0.1 Vector space

Definition 0.1.1. A vector space V over a field $\mathbb{K}$ is a set V with two operations called addition $+$ and multiplication $\cdot$ such that the following axioms are satisfied:

(1) (i) $u + v \in V$ for all $u, v \in V$. (Addition is closed)

(ii) $u + v = v + u$ for all $u, v \in V$. (Addition is commutative)

(iii) $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$. (Addition is associative)

(iv) There exists an element $0 \in V$, called the zero vector, such that $u + 0 = 0 + u = u$ for all $u \in V$.

(v) For all $u \in V$ there exists an element $-u \in V$, called the additive inverse of u, such that $u + (-u) = 0 = -u + u$.

(2) (i) $\alpha \cdot u \in V$ for all $u \in V$ and $\alpha \in \mathbb{K}$.

(ii) $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$ for all $u, v \in V$ and $\alpha \in \mathbb{K}$.

(iii) $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u.$ for all $u \in V$ and $\alpha, \beta \in \mathbb{K}$.

(iv) $(\alpha\beta) \cdot u = \alpha \cdot (\beta \cdot u)$ for all $u \in V$ and $\alpha, \beta \in \mathbb{K}$.

(v) For all $u \in V$ there exists an element $1 \in \mathbb{K}$, called the multiplicative identity of u, such that $1 \cdot u = u \cdot 1 = u$.

Example 0.1.2. Let $\mathbb{C}$ be the set of complex numbers. Define addition in $\mathbb{C}$ by

$$(a + bi) + (c + di) = (a + c) + (b + d)i \qquad \text{for all} \quad a, b, c, d \in \mathbb{R}, \tag{1}$$

and define scalar multiplication by

$$\alpha \cdot (a + bi) = \alpha a + \alpha bi \qquad \text{for all scalars } \alpha \in \mathbb{R}, \text{and for all } a, b \in \mathbb{R}. \tag{2}$$

Show that $(\mathbb{C}, +, \cdot)$ is a vector space over $\mathbb{R}$.

Solution : Let $u = a + bi,\ v = c + di,\ w = e + fi \in \mathbb{C}$, where $a, b, c, d, e, f \in \mathbb{R}$, we have

(1)

(i) The addition is closed :

$$\begin{aligned} u + v &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i \qquad \text{by (1).} \end{aligned}$$

Since $(a + c)$ and $(b + d)$ are real numbers then $u + v \in \mathbb{C}$.

(ii) The addition is commutative:

$$\begin{aligned} u+v &= (a+bi)+(c+di) \\ &= (a+c)+(b+d)i \qquad \text{by } (1), \\ &= (c+a)+(d+b)i \qquad \text{because addition on } \mathbb{R} \text{ is commutative}, \\ &= (c+di)+(a+bi) \qquad \text{by } (1), \\ &= v+u \end{aligned}$$

(iii) The addition is associative: we have to prove that $u+(v+w)=(u+v)+w$ for all $u,v,w\in\mathbb{C}$.

The left hand side (L.H.S):

$$\begin{aligned} u+(v+w) &= u+[(c+di)+(e+fi)] \\ &= (a+bi)+[(c+e)+(d+f)i] \qquad \text{by } (1), \\ &= [a+(c+e)]+[b+(d+f)]i \qquad \text{by } (1), \\ &= [(a+c)+e]+[(b+d)+f]i \qquad \text{because addition on } \mathbb{R} \text{ is associative.} \end{aligned}$$

The right hand side (R.H.S):

$$\begin{aligned} (u+v)+w &= [(a+bi)+(c+di)]+w \\ &= [(a+c)+(b+d)i]+(e+fi) \qquad \text{by } (1), \\ &= [(a+c)+e]+[(b+d)+f)]i \qquad \text{by } (1). \end{aligned}$$

Then L.H.S=R.H.S

(iv) The additive identity : For all $u=a+bi\in\mathbb{C}$, we have

$$\begin{aligned} (a+bi)+(0+0i) &= (a+0)+(b+0)i \qquad \text{by } (1), \\ &= a+bi \qquad \text{because } 0 \text{ is the additive identity in } \mathbb{R}. \end{aligned}$$

Then the additive identity of $\mathbb{C}$ is $(0+0i)$.

