1. **Natural numbers** ($\mathbb{N}$):

$$1, 2, 3, 4, 5, \ldots$$

2. **Integer numbers** ($\mathbb{Z}$):

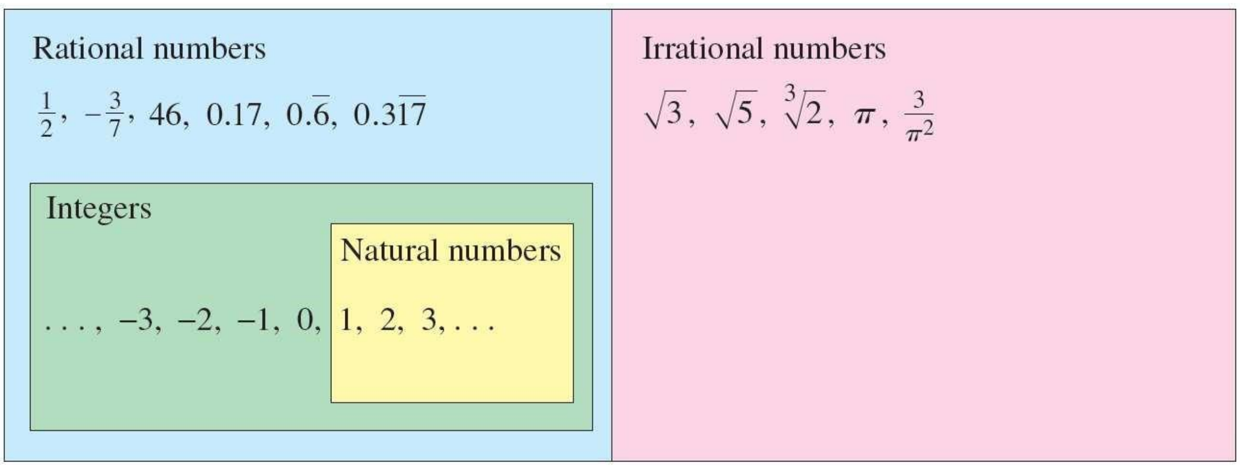
$$0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \ldots$$

3. **Rational numbers** ($\mathbb{Q}$):

$$r = \frac{m}{n}, \quad \text{where } m \in \mathbb{Z}, n \in \mathbb{N}$$

4. **Irrational Numbers** ($\mathbb{R}/\mathbb{Q}$):

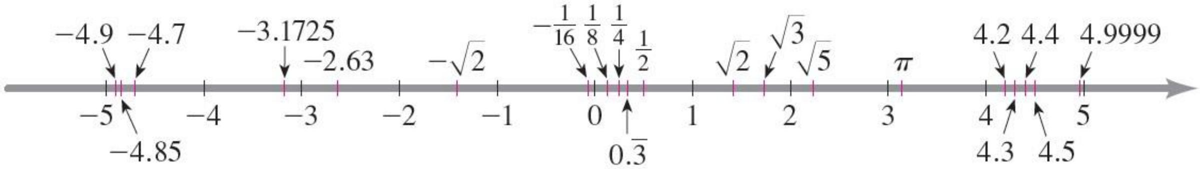
These are numbers that cannot be expressed as a ratio of integers.



5. **Real Numbers** ($\mathbb{R}$): The set of all real numbers is denoted by $\mathbb{R}$.

$$\mathbb{R} = \{x: -\infty < x < \infty\}$$

The real numbers can be represented by points on a line which is called a **coordinate line**, or a **real number line**, or simply a **real line**:



Properties of Real Numbers

PROPERTIES OF REAL NUMBERS

Property	Example	Description
Commutative Properties		
$a + b = b + a$	$7 + 3 = 3 + 7$	When we add two numbers, order doesn't matter.
$ab = ba$	$3 \cdot 5 = 5 \cdot 3$	When we multiply two numbers, order doesn't matter.
Associative Properties		
$(a + b) + c = a + (b + c)$	$(2 + 4) + 7 = 2 + (4 + 7)$	When we add three numbers, it doesn't matter which two we add first.
$(ab)c = a(bc)$	$(3 \cdot 7) \cdot 5 = 3 \cdot (7 \cdot 5)$	When we multiply three numbers, it doesn't matter which two we multiply first.
Distributive Property		
$a(b + c) = ab + ac$ $(b + c)a = ab + ac$	$2 \cdot (3 + 5) = 2 \cdot 3 + 2 \cdot 5$ $(3 + 5) \cdot 2 = 2 \cdot 3 + 2 \cdot 5$	When we multiply a number by a sum of two numbers, we get the same result as multiplying the number by each of the terms and then adding the results.

