{2,3},{},A]

Example 2: P({0,1})={{},{0},{1},{0,1}}

Classes of sets:

Collection of subset of a set with some properties
Example: Suppose A = { 1,2, 3} , let X be the class of subsets of A which contain exactly two elements of A. Then class
X = [{1,2},{1,3},{2,3}]

Cardinality

The cardinality of a set S, denoted **|S|**, is simply the number of elements a set has. So **|**{a,b,c,d**}| =** 4, and so on. The cardinality of a set need not be finite: some sets have infinite cardinality.

The cardinality of the power set

Theorem: If |A| = n then |P(A)| = 2^n (Every set with n elements has 2^n subsets)

Problem set

write the answers to the following questions.

1. |{1,2,3,4,5,6,7,8,9,0}|

2. |P({1,2,3})|

3. P({0,1,2})

4. P({1})

Answers

1. 10

2. 2^3=8

3. {{},{0},{1},{2},{0,1},{0,1,2},{0,2},{1,2}}

4. {{},{1}}

The Cartesian product

The Cartesian Product of two sets is the set of all tuples made from elements of two sets. We write the Cartesian Product of two sets A and B as A × B. It is defined as:

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

It may be clearer to understand from examples;

$$\{0, 1\} \times \{2, 3\} = \{(0, 2), (0, 3), (1, 2), (1, 3)\}$$

$$\{a, b\} \times \{c, d\} = \{(a, c), (a, d), (b, c), (b, d)\}$$

$$\{0, 1, 2\} \times \{4, 6\} = \{(0, 4), (0, 6), (1, 4), (1, 6), (2, 4), (2, 6)\}$$

Example:
If A = {1, 2, 3} and B = {x, y} then

A . B = {(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)}
B . A = {(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)}

It is clear that, the cardinality of the Cartesian product of two sets A and B is: $|A \times B| = |A||B|$

A Cartesian Product of two sets A and B can be produced by making tuples of each element of A with each element of B; this can be visualized as a grid (which *Cartesian* implies) or table: if, *e.g.*, A = { 0, 1 } and B = { 2, 3 }, the grid is

x		**A**	
		0	1
B	2	(0,2)	(1,2)
	3	(0,3)	(1,3)

Problem set

Answer the following questions:

1. {2,3,4}×{1,3,4}

2. {0,1}×{0,1}

3. |{1,2,3}×{0}|

4. |{1,1}×{2,3,4}|

Answers

1. {(2,1),(2,3),(2,4),(3,1),(3,3),(3,4),(4,1),(4,3),(4,4)}

2. {(0,0),(0,1),(1,0),(1,1)}

3. 3

4. 6

Partitions of set:

Let S be any nonempty set. A partition (∏) of S is a subdivision of S into nonoverlapping, nonempty subsets. A partition of S is a collection {Ai} of non-empty subsets of S such that:

1) Ai ≠∅, where i=1,2,3,……

2) The sets of {Ai } are mutually disjoint

orAi ∩ Aj = ∅ where i ≠ j.

3) $\cup A_i$= S, where $A_1 \cup A_2 \cup$ .... $\cup A_i$ = S

The partition of a set into five cells, A1, A2,A3,A4,A5, can be represented by Venn Diagram

Example 1:

let A = {1,2,3,n}

A1 = {1}, A2 = {3,n}, A3 = {2}

Π = {A1, A2, A3} is a partition on A because it satisfy the three above conditions.

Example 2 :

Consider the following collections of subsets of S = {1,2,3,4,5,6,7,8,9}

(i) [{1,3,5},{2,6},{4,8,9}]

(ii) [{1,3,5},{2,4,6,8},{5,7,9}]

(iii) [{1,3,5},{2,4,6,8},{7,9}]

Then

(i) is not a partition of S since 7 in S does not belong to any of the subsets.

(ii) is not a partition of S since {1,3,5} and {5,7,9} are not disjoint.

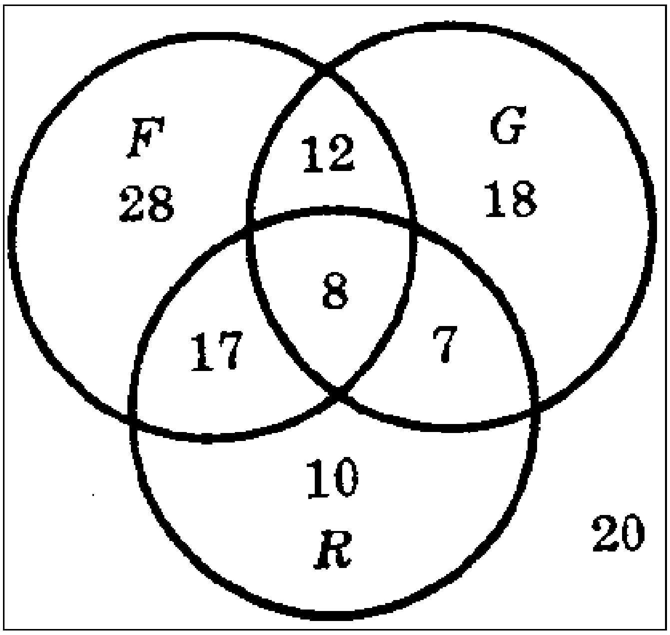
(iii) is a partition of S.

FINITE SETS, COUNTING PRINCPLE:

A set is said to be finite if it contains exactly m distinct elements where m denotes some nonnegative integer. Otherwise, a set is said to be infinite. For example, the

empty set ∅ and the set of letters of English alphabet are finite sets, whereas the set of even positive integers, {2,4,6,…..}, is infinite.
If a set A is finite, we let *n*(A) or #(A) denote the number of elements of A.
Example: If A ={1,2,a,w} then
n(A) = #(A) = |A| = 4
Lemma: If A and B are finite sets and disjoint Then $A \cup B$ is finite set and:
***n*(A ∪ B) = *n*(A) + *n*(B)**
Theorem (Inclusion–Exclusion Principle): Suppose A and B are finite sets. Then
$A \cup B$ and $A \cap B$ are finite and
$|A \cup B| = |A| + |B| - |A \cap B|$
That is, we find the number of elements in A or B (or both) by first adding n(A) and n(B) (inclusion) and then subtracting $n(A \cap B)$ (exclusion) since its elements were counted twice.
We can apply this result to obtain a similar formula for three sets:
Corollary:
If A,B,C are finite sets then
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
Example (1) :
A= {1,2,3}
B= {3,4}
C= {5,6}
$A \cup B \cup C$ = {1,2,3,4,5,6}
$|A \cup B \cup C| = 6$
|A| =3 , |B| = 2 , |C| = 2
$A \cap B$ = {3} , $|A \cap B| = 1$
$A \cap C$ = { } , $|A \cap C| = 0$
$B \cap C$ = { } , $|B \cap C| = 0$
$A \cap B \cap C$ ={} , $|A \cap B \cap C| = 0$
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
$|A \cup B \cup C| = 3 + 2 + 2 - 1 - 0 - 0 + 0 = 6$
Example (2):
Suppose a list A contains the 30 students in a mathematics class, and a list B contains the 35 students in an English class, and suppose there are 20 names on both lists. Find the number of students:
(a) only on list A
(b) only on list B
(c) on list $A \cup B$
Solution:
(a) List A has 30 names and 20 are on list B; hence 30 − 20 = 10 names are only on list A.
(b) Similarly, 35 − 20 = 15 are only on list B.
(c) We seek $n(A \cup B)$. By inclusion–exclusion,
$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 30 + 35 - 20 = 45.$
Example (3):

Suppose that 100 of 120 computer science students at a college take at least one of languages: French, German, and Russian and:
65 study French (F).
45 study German (G).
42 study Russian (R).
20 study French & German $F \cap G$.
25 study French & Russian $F \cap R$.
15 study German & Russian $G \cap R$.
Find the number of students who study:
1) All three languages ($F\cap G \cap R$)
2) The number of students in each of the eight regions of the Venn diagram



Solution:
$|F \cup G \cup R| = |F| + |G| + |R| - |F \cap G| - |F \cap R| - |G \cap R| + |F \cap G \cap R|$
$100 = 65 + 45 + 42 - 20 - 25 - 15 + |F \cap G \cap R|$
$100 = 92 + |F \cap G \cap R|$
$\therefore |F \cap G \cap R| = 8$ students study the 3 languages
$20 – 8 = 12$ $(F \cap G) - R$
$25 – 8 = 17$ $(F \cap R) - G$
$15 – 8 = 7$ $(G \cap R) - F$
65 – 12 – 8 – 17 = 28 students study French only
45 – 12 – 8 7 = 18 students study German only
42 – 17 – 8 7 = 10 students study Russian only
120 – 100 = 20 students do not study any language

Relations

Binary relation:

There are many relations in mathematics :"less than" , "is parallel to ","is a subset of", etc. These relations consider the existence or nonexistence of a certain connection between pairs of objects taken in a definite order. We define a relation simply in terms of ordered pairs of objects.