(v) The additive inverse : For all $u=a+bi\in\mathbb{C}$, we have

$$\begin{aligned} (a+bi)+\Big(-a+(-b)i\Big) &= \Big(a+(-a)\Big)+\Big(b+(-b)\Big)i \qquad \text{by } (1), \\ &= 0+0i \qquad \text{because } (-a) \text{ is the additive inverse of } a \text{ in } \mathbb{R}. \end{aligned}$$

Then the additive inverse of $a+bi\in\mathbb{C}$ is $-a+(-b)i$.

(2) Let $u=a+bi, v=c+di\in\mathbb{C}$ and $\alpha\in\mathbb{R}$.

(i) We have to prove that $\alpha\cdot u\in\mathbb{C}$.

$$\begin{aligned} \alpha\cdot u &= \alpha\cdot(a+bi) \\ &= \alpha a+\alpha bi \end{aligned}$$

Since $\alpha a, \alpha b \in \mathbb{R}$, then $\alpha \cdot u \in \mathbb{C}$.

(ii) We have to prove that $\alpha \cdot (u+v) = \alpha \cdot u + \alpha \cdot v$ for all $u, v \in \mathbb{C}$ and $\alpha \in \mathbb{R}$.

The left hand side (L.H.S) :

$$\begin{aligned}
\alpha \cdot (u+v) &= \alpha \cdot [(a+bi)+(c+di)] \\
&= \alpha \cdot [(a+c)+(b+d)i] && \text{by (1)} \\
&= \alpha(a+c)+\alpha(b+d)i && \text{by (2)} \\
&= (\alpha a+\alpha c)+(\alpha b+\alpha d)i && \text{because multiplication distributes over addition in } \mathbb{R}.
\end{aligned}$$

The right hand side (R.H.S) :

$$\begin{aligned}
\alpha \cdot u + \alpha \cdot v &= \alpha \cdot (a+bi) + \alpha \cdot (c+di) \\
&= (\alpha a+\alpha bi)+(\alpha c+\alpha di) && \text{by (2),} \\
&= (\alpha a+\alpha c)+(\alpha b+\alpha d)i && \text{by (1),}
\end{aligned}$$

Then L.H.S=R.H.S

(iii) We have to prove that $(\alpha+\beta)\cdot u = \alpha \cdot u + \beta \cdot u$ for all $u \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$.

The L.H.S :

$$\begin{aligned}
(\alpha+\beta)\cdot u &= (\alpha+\beta)\cdot(a+bi) \\
&= (\alpha+\beta)a+(\alpha+\beta)bi && \text{by (2),} \\
&= (\alpha a+\beta a)+(\alpha b+\beta b)i && \text{because multiplication distributes over addition in } \mathbb{R}.
\end{aligned}$$

The R.H.S :

$$\begin{aligned}
\alpha \cdot u + \beta \cdot u &= \alpha \cdot (a+bi) + \beta \cdot (a+bi) \\
&= (\alpha a+\alpha bi)+(\beta a+\beta bi) && \text{by (2),} \\
&= (\alpha a+\beta a)+(\alpha b+\beta b)i && \text{by (1).}
\end{aligned}$$

Then L.H.S=R.H.S

(iv) We have to prove that $(\alpha\ \beta)\cdot u = \alpha \cdot (\beta \cdot u)$ for all $u \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$.

The L.H.S :

$$\begin{aligned}
(\alpha\beta)\cdot u &= (\alpha\beta)\cdot(a+bi) \\
&= (\alpha\beta)a+(\alpha\beta)b\ i && \text{by (2),} \\
&= \alpha\beta a+\alpha\beta b\ i && \text{because multiplication is associative in } \mathbb{R}.
\end{aligned}$$

The R.H.S :

$$\begin{aligned}
\alpha \cdot (\beta \cdot u) &= \alpha \cdot [\beta \cdot (a+bi)] \\
&= \alpha \cdot [\beta a+\beta b\ i] && \text{by (2),} \\
&= \alpha\beta a+\alpha\beta b\ i && \text{by (2).}
\end{aligned}$$

Then L.H.S=R.H.S

(v) The multiplicative identity : we have to show that $1 \cdot u = u$ for all $u = a + bi \in \mathbb{C}$. (Note that, 1 represents scalar from the field $\mathbb{R}$ and NOT from the set $\mathbb{C}$).

$$
\begin{aligned}
1 \cdot u &= 1 \cdot (a + bi) \\
&= 1a + 1bi \qquad \text{by (2)}, \\
&= a + bi \\
&= u
\end{aligned}
$$

We have proved that all axioms hold in $\mathbb{C}$. Hence, $(\mathbb{C}, +, \cdot)$ is a vector space over $\mathbb{R}$.