Addition and Subtraction

The number 0 is special for addition; it is called the **additive identity** because

$$a + 0 = a$$

for any real number *a.* Every real number *a* has a **negative**, $-a$, that satisfies

$$a + (-a) = 0$$

Subtraction is the operation that undoes addition; to subtract a number from another, we simply add the negative of that number. By definition

$$a - b = a + (-b)$$

To combine real numbers involving negatives, we use the following properties.

Properties of Zero

Let a and b be real numbers, variables, or algebraic expressions.

1. $a + 0 = a$ and $a - 0 = a$
2. $a \cdot 0 = 0$
3. $\dfrac{0}{a} = 0, \qquad a \neq 0$
4. $\dfrac{a}{0}$ is undefined.
5. **Zero-Factor Property:** If $ab = 0$, then $a = 0$ or $b = 0$.

PROPERTIES OF NEGATIVES

Property	Example
1. $(-1)a = -a$	$(-1)5 = -5$
2. $-(-a) = a$	$-(-5) = 5$
3. $(-a)b = a(-b) = -(ab)$	$(-5)7 = 5(-7) = -(5 \cdot 7)$
4. $(-a)(-b) = ab$	$(-4)(-3) = 4 \cdot 3$
5. $-(a + b) = -a - b$	$-(3 + 5) = -3 - 5$
6. $-(a - b) = b - a$	$-(5 - 8) = 8 - 5$

Multiplication and Division

PROPERTIES OF FRACTIONS

Property	Example	Description
1. $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$	$\dfrac{2}{3} \cdot \dfrac{5}{7} = \dfrac{2 \cdot 5}{3 \cdot 7} = \dfrac{10}{21}$	When **multiplying fractions**, multiply numerators and denominators.
2. $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c}$	$\dfrac{2}{3} \div \dfrac{5}{7} = \dfrac{2}{3} \cdot \dfrac{7}{5} = \dfrac{14}{15}$	When **dividing fractions**, invert the divisor and multiply.
3. $\dfrac{a}{c} + \dfrac{b}{c} = \dfrac{a + b}{c}$	$\dfrac{2}{5} + \dfrac{7}{5} = \dfrac{2 + 7}{5} = \dfrac{9}{5}$	When **adding fractions** with the **same denominator**, add the numerators.
4. $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$	$\dfrac{2}{5} + \dfrac{3}{7} = \dfrac{2 \cdot 7 + 3 \cdot 5}{35} = \dfrac{29}{35}$	When **adding fractions** with **different denominators**, find a common denominator. Then add the numerators.
5. $\dfrac{ac}{bc} = \dfrac{a}{b}$	$\dfrac{2 \cdot 5}{3 \cdot 5} = \dfrac{2}{3}$	**Cancel** numbers that are **common factors** in the numerator and denominator.
6. If $\dfrac{a}{b} = \dfrac{c}{d}$, then $ad = bc$	$\dfrac{2}{3} = \dfrac{6}{9}$, so $2 \cdot 9 = 3 \cdot 6$	**Cross multiply**.

Other properties of Real Numbers

Let x,y, and z be real numbers, then:

1. Precisely one of $x < y$ or $x = y$ or $x > y$ holds (Trichotomy Law).

2. If $x < y$ and $y < z$, then $x < z$ (Transitive Law).

3. If $x < y$ then $x+z < y+z$ (Addition Law for Order).

4. If $x < y$ and $z > 0$, then $xz < yz$ (Multiplication Law for Order).

   If $x < y$ and $z < 0$, then $xz > yz$

5. If $x>y$ then $-x<-y$

6- If $x>y$ then $\frac{1}{x} < \frac{1}{y}$

Sets and Intervals

A **set** is a collection of objects, and these objects are called the **elements** of the set.

If S is a set, the notation $a \in S$ means that a is an element of S

Some sets can be described by listing their elements within braces. For instance, the set A that consists of all positive integers less than 7 can be written as

$$A = \{1,2,3,4,5,6\}$$

We could also write A in **set-builder notation** as

$$A = \{x \mid x \text{ is an integer and } 0 < x < 7\}$$

which is read "A is the set of all x such that x is an integer and $0 < x < 7$. Certain sets of real numbers, called **intervals**, occur frequently in calculus and correspond geometrically to line segments.