Product sets:

Consider two arbitrary sets A and B. The set of all ordered pairs (a,b) where $a\in A$ and $b\in B$ is called the product, or cartesian product, of A and B.

$A \times B = \{(a,b) : a\in A \text{ and } b\in B\}$

Example: Let A = {1,2} and B = {a ,b ,c} then

A × B = {(1,a), (1,b),(1,c),(2,a),(2,b),(2,c)}

Also, A × A = {(1, 1), (1, 2), (2, 1), (2, 2)}

- The order in which the sets are considered is important, so $A\times B \neq B \times A$.

Let A and B be sets. A binary relation, R, from A to B is a subset of A×B. If $(x,y) \in R$, we say that x is R-related to y and denote this by xRy

if)x,y) $\notin R$, we write x

y and say that x is not R-related to y .

if R is a relation from A to A ,i.e. R is a subset of A × A, then we say that R is a relation on A.

The **domain** of a relation R is the set of all first elements of the ordered pairs which belong to R, and the **range** of R is the set of second elements.

Example 1:

Let A = {1, 2, 3, 4}. Define a relation R on A by writing $(x, y) \in R$ if $x < y$. Then

R = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}.

Example 2:

let A = {1,2,3} and R = {(1,2),(1,3),(3,2)}. Then R is a relation on A since it is a subset of A×A with respect to this relation:

1R2, 1R3, 3R2 but $(1,1)\notin R$ & $(2,1)\notin R$

The domain of R is {1,3} and

The range of R is {2,3}

Example 3:

Let A = {1, 2, 3}. Define a relation R on A by writing $(x, y) \in R$, such that $a\geq b$, list the element of R

$aRb \leftrightarrow a\geq b$, $a,b\in A$

$\therefore$ R = {(1,1),(2,1), (2,2), (3,1), (3,2), (3,3)}.

Example 4:

A relation on the set Z of integers is “m divides n.” A common notation for this relation is to write m|n when m divides n. Thus 6 | 30 but 7*25.

Representation of relations:

1) By language

2) By ordered pairs

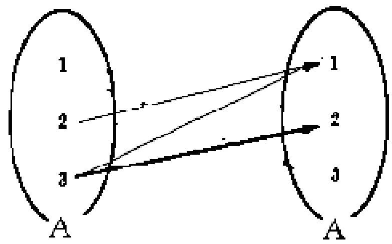
3) By arrow form
4) By matrix form
5) By coordinates
6) By graph form

Example:
Let A = {1,2,3}, the relation R on A such that: aRb ↔ a>b; a,b∈ A

1) By language:
R={(a,b) : a,b∈ A and aRb ↔ a>b}

2) By ordered pairs
R = {(2,1),(3,1),(3,2)}

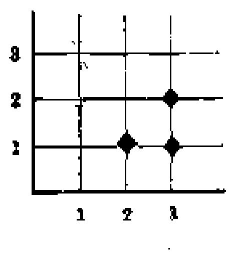
3) By arrow form



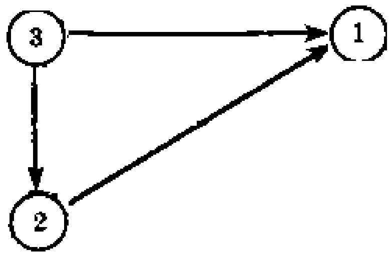
4) By matrix form

	1	2	3
1	0	0	0
2	1	0	0
3	1	1	0

5) By coordinates



6) By graph form



TYPES OF RELATIONS:
Properties of relations:

Let R be a relation on the set A

1) Reflexive : R is reflexive if : $\forall$ a $\in$ A$\rightarrow$ aRa or (a,a) $\in$ R ; $\forall$ a, b $\in$ A. . Thus R is not reflexive if there exists a $\in$ A such that (a, a) $\notin$ R.

2) Symmetric :aRb$\rightarrow$ bRa$\forall$ a,b$\in$ A. if whenever *(a, b)* $\in$*R* then *(b, a)* $\in$*R.*
Thus *R* is not symmetric if there exists *a, b* $\in$*A* such that *(a, b)* $\in$*R* but *(b, a)* $\notin$ *R.*

3) Transitive :aRb$\wedge$ bRc$\rightarrow$ aRc. that is, if whenever *(a, b), (b, c)* $\in$*R* then*(a, c)* $\in$*R.* Thus *R* is not transitive if there exist *a, b, c* $\in$*R* such that *(a, b), (b, c)* $\in$*R* but *(a, c)* $\notin$ *R.*

4) Equivalence relation : it is Reflexive & Symmetric & Transitive. That is, *R* is an equivalence relation on *S* if it has the following three properties:
a - For every *a* $\in$*S, aRa*.
b- If *aRb*, then *bRa*.
c- If *aRb*and *bRc*, then *aRc*.

5) Irreflexive :$\forall$ a $\in$ A (a,a) $\notin$ R

6) AntiSymmetric : if aRb and bRa$\rightarrow$ a=b
the relations $\geq$,$\leq$ and $\subseteq$ are antisymmetric

Example 5:
Consider the relation of C of set inclusion on any collection of sets:
1) A $\subset$ A for any set, so $\subset$ is reflexive
2) A $\subset$ B dose not imply B $\subset$ A, so $\subset$ is not symmetric
3) If A $\subset$ B and B $\subset$ C then A$\subset$ C, so $\subset$ is transitive
4) $\subset$ is reflexive, not symmetric & transitive, so $\subset$ is not equivalence relations
5) A $\subset$ A, so $\subset$ is not Irreflexive
6) If A $\subset$ B and B $\subset$ A then A = B, so $\subset$ is anti-symmetric

Example 6:
If A ={1,2,3} and R={(1,1),(1,2),(2,1),(2,3)}Is R equivalence relation ?

1) 2 is in A but $(2,2) \notin R$, so R is not reflexive
2) $(2,3) \in R$ but $(3,2) \notin R$, so R is not symmetric
3) $(1,2) \in R$ and $(2,3) \in R$ but $(1,3) \notin R$, so R is not transitive
So R is not Equivalence relation

Example 7 :
What is the properties of the relation =?
1) a=a for any element $a \in A$, so = is reflexive
2) If a = b then b = a, so = is symmetric
3) If a = b and b = c then a = c, so = is transitive
4) = is (reflexive + symmetric + transitive), so = is equivalence
5) a = a, so = is not Irreflexive
6) If a = b and b = a then a = b, so = is anti-symmetric

Remark:
The properties of being symmetric and being ant symmetric are not negatives of each other. For example, the relation $R = \{(1, 3), (3, 1), (2, 3)\}$ is neither symmetric nor antisymmetric. On the other hand, the relation $R = \{(1, 1), (2, 2)\}$ is both symmetric and antisymmetric.

-Reflexive Closures
Let *R* be a relation on a set *A*. Then:
$R \cup \{(a, a) \mid a \in A\}$ is the reflexive closure of *R*. In other words, **reflexive(*R*)** is obtained by simply adding to *R* those elements *(a, a)* in the diagonal which do not already belong to *R*.

-Symmetric Closures
$R \cup R^{-1}$ is the symmetric closure of *R*. in other words, **symmetric(*R*)** is obtained by adding to *R* all pairs *(b, a)* whenever *(a, b)* belongs to *R*.