Example 0.1.3. Let $M_{2\times 2}(\mathbb{R}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}\right\}$ be the set of all two by two matrices with entries in $\mathbb{R}$. For $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in M_{2\times 2}$ and $\alpha \in \mathbb{R}$, addition and scalar multiplication of matrices defined by

$$A + B = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{pmatrix} \tag{3}$$

$$\alpha \cdot A = \alpha \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} \alpha a_1 & \alpha a_2 \\ \alpha a_3 & \alpha a_4 \end{pmatrix}. \tag{4}$$

Prove that $(M_{2\times 2}, +, \cdot)$ is a vector space over $\mathbb{R}$.

Solution: Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \in M_{2\times 2}$.

(1)

(i)

$$A + B = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{pmatrix} \quad \text{by (3).}$$

Since $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are real numbers, then $a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4 \in \mathbb{R}$. Hence, $A + B \in M_{2\times 2}(\mathbb{R})$.

(ii) We have to show that $A + B = B + A$ for all $A, B \in M_{2\times 2}$.

$$\begin{aligned}
A + B &= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \\
&= \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{pmatrix} \quad \text{by (3),} \\
&= \begin{pmatrix} b_1 + a_1 & b_2 + a_2 \\ b_3 + a_3 & b_4 + a_4 \end{pmatrix} \quad \text{because addition on } \mathbb{R} \text{ is commutative} \\
&= \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} + \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad \text{by (3),} \\
&= B + A
\end{aligned}$$

(iii) We have to show that $A + (B + C) = (A + B) + C$ for all $A, B, C \in M_{2\times 2}$.

The L.H.S:

$$\begin{aligned}
A+(B+C) &= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \left[\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} + \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}\right] \\
&= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} b_1+c_1 & b_2+c_2 \\ b_3+c_3 & b_4+c_4 \end{pmatrix} \qquad \text{by (3)}, \\
&= \begin{pmatrix} a_1+(b_1+c_1) & a_2+(b_2+c_2) \\ a_3+(b_3+c_3) & a_4+(b_4+c_4) \end{pmatrix} \qquad \text{by (3)}, \\
&= \begin{pmatrix} (a_1+b_1)+c_1 & (a_2+b_2)+c_2 \\ (a_3+b_3)+c_3 & (a_4+b_4)+c_4 \end{pmatrix} \qquad \text{because addition on } \mathbb{R} \text{ is associative.}
\end{aligned}$$

The R.H.S:

$$\begin{aligned}
(A+B)+C &= \left[\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\right] + \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \\
&= \begin{pmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{pmatrix} + \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \qquad \text{by (3)}, \\
&= \begin{pmatrix} (a_1+b_1)+c_1 & (a_2+b_2)+c_2 \\ (a_3+b_3)+c_3 & (a_4+b_4)+c_4 \end{pmatrix} \qquad \text{by (3)}.
\end{aligned}$$

Then L.H.S= R.H.S

(iv) For all $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in M_{2\times 2}$, we have

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

Then the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the additive identity.

(v) For all $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in M_{2\times 2}$, we have $(-A) = \begin{pmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{pmatrix} \in M_{2\times 2}$, where

$$A+(-A) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the matrix $(-A)$ is the additive inverse for the matrix A.

(2)

(i) We have to show that $\alpha \cdot A \in M_{2\times 2}(\mathbb{R})$ for all $A \in M_{2\times 2}$ and $\alpha \in \mathbb{R}$.

$\alpha \cdot A = \alpha \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} \alpha a_1 & \alpha a_2 \\ \alpha a_3 & \alpha a_4 \end{pmatrix}$ by (4).

Since $\alpha, a_1, a_2, a_3, a_4$ are real numbers then $\alpha a_1, \alpha a_2, \alpha a_3, \alpha a_4 \in \mathbb{R}$.