If $a < b$, then the **open interval** from a to b consists of all numbers between a and b and is denoted by the symbol (a,b). $(a, b) = \{x: a < x < b\}$

The **closed interval** from a to b includes the endpoints and is denoted $[a,b]$. Using set-builder notation, we can write

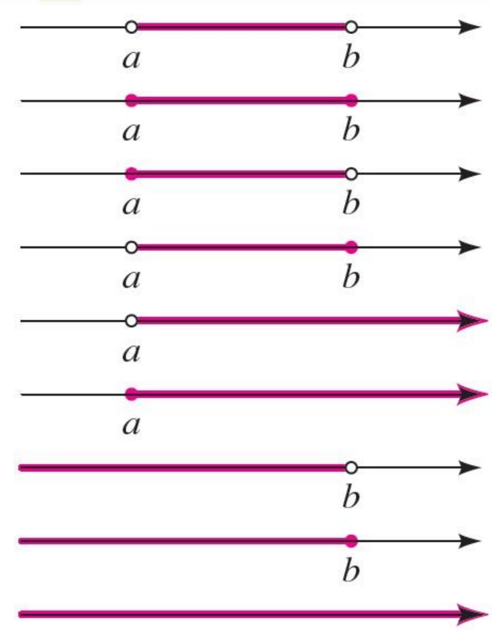
$[a, b] = \{x: a \leq x \leq b\}$

The **half open from the left** or **half closed from the right** (a,b] $(a, b] = \{x: a < x \leq b\}$

The half open from the right or **half closed from the left** [a,b), $[a, b) = \{X: a \leq x < b\}$

The following table lists the possible types of intervals.

Notation	Set description	Graph
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \leq x \leq b\}$	
$[a, b)$	$\{x \mid a \leq x < b\}$	
$(a, b]$	$\{x \mid a < x \leq b\}$	
(a, ∞)	$\{x \mid a < x\}$	
$[a, \infty)$	$\{x \mid a \leq x\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \leq b\}$	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

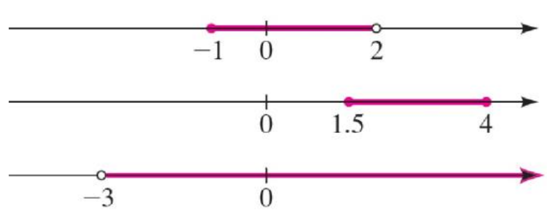


EXAMPLE: Express each interval in terms of inequalities, and then graph the interval:

(a) $[-1, 2) = \{x \mid -1 \leq x < 2\}$

(b) $[1.5, 4] = \{x \mid 1.5 \leq x \leq 4\}$

(c) $(-3, \infty) = \{x \mid -3 < x\}$



Inequalities: Find the set of all the following inequalities:

$Example(1)$: $\frac{7}{x} > 2, x \neq 0$, $x \in \mathbb{R}$

The first case : if x>0

$7 > 2x \Rightarrow \frac{7}{2} > x \Rightarrow x < \frac{7}{2}$

The solution of the first case is $(0, \frac{7}{2})$

The second case : if x<0

$\frac{7}{x} > 2 \Rightarrow 7 < 2x \Rightarrow \frac{7}{2} < x$

The solution of the second case is = ∅

Example(2): (x+3)(x+4)>0

Sol:

1^{st} case: (x+3)>0 ∧ (x+4)>0

x>-3 ∧ x>-4

sol. Set 1^{st} case =(-3,∞)

2^{st} case: (x+3)<0 ∧ (x+4)<0

X<-3 ∧ x<-4

sol. Set 2^{st} case =(-∞,-4)

sol. set=(-3, ∞) U (-∞,-4)=R/[-4,-3]

H.W: Find the sets of the following inequalities:

1- 2+3x<5x+8

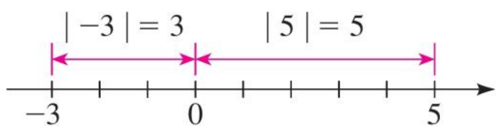
2- 4<3x-2≤ 10

3 - $\frac{x}{x-3} < 4$, $x \neq 3$, $x \in R$

4- $\frac{x-1}{x^2+x-6} < 0$

Absolute Value and Distance

The **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line.