EXAMPLE :
Consider the relation $R = \{(1, 1), (1, 3), (2, 4), (3, 1), (3, 3), (4, 3)\}$ on the set $A = \{1, 2, 3, 4\}$.Then
reflexive$(R) = R \cup \{(2, 2), (4, 4)\}$ and
symmetric$(R) = R \cup \{(4, 2), (3, 4)\}$

-Transitive Closure
R* is the transitive closure of *R, where:*
$R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$ and $R^2 = R \circ R$ and $R^n = R^{n-1} \circ R$

Theorm: Suppose *A* is a finite set with *n* elements and Let *R* be a relation on a set *A* with *n* elements. Then : transitive $(R) = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$

EXAMPLE :
Consider the relation $R = \{(1, 2), (2, 3), (3, 3)\}$ on $A = \{1, 2, 3\}$. Then:

$R^2 = R \circ R = \{(1, 3), (2, 3), (3, 3)\}$ and
$R^3 = R^2 \circ R = \{(1, 3), (2, 3), (3, 3)\}$ then
transitive$(R) = \{(1, 2), (2, 3), (3, 3), (1, 3)\}$

Inverse relations:
$R^{-1} = \{(b,a) : (a,b) \in R\}$
Example 1 :
Let R be the following relation on A ={1,2,3}
R = {(1,2),(1,3),(2,3)}
$\therefore R^{-1} = \{(2,1),(3,1),(3,2)\}$
The matrix for R :

$$MR = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

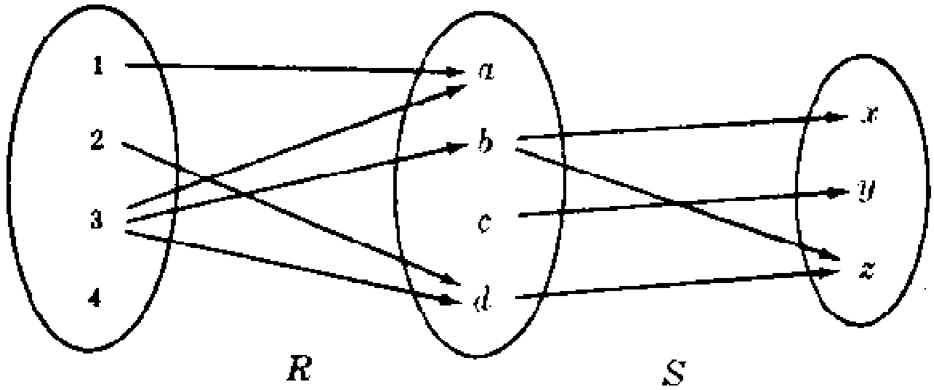
And

$$MR^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

MR^{-1} is the transpose of matrix R

Composition of relations:
Let A, B, C be sets and let :
$R : A \to B$ ($R \subset A \times B$)
$S : B \to C$ ($S \subset B \times C$)
There is a relation from A to C denoted by
$R \circ S$ (composition of R and S) : $A \to C$
$R \circ S = \{(a,c) : \exists\, b \in B$ for which $(a,b) \in R$ and $(b,c) \in S\}$

Example : let A ={1,2,3,4}
B = {a, b, c, d}
C = {x, y, z}
R = {(1,a),(2,d),(3,a),(3,d),(3,b)}
S = {(b,x),(b,z),(c,y),(d,z)}
Find $R \circ S$?
Solution :
1) The first way by arrow form



There is an arrow (path) from 2 to d which is followed by an arrow from d to z
2Rd and dSz⇒ 2(R ° S) z
and 3(R◦S)x and 3(R◦S)z
so R ° S = {(3,x),(3,z),(2,z)}

2) The second way by matrix:

$$MR = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad MS = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

R ° S = MR .MS =

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

R ° S = {(2,z),(3,x),(3,z)}

Theorem 2.1:
Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C,andT is a relation from C to D. Then
(R ◦ S) ◦ T = R ◦ (S ◦ T)

n-ARY RELATIONS

All the relations discussed above were binary relations. By an *n-ary relation*, we mean a set of ordered *n*-tuples. For any set *S*, a subset of the product set *Sn*is called an *n*-ary relation on *S*. In particular, a subset of S3 is called a *ternary relation* on *S*.

EXAMPLE

(a) Let *L* be a line in the plane. Then "betweenness" is a ternary relation *R* on the points of *L*; that is, $(a, b, c) \in R$, if *b* lies between *a* and*c* on *L*.

(b) The equation $x2 + y2 + z2 = 1$ determines a ternary relation *T* on the set **R** of real numbers. That is, a triple (x, y, z) belongs to *T* if (x, y, z) satisfies the equation, which means (x, y, z) is the coordinates of a point in **R**3 on the sphere *S* with radius 1 and center at the origin $O = (0, 0, 0)$.

Function:

Function is an important class of relation.

Definition:

Let A,B be two nonempty sets, a function F: A□B is a rule which associates with **each** element of A a**unique** element in B.

The set A is called the **domain** of the function, and the set B is called the **range** of the function.

Example 1:

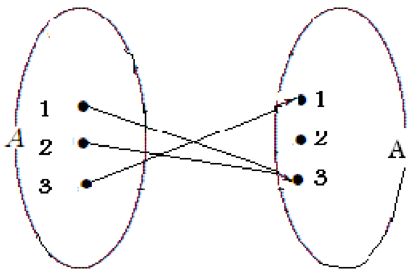
Consider the function $f(x) = x^3$, i.e., *f* assigns to each real number its cube. Then the image of 2 is 8, and so we may write $f(2) = 8$.

Example2 :

consider the following relation on the set A={1,2,3}

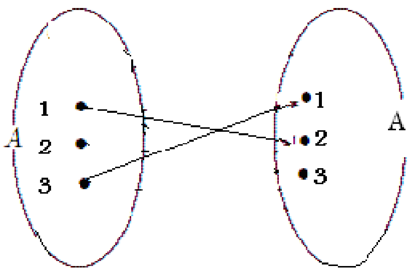
F = {(1,3),(2,3),(3,1)}

F is a function



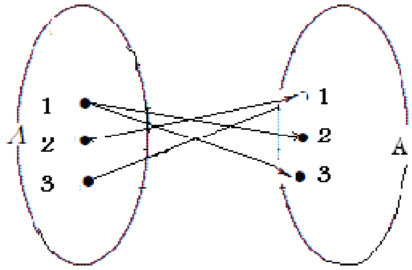
G = {1,2},(3,1)}

G is not a function from A to A



H = {(1,3),(2,1),(1,2),(3,1)}

H is not a function

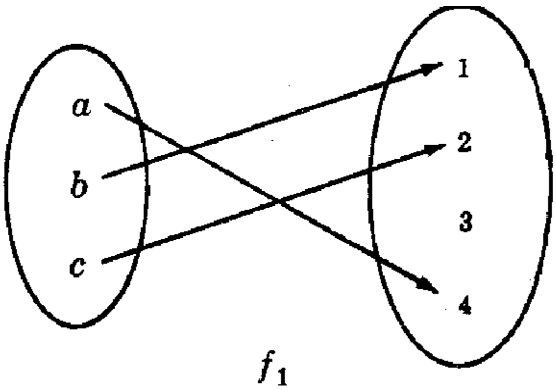


One-to-one ,onto and invertible functions :

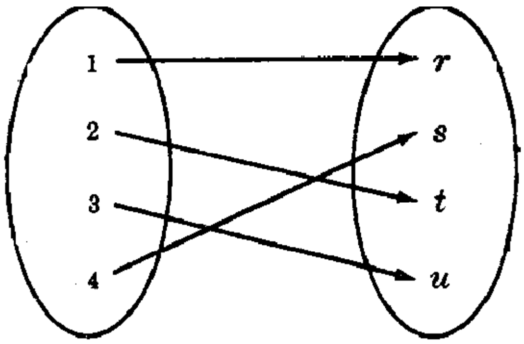
1) One –to-one : a function F:A→B is said to be one-to-one if different elements in the domain (A) have distinct images.
Or If F(a) =F(a') ⇒ a = a'

2) Onto : F:A→B is said to be an onto function if each element of B is the image of some element of A.
∀ b∈B∃ a ∈ A**:**F(a) = b

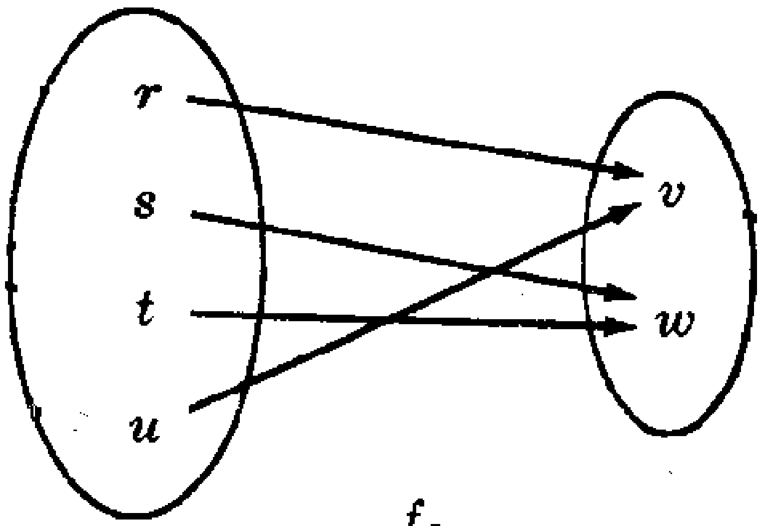
3) Invertible (One-to-one correspondence)
F:A→ B is invertible if its inverse relation f_{-1} is a function F:B →A
F:A→ B is invertible if and only if F is **both** one-to-one and onto
F^{-1}:{(b,a) ∀ (a,b) ∈ F}



one to one but not onto (3∈ B but it is not the image under f1)