Hence, $\alpha \cdot A \in M_{2\times 2}(\mathbb{R})$.

(ii) We have to show that $\alpha \cdot (A+B) = \alpha \cdot A + \alpha \cdot B$ for all $A, B \in M_{2\times 2}$ and $\alpha \in \mathbb{R}$.

The L.H.S:

$$\begin{aligned}
\alpha \cdot (A+B) &= \alpha \cdot \left[\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\right] \\
&= \alpha \cdot \begin{pmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{pmatrix} && \text{by (3),} \\
&= \begin{pmatrix} \alpha(a_1+b_1) & \alpha(a_2+b_2) \\ \alpha(a_3+b_3) & \alpha(a_4+b_4) \end{pmatrix} && \text{by (4),} \\
&= \begin{pmatrix} \alpha a_1+\alpha b_1 & \alpha a_2+\alpha b_2 \\ \alpha a_3+\alpha b_3 & \alpha a_4+\alpha b_4 \end{pmatrix} && \text{because multiplication distributes over addition in } \mathbb{R}.
\end{aligned}$$

The R.H.S:

$$\begin{aligned}
\alpha \cdot A + \alpha \cdot B &= \alpha \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \alpha \cdot \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \\
&= \begin{pmatrix} \alpha a_1 & \alpha a_2 \\ \alpha a_3 & \alpha a_4 \end{pmatrix} + \begin{pmatrix} \alpha b_1 & \alpha b_2 \\ \alpha b_3 & \alpha b_4 \end{pmatrix} && \text{by (4),} \\
&= \begin{pmatrix} \alpha a_1+\alpha b_1 & \alpha a_2+\alpha b_2 \\ \alpha a_3+\alpha b_3 & \alpha a_4+\alpha b_4 \end{pmatrix} && \text{by (3).}
\end{aligned}$$

Then L.H.S= R.H.S

(iii) We have to show that $(\alpha+\beta) \cdot A = \alpha \cdot A + \beta \cdot A$ for all $A \in M_{2\times 2}$ and $\alpha, \beta \in \mathbb{R}$.

The L.H.S:

$$\begin{aligned}
(\alpha+\beta) \cdot A &= (\alpha+\beta) \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \\
&= \begin{pmatrix} (\alpha+\beta)a_1 & (\alpha+\beta)a_2 \\ (\alpha+\beta)a_3 & (\alpha+\beta)a_4 \end{pmatrix} && \text{by (4),} \\
&= \begin{pmatrix} \alpha a_1+\beta a_1 & \alpha a_2+\beta a_2 \\ \alpha a_3+\beta a_3 & \alpha a_4+\beta a_4 \end{pmatrix} && \text{because multiplication distributes over addition in } \mathbb{R}.
\end{aligned}$$

The R.H.S:

$$\begin{aligned}
\alpha \cdot A + \beta \cdot A &= \alpha \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \beta \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \\
&= \begin{pmatrix} \alpha a_1 & \alpha a_2 \\ \alpha a_3 & \alpha a_4 \end{pmatrix} + \begin{pmatrix} \beta a_1 & \beta a_2 \\ \beta a_3 & \beta a_4 \end{pmatrix} && \text{by (4),} \\
&= \begin{pmatrix} \alpha a_1+\beta a_1 & \alpha a_2+\beta a_2 \\ \alpha a_3+\beta a_3 & \alpha a_4+\beta a_4 \end{pmatrix} && \text{by (3).}
\end{aligned}$$

Then L.H.S = R.H.S

(iv) We have to show that $(\alpha\beta)\cdot A = \alpha\cdot(\beta\cdot A)$ for all l $A \in M_{2\times 2}$ and $\alpha, \beta \in \mathbb{R}$.