Distance is always positive or zero, so we have $|a| \geq 0$ for every number a. Remembering that $-a$ is positive when a is negative, we have the following definition.

DEFINITION OF ABSOLUTE VALUE

If a is a real number, then the **absolute value** of a is

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

EXAMPLES:

(a) $|3| = 3$ (b) $|-3| = -(-3) = 3$ (c) $|0| = 0$ (d) $|3 - \pi| = -(3 - \pi) = \pi - 3$

Properties of absolute value

1-$|x| < a \ \ iff \ \ -a < x < a$
2-$|x| \leq a \ \ iff -a \leq x \leq a$
3-$|x| > a \ \ iff \ \ x > a \ \ or \ x < -a$
4-$|x| \geq a \ \ iff \ \ x \geq a \ \ or \ x \leq -a$

PROPERTIES OF ABSOLUTE VALUE

Property	Example	Description
1. $\lvert a\rvert \geq 0$	$\lvert -3\rvert = 3 \geq 0$	The absolute value of a number is always positive or zero.
2. $\lvert a\rvert = \lvert -a\rvert$	$\lvert 5\rvert = \lvert -5\rvert$	A number and its negative have the same absolute value.
3. $\lvert ab\rvert = \lvert a\rvert\lvert b\rvert$	$\lvert -2\cdot 5\rvert = \lvert -2\rvert\lvert 5\rvert$	The absolute value of a product is the product of the absolute values.
4. $\left\lvert \frac{a}{b}\right\rvert = \frac{\lvert a\rvert}{\lvert b\rvert}$	$\left\lvert \frac{12}{-3}\right\rvert = \frac{\lvert 12\rvert}{\lvert -3\rvert}$	The absolute value of a quotient is the quotient of the absolute values.

DISTANCE BETWEEN POINTS ON THE REAL LINE

If a and b are real numbers, then the **distance** between the points a and b on the real line is

$$d(a, b) = |b - a|$$

Ex: Find the set of all the following inequality:

$|7-4x| \geq 1$

Sol: $7-4x \geq 1$

$x \leq \frac{3}{2}$

Or

$7-4x \leq -1$

$x \geq 2$

Sol. Set:=$(\infty, \frac{3}{2}] \cup [2, \infty)$=R/$(\frac{3}{2}, 2)$

H.W: Find the set of each of the following inequalities:

1-$|x-4| < 5$

2-$|2x-5| = 7$

3-$|x-2| = |3x+4|$

4-$|2x-5| > 3$

5-$\left|\frac{3x+8}{2x-3}\right| \leq 4$

6-$|x^2+2x-24| > 0$

Note: Let a be a real number, $|a|$ can also be defined as follows:

$$|a| = \sqrt{(a)^2}$$

Theorem: Let a and b be real numbers then:

1-$|a| = |-a|$

2-$|a.b| = |a|\,.\,|b|$

3-$\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \quad , b \neq 0$

4-$|a+b| \leq |a| + |b|$

5-$|a-b| \leq |a| + |b|$

6-$|a-b| \geq |a| - |b|$

7-$|a+b| \geq |a| - |b|$

Proof:

1-$|a| = \sqrt{(a)^2} = \sqrt{(-a)^2} = |-a|$

2-$|a.b| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}.\sqrt{b^2} = |a||b|$

3-$\left|\frac{a}{b}\right| = \sqrt{(\frac{a}{b})^2} = \sqrt{\frac{a^2}{b^2}} = \frac{\sqrt{a^2}}{\sqrt{b^2}} = \frac{|a|}{|b|}$

4-$|a+b| = \sqrt{(a+b)^2} = \sqrt{a^2+2ab+b^2}$

$$\leq \sqrt{a^2 + 2\sqrt{a^2}.\sqrt{b^2} + b^2}$$

$$= \sqrt{(\sqrt{a^2} + \sqrt{b^2})^2} = \sqrt{a^2} + \sqrt{b^2} = |a| + |b|$$

5-$|a-b| \leq |a| + |b|$

$|a-b| = |a+(-b)| \leq |a| + |-b| = |a| + |b|$

6-$|a-b| \geq |a| - |b|$

$|a| = |a+b-b| \leq |a-b| + |b| \rightarrow |a-b| \geq |a| - |b|$

7-$|a+b| \geq |a| - |b|$

$|a| = |a+b-b| \leq |a+b| + |-b| \rightarrow |a+b| \geq |a| - |b|$