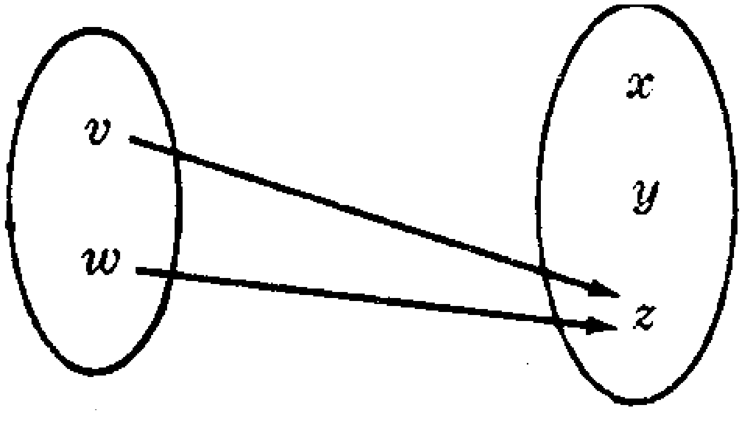


both one to one & onto
(or one to one correspondence between A and B)



f_3

not one to one & onto



not one to one & not onto

Graph of a function:

By a *real polynomial function*, we mean a function *f*: **R** → **R** of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

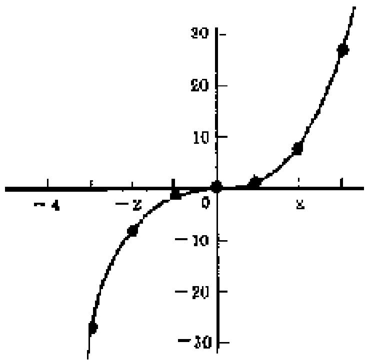
where the a_i are real numbers. Since **R** is an infinite set, it would be impossible to plot each point of the graph. However, the graph of such a function can be approximated by first plotting some of its points and then drawing a smooth curve though these points. The table points are usually obtained from a table where various values are assigned to x and the corresponding value of f(x) computed.

Example 1 : let f:R→R and f(x)= x^3 , find f(x)

f(3) = 3^3 = 27

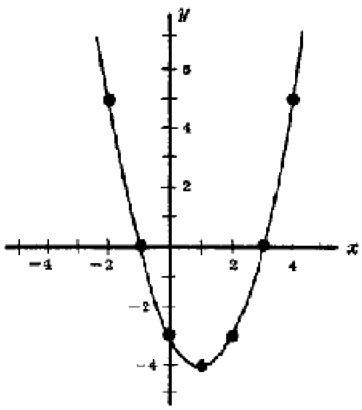
f(-2) = $(-2)^3$ = -8

x	$f(x)$
−3	−27
-2	8
1	−1
0	0
1	1
2	8
3	27



Example 2: let f: R→R and $f(x) = x^2 - 2x - 3$, , find f(x)

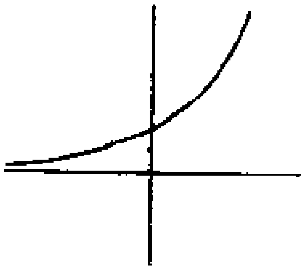
x	$f(x)$
−2	5
−1	0
0	−3
1	−4
2	−3
3	0
4	5



Geometrical Characterization of One-to-One and Onto Functions

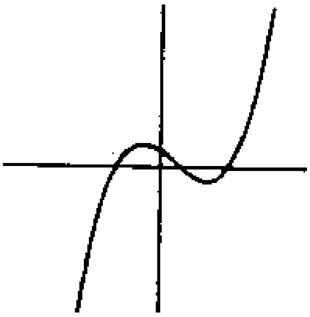
For the functions of the form f : R →R. the graphs of such functions may be plotted in the Cartesian plane and functions may be identified with their graphs, so the concepts of being one-to-one and onto have some geometrical meaning :

(1) f :R →R is said to be one-to-one if there are no 2 distinct pairs (a_1,b) and (a_2,b) in the graph one-to-one or if each horizontal line intersects the graph of f in at most one point.



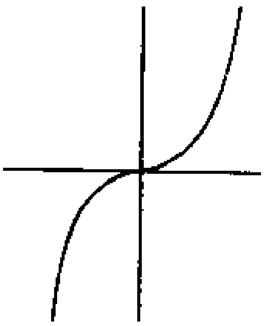
$f_2(x) = 2^x$

(2) f :R →R is an onto function if each horizontal line intersects the graph of f at one or more points (at least once)

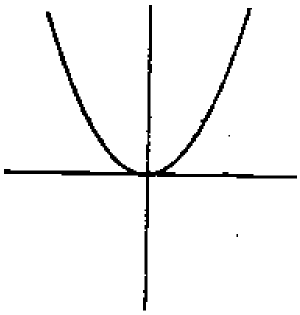


$f_3(x) = x^3 - 2x^2 - 5x + 6$

(3) if f is both one-to-one and onto, i.e. invertible, then each horizontal line will intersect the graph of f at exactly one point.



$f_4(x) = x^3$



$f_1(x) = x^2$

F(x) NOT (ONE-TO-ONE) & NOT (ONTO)

Factorial Function

The product of the positive integers from 1 to *n*, inclusive, is called "*n* factorial" and is usually denoted by *n*!. That is,

$n! = n(n-1)(n-2) \cdot \cdot \cdot 3 \cdot 2 \cdot 1$

where 0! = 1, so that the function is defined for all nonnegative integers. Thus:

0! = 1,1! = 1,

2! = 2.1 = 2,3! = 3.2.1 = 6,

4! = 4.3.2.1 = 24 5! = 5.4.3.2.1 = 120

6! = 6.5.4.3.2.1 = 720

This is true for every positive integer *n*; that is,

$n! = n \cdot (n-1)!$

Accordingly, the factorial function may also be defined as follows:

Definition of Factorial Function:

(a) If $n = 0$, then $n! = 1$.

(b) If $n > 0$, then $n! = n \cdot (n-1)!$

The definition of *n*! is recursive, since it refers to itself when it uses $(n-1)!$. However:

(1) The value of *n*! is explicitly given when $n = 0$ (thus 0 is a base value).

(2) The value of *n*! for arbitrary *n* is defined in terms of a smaller value of *n* which is closer to the base value 0.

Accordingly, the definition is not circular, or, in other words, the function is well-defined.

EXAMPLE :the 4! Can be calculated in 9 steps using the recursive definition .

(1) $4! = 4 \cdot 3!$

(2) $3! = 3 \cdot 2!$

(3) $2! = 2 \cdot 1!$

(4) $1! = 1 \cdot 0!$

(5) $0! = 1$

(6) $1! = 1 \cdot 1 = 1$

(7) $2! = 2 \cdot 1 = 2$

(8) $3! = 3 \cdot 2 = 6$

(9) $4! = 4 \cdot 6 = 24$

Fibonacci Sequence

The Fibonacci sequence (usually denoted by F0, F1, F2, . . .) is as follows:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, . . .

That is, F0 = 0 and F1 = 1 and each succeeding term is the sum of the two preceding terms. For example, the next two terms of the sequence are

34 + 55 = 89 and 55 + 89 = 144

Fibonacci Sequence can be defined:

(a) If n = 0, or n = 1, then $F_n = n$.

(b) If n > 1, then $F_n = F_{n-2} + F_{n-1}$.

Where : The base values are 0 and 1, and the value of F_n is defined in terms of smaller values of n which are closer to the base values.

Accordingly, this function is well-defined.

Vectors:-

vector, u, means a list (or n-tuple) of numbers:

$$u = (u_1, u_2, \ldots, u_n)$$

whereu_i are called the components of u. If all the u_i are zero i.e., $u_i = 0$, then u is called the zero vector.

Given vectors u and v are equal i.e., u = v, if they have the same number of components and if corresponding components are equal.