The L.H.S:

$$\begin{aligned}
(\alpha\beta)\cdot A &= (\alpha\beta)\cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \\
&= \begin{pmatrix} (\alpha\beta)a_1 & (\alpha\beta)a_2 \\ (\alpha\beta)a_3 & (\alpha\beta)a_4 \end{pmatrix} && \text{by (4),} \\
&= \begin{pmatrix} \alpha(\beta a_1) & \alpha(\beta a_2) \\ \alpha(\beta a_3) & \alpha(\beta a_4) \end{pmatrix} && \text{because multiplication on } \mathbb{R} \text{ is associative.}
\end{aligned}$$

The R.H.S:

$$\begin{aligned}
\alpha\cdot(\beta\cdot A) &= \alpha\cdot\left[\beta\cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\right] \\
&= \alpha\cdot \begin{pmatrix} \beta a_1 & \beta a_2 \\ \beta a_3 & \beta a_4 \end{pmatrix} && \text{by (4),} \\
&= \begin{pmatrix} \alpha(\beta a_1) & \alpha(\beta a_2) \\ \alpha(\beta a_3) & \alpha(\beta a_4) \end{pmatrix} && \text{by (4).}
\end{aligned}$$

Then L.H.S= R.H.S

(v) For all $A \in M_{2\times 2}$, we have $1 \in \mathbb{R}$ such that

$$1\cdot A = 1\cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 1a_1 & 1a_2 \\ 1a_3 & 1a_4 \end{pmatrix} = A.$$

Then $1 \in \mathbb{R}$ is the multiplicative identity .

Example 0.1.4. Let $V = \{x \in \mathbb{R} \mid x > 0\}$. For $x, y \in V$ and $\alpha \in \mathbb{R}$, we define addition and scalar multiplication as following

$$\begin{aligned} x \oplus y &= xy, \\ \alpha \otimes x &= x^{\alpha}. \end{aligned}$$

Show that $(V, \oplus, \otimes)$ is a vector space over $\mathbb{R}$.

Example 0.1.5. Is the set $V = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b > 0 \right\}$ with the usual addition and scalar multiplication of matrices define a vector space over $\mathbb{R}$?

Solution: Let $\alpha = -2 \in \mathbb{R}$, then $\alpha \begin{bmatrix} a \\ b \end{bmatrix} = -2 \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2a \\ -2b \end{bmatrix} \notin V$.

Since $a, b > 0$ then $-2a, -2b < 0$.

Proposition 0.1.6. *Let V be a vector space over $\mathbb{K}$, then we have*

(1) The additive identity, $0 \in V$, is unique.

(2) The additive inverse, $(-u) \in V$, for $u \in V$ is unique.

(3) For all $u \in V$ we have $0 \cdot u = 0$.

(4) For all $u \in V$ we have $(-1) \cdot u = -u$.

(5) For all $u, v, w \in V$, if $u + v = u + w$ then $v = w$.

(6) For all $u, v \in V$, the equation $u + x = v$ has a unique solution $x = v - u \in V$.

(7) For all $u \in V$, we have $-(-u) = u$.

0.2 Subspace

In this section we suppose that $(V, +, \cdot)$ is a vector space over $\mathbb{K}$.

Definition 0.2.1. A non-empty subset U of V is called a subspace of V if $(U, +, \cdot)$ is a vector space over $\mathbb{K}$.

Proposition 0.2.2. *Anon-empty subset U of a vector space V over $\mathbb{K}$ is a subspace of V if and only if the following conditions are satisfied:*

(1) $0 \in U$.

(2) For all $u, v \in U$, we have $u + v \in U$.

(3) For all $u \in U$ and $\alpha \in \mathbb{K}$, we have $\alpha \cdot u \in U$.

Remark 0.2.3. Every vector space V has two subspaces namely V and $\{0\}$. Any other subspace of V is called a proper subspace of V.

Example 0.2.4. Show that which of these sets are subspace of $\mathbb{R}^3$

(1) $U = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$.

(2) $U = \{(x, y, 1) \mid x, y \in \mathbb{R}\}$.

.

Proposition 0.2.5. *If W_1 and W_2 are subspaces of V, then $W_1 \cap W_2$ is a subspace of V.*

Proof. We have to satisfy the three conditions in Proposition 0.2.2.