Addition of Two Vectors

If two vectors, u and v, have the number of components, their sum, u + v, is the vector obtained by adding corresponding components from u and v.

$$\begin{aligned} u + v &= (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) \\ &= (u_1 + v_1 + u_2 + v_2, \ldots, u_n + v_n) \end{aligned}$$

Definition.*Scalar multiplication* of a vector $y = (y1, y2, \ldots ,yk)$ and a scalar _ is defined to be a new
vectorz = $(z1, z2, \ldots , zk)$, written $z = \_y$ or $z = y\_$, whose components are given by $z j = \_y j$.
Definition.*Vector addition* of two *k*-dimensional vectors $x = (x1, x2, \ldots ,xk)$ and $y = (y1, y2, \ldots , yk)$
is defined as a newvector $z = (z1, z2, \ldots , zk)$, denoted $z = x+y$, with components given by $z j = x j+y j$.
As an example of scalar multiplication, consider
4(3, 0,−1, 8) = (12, 0,−4, 32),
and for vector addition,
(3, 4, 1,−3) + (1, 3,−2, 5) = (4, 7,−1, 2).
Using both operations, we can make the following type of calculation:
$(1, 0)x_1 + (0, 1)x_2 + (-3,-8)x_3 = (x_1, 0) + (0, x_2) + (-3x_3,-8x_3)$
$= (x_1 - 3x_3, x_2 - 8x_3)$.
It is important to note that *y* and *z* must have the same dimensions for vector addition and vector
comparisons. Thus (6, 2,−1) + (4, 0) is *not* defined, and (4, 0,−1) = (4, 0) makes *no* sense at all.

Matrix

A *matrix* is a rectangular array of numbers or other mathematical objects for which operations such as addition and multiplication are defined. Most commonly, a matrix over a field*F* is a rectangular array of scalars each of which is a member of *F*. Most of this article focuses on *real* and *complex matrices*, i.e., matrices whose elements are real numbers or complex numbers, respectively. More general types of entries are discussed below. For instance, this is a real matrix:

$$\mathbf{A} = \begin{bmatrix} -1.3 & 0.6 \\ 20.4 & 5.5 \\ 9.7 & -6.2 \end{bmatrix}.$$

The numbers, symbols or expressions in the matrix are called its *entries* or its *elements*. The horizontal and vertical lines of entries in a matrix are called *rows* and *columns*, respectively.

Size

The size of a matrix is defined by the number of rows and columns that it contains. A matrix with m rows and n columns is called an $m \times n$ matrix or m-by-n matrix, while m and n are called its *dimensions*. For example, the matrix **A** above is a 3×2 matrix.

Matrices which have a single row are called *row vectors*, and those which have a single column are called *column vectors*. A matrix which has the same number of rows and columns is called a *square matrix*. A matrix with an infinite number of rows or columns (or both) is called an *infinite matrix*. In some contexts, such as computer algebra programs, it is useful to consider a matrix with no rows or no columns, called an *empty matrix*.

Name	Size	Example	Description
Row vector	$1 \times n$	$\begin{bmatrix} 3 & 7 & 2 \end{bmatrix}$	A matrix with one row, sometimes used to represent a vector
Column vector	$n \times 1$	$\begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$	A matrix with one column, sometimes used to represent a vector
Square matrix	$n \times n$	$\begin{bmatrix} 9 & 13 & 5 \\ 1 & 11 & 7 \\ 2 & 6 & 3 \end{bmatrix}$	A matrix with the same number of rows and columns, sometimes used to represent a linear transformation from a vector space to itself, such as reflection, rotation, or shearing.

Notation

Matrices are commonly written in box brackets or large parentheses:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m\times n}.$$

The specifics of symbolic matrix notation vary widely, with some prevailing trends. Matrices are usually symbolized using upper-case letters (such as **A** in the examples above), while the corresponding lower-case letters, with two subscript indices (e.g., a_{11}, or $a_{1,1}$), represent the entries. In addition to using upper-case letters to symbolize matrices

Matrix addition

Two matrices must have an equal number of rows and columns to be added. The sum of two matrices A and B will be a matrix which has the same number of rows and columns as do A and B. The sum of A and B, denoted A + B, is computed by adding corresponding elements of A and B

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{\mathbf{1}n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

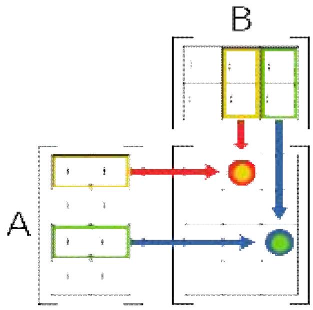
For example:

$$\begin{bmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 7 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 3+0 \\ 1+7 & 0+5 \\ 1+2 & 2+1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 8 & 5 \\ 3 & 3 \end{bmatrix}$$

We can also subtract one matrix from another, as long as they have the same dimensions. **A** − **B** is computed by subtracting corresponding elements of **A** and **B**, and has the same dimensions as **A** and **B**. For example:

$$\begin{bmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 7 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1-0 & 3-0 \\ 1-7 & 0-5 \\ 1-2 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -6 & -5 \\ -1 & 1 \end{bmatrix}$$

Matrix multiplication



Schematic depiction of the matrix product **AB** of two matrices **A** and **B**.

Multiplication of two matrices is defined if and only if the number of columns of the left matrix is the same as the number of rows of the right matrix. If **A** is an *m*-by-*n* matrix and **B** is an *n*-by-*p* matrix, then their *matrix product***AB** is the *m*-by-*p* matrix whose entries are given by dot product of the corresponding row of **A** and the corresponding column of **B**:

$$[\mathbf{AB}]_{i,j} = A_{i,1}B_{1,j} + A_{i,2}B_{2,j} + \cdots + A_{i,n}B_{n,j} = \sum_{r=1}^{n} A_{i,r}B_{r,j},$$

where $1 \leq i \leq m$ and $1 \leq j \leq p$. For example, the underlined entry 2340 in the product is calculated as $(2 \times 1000) + (3 \times 100) + (4 \times 10) = 2340$:

$$\begin{bmatrix} \underline{2} & \underline{3} & \underline{4} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \underline{1000} \\ 1 & \underline{100} \\ 0 & \underline{10} \end{bmatrix} = \begin{bmatrix} 3 & \underline{2340} \\ 0 & 1000 \end{bmatrix}.$$

Matrix multiplication satisfies the rules (**AB**)**C** = **A**(**BC**) (associativity), and (**A**+**B**)**C** = **AC**+**BC** as well as **C**(**A**+**B**) = **CA**+**CB** (left and right distributivity), whenever the size of the matrices is such that the various products are defined.[14] The product **AB** may be defined without **BA** being defined, namely if **A** and**B** are *m*-by-*n* and *n*-by-*k* matrices, respectively, and $m \neq k$. Even if both products are defined, they need not be equal, i.e., generally

$$\mathbf{AB} \neq \mathbf{BA},$$

i.e., *matrix multiplication is not commutative*, in marked contrast to (rational, real, or complex) numbers whose product is independent of the order of the factors. An example of two matrices not commuting with each other is:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix},$$

whereas

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}.$$

Besides the ordinary matrix multiplication just described, there exist other less frequently used operations on matrices that can be considered forms of multiplication

transpose of a matrix**A** is another matrix $\mathbf{A}^T$ (also written $\mathbf{A}'$, $\mathbf{A}^{tr}$, ${}^t\mathbf{A}$ or $\mathbf{A}^t$) created by any one of the following equivalent actions:

- reflect **A** over its main diagonal (which runs from top-left to bottom-right) to obtain $\mathbf{A}^T$
- write the rows of **A** as the columns of $\mathbf{A}^T$
- write the columns of **A** as the rows of $\mathbf{A}^T$

Formally, the *i*th row, *j*th column element of $\mathbf{A}^T$ is the *j*th row, *i*th column element of **A**:

$$[\mathbf{A}^T]_{ij} = [\mathbf{A}]_{ji}$$

If **A** is an $m \times n$ matrix then $\mathbf{A}^T$ is an $n \times m$ matrix.