(1) Since W_1 and W_2 are subspaces of V, then $0 \in W_1$ and $0 \in W_2$.
Hence,
$$0 \in W_1 \cap W_2.$$

(2) Let $u, v \in W_1 \cap W_2$, then $u, v \in W_1$ and $u, v \in W_2$.
Since W_1 and W_2 are subspaces of V, then $u + v \in W_1$ and $u + v \in W_2$.
Hence,
$$u + v \in W_1 \cap W_2.$$

(3) Let $\alpha \in \mathbb{K}$ and $u \in W_1 \cap W_2$, then $u \in W_1$ and $u \in W_2$.
Since W_1 and W_2 are subspaces of V then $\alpha \cdot u \in W_1$ and $\alpha \cdot u \in W_2$.
Hence,
$$\alpha \cdot u \in W_1 \cap W_2.$$

□

Example 0.2.6. Show that if W_1 and W_2 are subspaces of a vector space V, then $W_1 \cup W_2$ is NOT a subspace of V.

To prove this, we have $W_1 = \{(a, 0) \mid a \in \mathbb{R}\}$ and $W_2 = \{(0, b) \mid b \in \mathbb{R}\}$ are both subspaces of $\mathbb{R}^2$. But $W_1 \cup W_2$ is not a subspace of $\mathbb{R}^2$ because $(1, 0) \in W_1 \cup W_2$ and $(0, 1) \in W_1 \cup W_2$ while $(1, 0) + (0, 1) = (1, 1) \notin W_1 \cup W_2$.

0.3 Linear Combinations and Span

Definition 0.3.1. Let $v_1, v_2, \cdots, v_n$ be vectors in a vector space V over $\mathbb{K}$. A linear combination of these vectors is any expression of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

for some scalars $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{K}$.

Example 0.3.2. Consider the vector space $\mathbb{R}^2$. The vector $v = (-7, -13)$ is a linear combination of $v_1 = (-2, 1)$ and $v_2 = (1, 5)$, where

$$v = 2v_1 + (-3)v_2.$$

Example 0.3.3. Consider the vector space $\mathbb{R}^2$. The vector $v = (1, -3)$ is a linear combination of $v_1 = (0, 1)$, $v_2 = (2, -1)$, $v_3 = (1, -2)$ and $v_4 = (0, 3)$ where

$$v = (-2)v_1 + (0)v_2 + 1v_3 + (\frac{1}{3})v_4.$$

Sometimes we cannot write a vector v in a vector space V as a linear combination of $v_1, v_2, \cdots, v_n \in V$, as explained in this example.

Example 0.3.4. Let $v_1 = (2, 5, 3), v_2 = (1, 1, 1)$, and $v = (4, 2, 0)$. Because there exist no scalars $\alpha_1, \alpha_2 \in \mathbb{K}$ such that $v = \alpha_1 v_1 + \alpha_2 v_2$ then v is not a linear combination of v_1 and v_2.

Definition 0.3.5. Let V be a vector space over $\mathbb{K}$, and let $S = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. We say that S spans V, or S generates V, if every vector v in V can be written as a linear combination of vectors in S. That is, for all $v \in V$, we have

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

for some scalars $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{K}$.

Example 0.3.6. Show that the set $S = \{(0,1,1),(1,0,1),(1,1,0)\}$ spans $\mathbb{R}^3$ and write the vector $(2,4,8)$ as a linear combination of vectors in S.

Solution:

A vector in $\mathbb{R}^3$ has the form $v = (x, y, z)$.

Hence we need to show that, for some scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, every such v can be written as

$$\begin{aligned}(x, y, z) &= \alpha_1(0,1,1) + \alpha_2(1,0,1) + \alpha_3(1,1,0) \\ &= (\alpha_2 + \alpha_3, \alpha_1 + \alpha_3, \alpha_1 + \alpha_2)\end{aligned}$$

This give us system of equations

$$\begin{aligned} x &= \alpha_2 + \alpha_3 \\ y &= \alpha_1 + \alpha_3 \\ z &= \alpha_1 + \alpha_2 \end{aligned}$$

This system of equations can be written in matrix form

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

We can write it as $A\alpha = b$. Since $det(A) = 2$ then this system has a solution.

Now, to write $(2,4,8)$ as a linear combination of vectors in S, we find that

$$A^{-1} = \begin{bmatrix} -0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \end{bmatrix}$$

Then

$$\alpha = A^{-1}b$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} -0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

So, $\alpha_1 = 5, \alpha_2 = 3, \alpha_3 = -1$, and

$$(2,4,8) = 5(0,1,1) + 3(1,0,1) + (-1)(1,1,0).$$