Examples

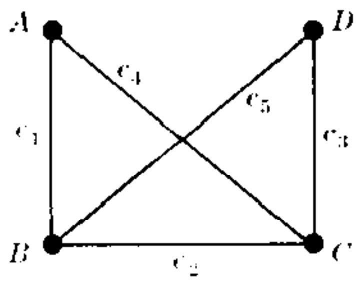
$$\begin{bmatrix} 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

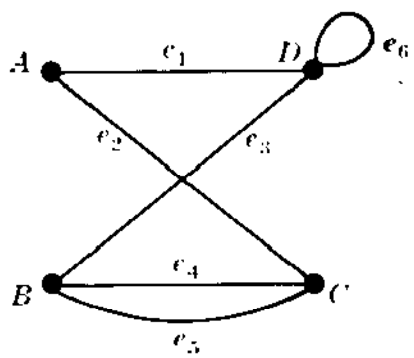
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Graphs:

A graph G consists of two things:
(i) A set V whose elements are vertices, points or nodes.
(ii) A set E of unordered pairs of distinct vertices called edges.
We denote such a graph by G(V,E) .
Vertices u and v are said to be adjacent if there is an edge {u,v}.
Graphs are the most useful model with computer science such as logical design, formal languages, communication network, compiler writing and retrieval.
G(V,E)
$V = \{V_1, V_2, V_3, V_4\}$
$E = \{e_1, e_2, e_3, e_4\}$
$E = \{(v_1,v_2),(v_2,v_3),(v_3,v_1),(v_3,v4)\}$



(a) Graph



(b) Multigraph

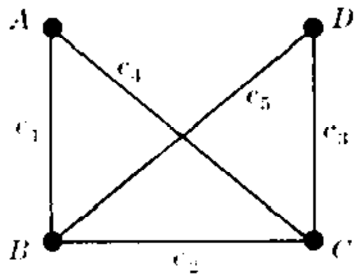
For example we have in (a) the graph G(V,E) where (i) V consists of four vertices A, B, C, D ; and (ii) E consists of five edges e_1 ={A,B}, e_2 = {B,C}, e_3 = {C, D}, e_4 = {A, C}, e_5 = {B, D}.
The diagram in (b) is not a graph but a multigraph . The reason is that e4 and e_5 are multiple edges, i.e. edges connecting the same endpoints, and e6 is a loop, i.e. an edge whose endpoints are the same vertex. The definition of a graph does not permit such multiple edges or loops.
Let G(V,E) be a graph. Let V' be a subset of V and let E' be subset of E whose end-points belong to V'. Then G(V',E') is a graph and is called a subgraph of G(V,E). If E' contains all the edges of E whose endpoints lie in V', then G(V',E') is called the subgraph generated by V'.

Degree :

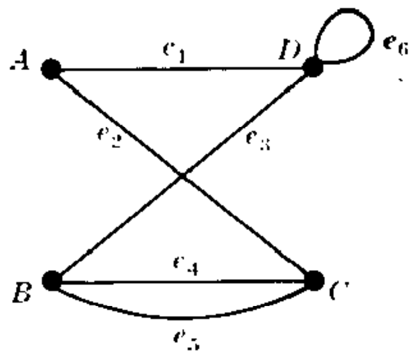
The degree of a vertex v, written deg(v), is equal to the number of edges which are incident on v. since each edge is counted twice in counting the degrees of the vertices of a graph, we have the following result.
Theorem: The sum of the degrees of the vertices of a graph is equal to twice the number of edges.

For example, in the figure (a) we have
deg(A) = 2,
deg(B) = 3,
deg(C) = 3,
deg(D) = 2

The sum of the degrees equals ten which, as expected, is twice the number of edges.
A vertex is said to be **even** or **odd** according as its degree is an even or odd number. Thus A and D are even vertices whereas B and C are odd vertices.
This theorem also holds for multigraphs where a loop is counted twice towards the degree of its endpoint. For example, in Fig (b) we have deg (D) = 4 since the edge e_6 is counted twice; hence D is an even vertex
A vertex of degree zero is called an isolated vertex.



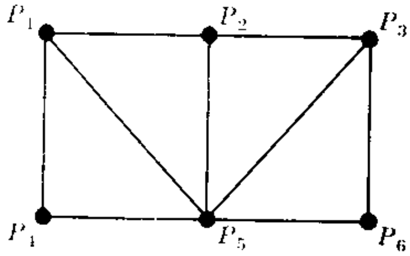
(a) Graph



(b) Multigraph

Connectivity

A **walk** in a multigraph consists of an alternating sequence of vertices and edges of the form
$v_0, e_1, v_1, e_2, v_2, \ldots\ldots, e_{n-1}, v_{n-1}, e_n, v_n$
Length of walk: is the number n of edges.
Closed walk: the walk is said to be closed if $v_0 = v_n$. Otherwise, we say that the walk is from v_0 to v_n.
Trail: is a walk in which all edges are distinct.
Path: is a walk in which all vertices are distinct.
Cycle: is a closed walk such that all vertices are distinct except $v_1 = v_n$
Example: Consider the following graph, then



(P_4, P_1, P_2, P_5, P_1, P_2, P_3, P_6)
is a walk from P_4 to P_6. It is not a trail since the edge {P_1,P_2} is used twice.
The sequence: (P_4, P_1, P_5, P_3, P_5, P_6)
Is not a walk since there is no edge {P_2, P_6}.
The sequence: (P_4, P_1, P_5, P_2, P_3, P_5, P_6)
Is a trail since no edge is used twice; but it is not a path since the vertex P_5 is used twice.
The sequence: (P_4, P_1, P_5, P_3, P_6)
Is a path from P_4 to P6.
The shortest path from P_4 to P_6 is (P_4, P_5, P_6) which has length 2 (2 edges only)
The distance between vertices u & v d(u,v) is the length of the shortest path
$d(P_4,P_6) = 2$

Connectivity, Connected Components

A graph G is connected if there is a path between any two of its vertices. The graph in Fig.(4) is connected, but the graph in Fig. (5) is not connected since, for example, there is no path between vertices D and E.
Suppose G is a graph. A connected subgraph H of G is called a connected component of G if H is not contained in any larger connected subgraph of G. It is clear that any graph G can be partitioned into its connected components. For example, the graph G in Fig. (5) has three connected components, the subgraphs induced by the vertex sets {A,C,D}, {E,F}, and {B}.
The vertex B in Fig. (5) is called an isolated vertex since B does not belong to any edge or, in other words, deg(B) = 0

Distance

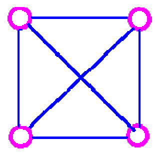
Consider a connected graph G. The distance between vertices u and v in G, written d(u,v),is the length of the shortest path between u and v .

PlanarGraphs

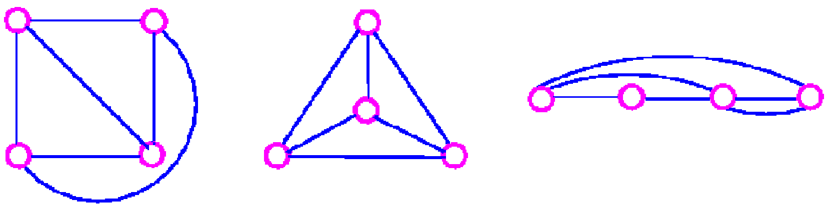
A graph G is planar if it can be drawn in the plane in such a way that no two edges meet each other except at a vertex to which they are incident. Any such drawing is called a plane drawing of G.

For example, the graph K_4 is planar, since it can be drawn in the plane without edges crossing.

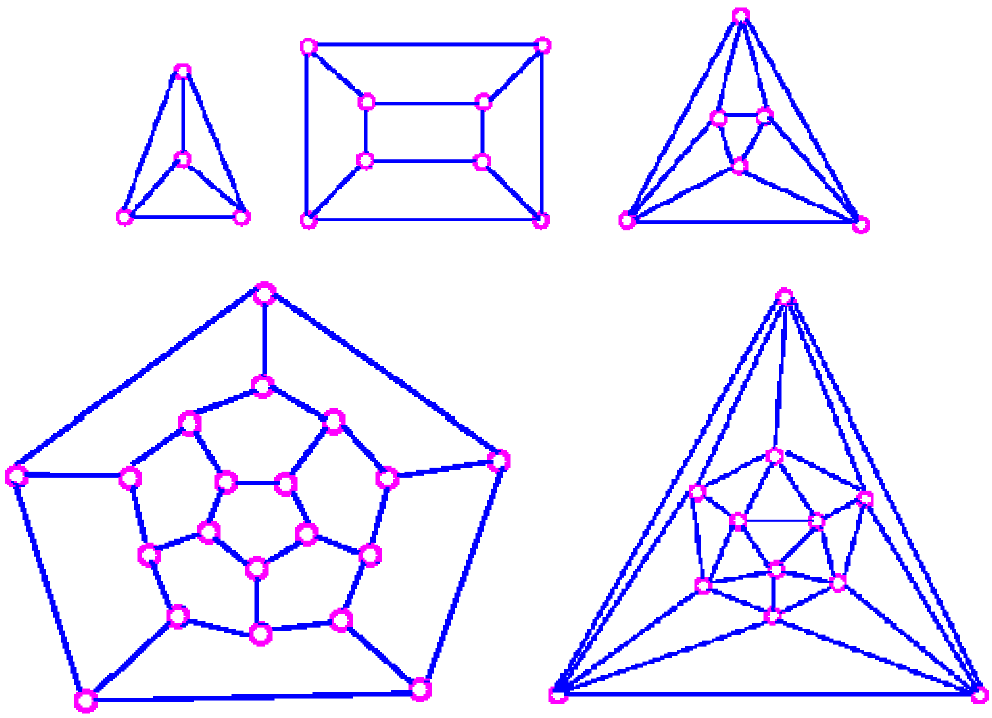
K_4



The three plane drawings of K_4 are:



The five Platonic graphs are all planar.

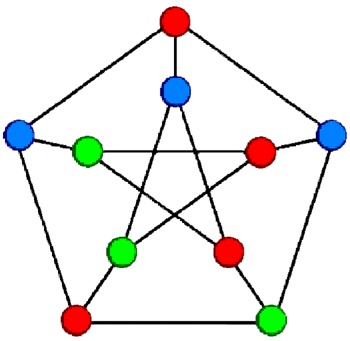


On the other hand, the complete bipartite graph $K_{3,3}$ is not planar, since every drawing of $K_{3,3}$ contains at least one crossing. why? because $K_{3,3}$ has a cycle which must appear in any plane drawing.

Graph coloring

it is a special case of graph labeling; it is an assignment of labels traditionally called "colors" to elements of a graph subject to certain constraints. In its simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color; this is called a **vertex coloring**. Similarly, an edge coloring assigns a color to each edge so that no two adjacent edges share the same color, and a **face coloring** of a planar graph assigns a color to each face or region so that no two faces that share a boundary have the same color.

Vertex coloring is the starting point of the subject, and other coloring problems can be transformed into a vertex version. For example, an edge coloring of a graph is just a vertex coloring of its line graph, and a face coloring of a plane graph is just a vertex coloring of its dual. However, non-vertex coloring problems are often stated and studied *as is*. That is partly for perspective, and partly because some problems are best studied in non-vertex form, as for instance is edge coloring.



Tree:

Tree is a connected graph with no cycle

Theorem:

Let G be graph with more than one vertex. Then the following are equivalence:

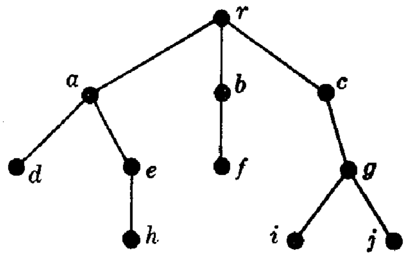
1) G is a tree.

2) G is cycle-free with (n-1) edges.

3) G is connected and has (n-1) edges. (i.e: if any edge is deleted then the resulting graph is not connected)

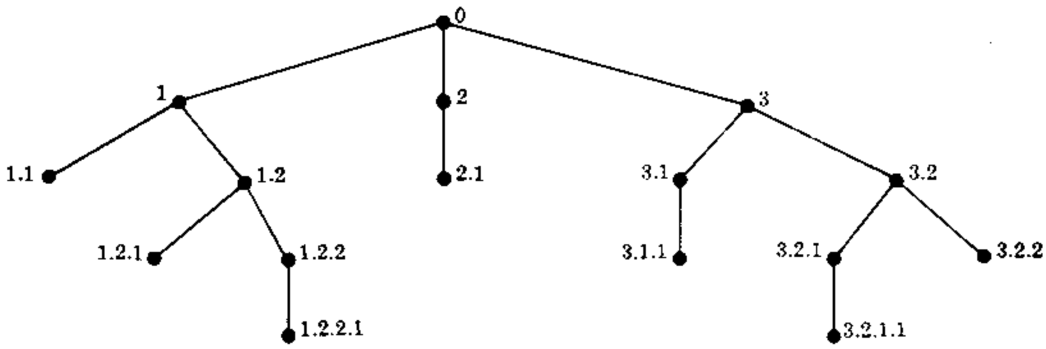
Rooted tree:

A rooted tree R consists of a tree graph together with vertex r called the root of the tree.

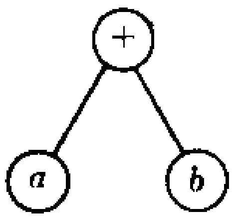


Height or depth: The number of levels of a tree
Leaves: The vertices of the tree that have no child (vertices with degree one)
Order Rooted Tree (ORT): Whenever draw the digraph of a tree, we assume some ordering at each level, by arranging children from left to right.

Degree of tree: The largest number of children in the vertices of the tree
Binary tree : every vertex has at most 2 children
Any algebraic expression involving bi



nary operations +, -, ×, ÷ can be represented by an order rooted tree (ORT)
the binary rooted tree for a+b is :



The variable in the expression a & b appear as leaves and the operations appear as the other vertices.

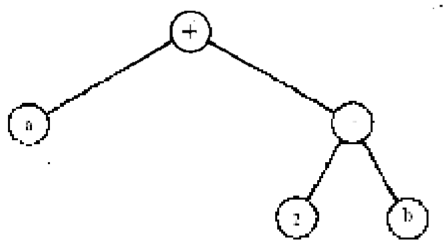
Polish notation:

The polish notation form of an algebraic expression represents the expression unambiguously with out the need for parentheses

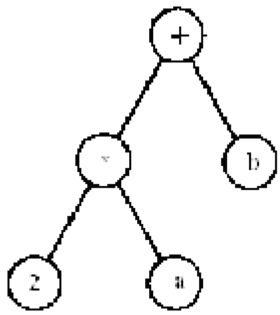
1) a + b (infix)
2) + a b (prefix)
3) a b + (postfix)

example 1:
infix polish notation is : a + b
prefix polish notation : + a b

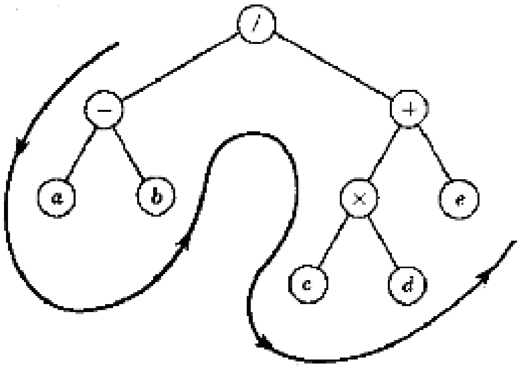
example 2:
infix polish notation is : a + 2 * b
prefix polish notation : + a * 2 b



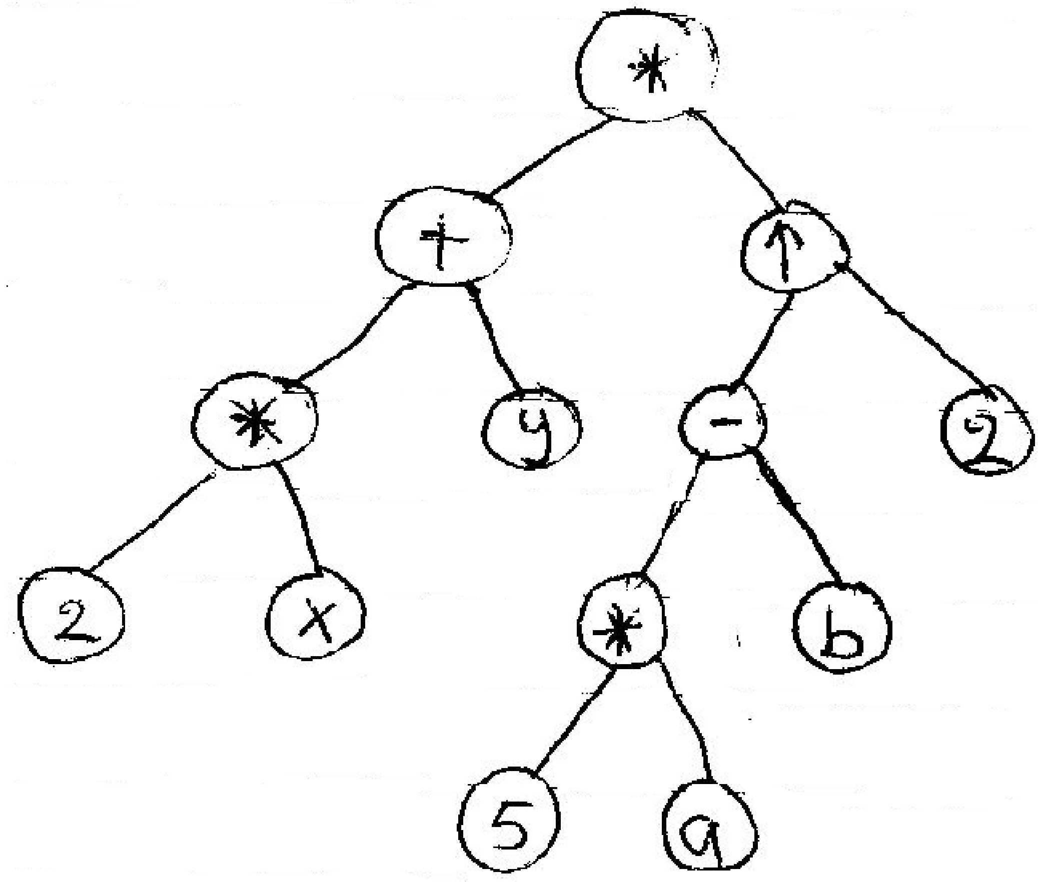
example 3:
infix polish notation is : 2 * a + b
prefix polish notation : + * 2 a b



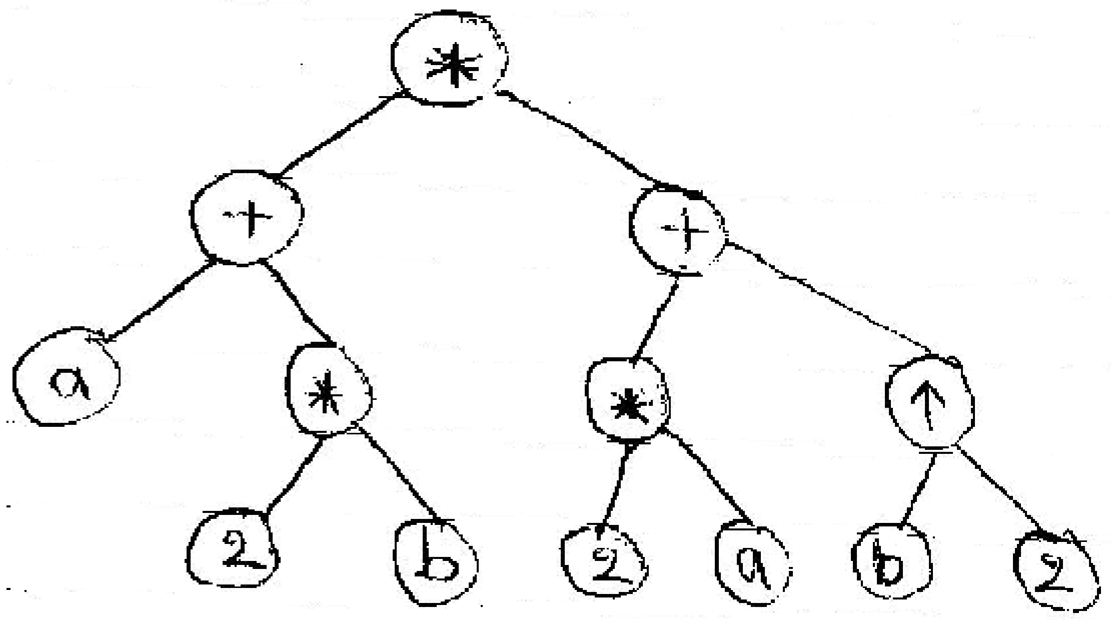
example 4:
infix polish notation is : (a – b) / (c * d) + e)
prefix polish notation : / - a b + * c d e



example 5:
infix polish notation is : (2 * x + y) (5 * a – b)^2
prefix polish notation : * + * 2 x y ^ - * 5 a b 2



example 6:
infix polish notation is : (a + 2 * b) (2 * a + b^2)
prefix polish notation : * + a * 2 b + * 2 a ^ b 2



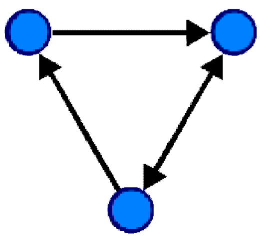
Directed graph

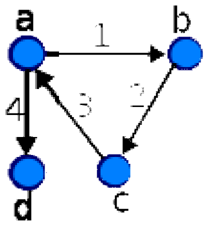
It is a graph, or set of vertices connected by edges, where the edges have a direction associated with them. In formal terms, a directed graph is an ordered pair G = (V, A) (sometimes G = (V, E)) where

- V is a set whose elements are called vertices, nodes, or points;
- A is a set of ordered pairs of vertices, called arrows, directed edges (sometimes simply edges with the corresponding set named E instead of A), directed arcs, or directed lines.

It differs from an ordinary or undirected graph, in that the latter is defined in terms of unordered pairs of vertices, which are usually called edges, arcs, or lines.

A directed graph is called a simple digraph if it has no multiple arrows (two or more edges that connect the same two vertices in the same direction) and no loops (edges that connect vertices to themselves). A directed graph is called a directed multigraph or multidigraph if it may have multiple arrows (and sometimes loops). In the latter case the arrow set forms a multiset, rather than a set, of ordered pairs of vertices.





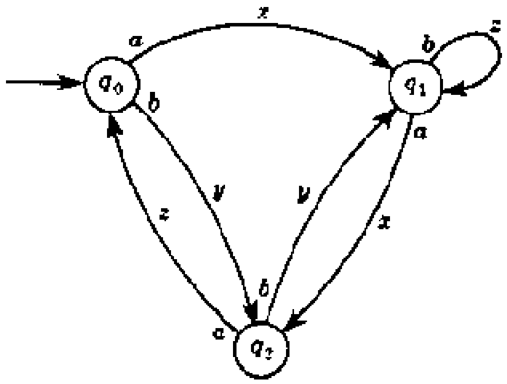
	1	2	3	4
a	1	0	-1	1
b	-1	1	0	0
c	0	-1	1	0
d	0	0	0	-1

Finite state machines (FSM):

We may view a digital computer as a machine which is in a certain “internal state” at any given moment. The computer “reads” an input symbol, and then “prints” an output symbol and changes its “state”. The output symbol depends solely upon the input symbol and the internal state of the machine, and the internal state of the machine depends solely upon the preceding state of the machine and the preceding input symbol.

A finite state machine FSM (or complete sequential machine) M consists of five things:
(1) A finite set A of input symbols.
(2) A finite set S of internal states.
(3) A finite set Z of output symbols.
(4) A next-state function f
$f: S \times A \rightarrow S$
(5) An output function g
$g: S \times A \rightarrow Z$
This machine M is denoted by M = (A, S, Z, q_0, f, g) where q_0 is the initial state.

Example 1: The following defines a FSM with two input symbols, three internal states and three output symbols:
(1) A = {a, b}
(2) S = {q_0, q_1, q_2}
(3) Z = {x, y, z}
(4) Next-state function f: S x A→ S defined by :
$f(q_0, a) = q_1 f(q_1,a) = q_2 f(q_2,a) = q_0$
$f(q_0, b) = q_2$ $f(q_1, b) = q_1$ $f(q_2, b) = q_1$
(5) Output function g: S x A → Z defined by
$g(q_0, a) = x$ $g(q_1,a) = x$ $g(q_2,a) = z$
$g(q_0, b) = y$ $g(q_1, b) = z$ $g(q_2, b) = y$
There are two ways of representing a finite state machine in compact form. One way is by a table called the **state table** of machine, and the other way is by a labeled directed graph called the **state diagram** of the machine.



	a	b
q_0	q_1, x	q_2, y
q_1	q_2, x	q_1, z
q_2	q_0, z	q_1, y

Example 2:

If the input string**: abaab ,**is given to the machine in example (1), and suppose q_0 is the initial state of the machine.

We calculate the string of states and the string of output symbols from the state diagram by beginning at the vertex q_0 and following the arrows which are labeled with the input symbols:

$$q_0 \xrightarrow{a,\,x} q_1 \xrightarrow{b,\,z} q_1 \xrightarrow{a,\,x} q_2 \xrightarrow{a,\,z} q_0 \xrightarrow{b,\,y} q_2$$

This yields the following strings of states and output symbols:

State : q_0 $q_1 q_1$ q_2 q_0 q_2

Output symbols : x z x z y