Chapter Two: Methods for solving 1st order Ordinary Differential Equations

We will study some methods that can be used to find the solutions of the first order equations which take the form $y' = f(t, y)$

1- Separable Equations:

Finding a way to separate the variables can be considered the best method to attempt first when trying to solve a differential equation. Even if we use one of the methods that we will discuss later for a given differential equation, we will invariably end up with the same integral to solve. Formally, a differential equation is separable if it can be written as:

$$\frac{dy}{dt} = a(t)b(y)$$

where $a, b: \mathbb{R} \to \mathbb{R}$ are continuous functions.

Solving Method:

The approach of solving a first order ODE using **separation of variables** is as follows:

Step 0: If required, we can first determine whether or not that the ODE is separable by using the theorem below:

Theorem: The DE $y' = f(t, y)$ is separable if and only if $f(t, y)\frac{\partial^2 f}{\partial t \partial y} = \frac{\partial f}{\partial t}\frac{\partial f}{\partial y}$

Step 1: Rewrite (if necessary) the equation in the required form:

$$\frac{dy}{dt} = a(t)b(y)$$

Step 2: Find the general solution as follows:

$$\int \frac{dy}{b(y)} = \int a(t)dt$$

Step 3: If the initial conditions are given, solve to find the unique solution.

EXAMPLE: Verify that the IVP below is separable and find its solution:

$$y' = -2ty \, , y(0) = 1$$

Solution:

Step 0: To verify that the IVP above is separable, we set that $y' = f(t, y)$ and we have to prove that: $f(t,y)\frac{\partial^2 f}{\partial t \partial y} = \frac{\partial f}{\partial t}\frac{\partial f}{\partial y}$

So $f(t,y) = -2ty \, , \frac{\partial f}{\partial t} = -2y \, , \quad \frac{\partial f}{\partial y} = -2t \, , and \ \frac{\partial^2 f}{\partial t \partial y} = -2$

$f(t,y)\frac{\partial^2 f}{\partial t \partial y} = -2ty \, . -2 = 4ty$ is equal to $\frac{\partial f}{\partial t}\frac{\partial f}{\partial y} = (-2y).(-2t) = 4ty$

Step 1: $y' = -2ty \quad \rightarrow \quad \frac{dy}{dt} = -2ty \quad (a(t) = -2t, \ b(y) = y)$

Step2: $\rightarrow \quad \frac{dy}{y} = -2tdt$

$\rightarrow \quad \int \frac{dy}{y} = \int -2tdt \quad \rightarrow \quad Ln(y) = -\mathrm{t}^2 + \mathrm{c} \quad \rightarrow \quad e^{Ln(y)} = e^{-\mathrm{t}^2+\mathrm{c}}$

The general solution is $y = e^{-\mathrm{t}^2+\mathrm{c}}$

Step 3: $y = e^{-\mathrm{t}^2+\mathrm{c}}$ and $y(0) = -1$

So $y(0) = e^{-0^2+\mathrm{c}} = e^{\mathrm{c}} = 1 \quad \rightarrow \quad c = o$

Finally, $y = e^{-\mathrm{t}^2+\mathrm{c}} = e^{-\mathrm{t}^2+0} = e^{-\mathrm{t}^2}$ is the solution of the above IVP.

To verify the solution:

$y = e^{-\mathrm{t}^2} \rightarrow y' = e^{-\mathrm{t}^2}(-2t) = -2ty$ correct (☺👍)

Homework: verify that each of the following DEs are separable and use separation of variables method to find the solution of each one:

1- $y' = 1 + y$

2- $\frac{dy}{dt} = -\frac{y}{t}$, $y(0) = -4$

3- $\frac{1}{y^2}\frac{dy}{dx} + x^2 = 0$

4- $\frac{dy}{dt} = \frac{t(e^{t^2}+2)}{6y^2}$, $y(0) = 1$

5- $(y + y^2)dt - tdy = 0,$ $y(1) = 2$

Hint: $\frac{rx+sy}{xy}$ can be transferred into $\frac{A}{x} + \frac{B}{y}$ for some real numbers A and B

EXAMPLE: Check if the ODE below is separable and find its solution:

$$(t - y)dt + tdy = 0$$

Solution: Step 0: First we need to rewrite the ODE in the form $y' = f(t, y)$

$$(t - y)dt + tdy = 0 \quad \rightarrow tdy = -(t - y)dt \quad \rightarrow tdy = (y - t)dt$$

$$\rightarrow \frac{dy}{dt} = \frac{y - t}{t} = \frac{y}{t} - 1 \quad \rightarrow \quad y' = \frac{y}{t} - 1 = f(t, y)$$

To determine if the ODE above is separable, we set that $y' = f(t, y)$ and we have to prove that: $f(t,y)\frac{\partial^2 f}{\partial t \partial y} = \frac{\partial f}{\partial t}\frac{\partial f}{\partial y}$

So $f(t, y) = \frac{y}{t} - 1$, $\frac{\partial f}{\partial t} = -\frac{y}{t^2}$, $\frac{\partial f}{\partial y} = \frac{1}{t}$, and $\frac{\partial^2 f}{\partial t \partial y} = -\frac{1}{t^2}$

$f(t,y)\frac{\partial^2 f}{\partial t \partial y} = \left(\frac{y}{t} - 1\right). -\frac{1}{t^2} = -\frac{y}{t^3} + \frac{1}{t^2}$ is **NOT** equal to $\frac{\partial f}{\partial t}\frac{\partial f}{\partial y} = -\frac{y}{t^2} . \frac{1}{t} = -\frac{y}{t^3}$

Therefore, the ODE is not separable.

<u>QUESTION</u>: Is it possible to overcome this problem, and make the above ODE is separable?

<u>ANSWER</u>: **YES**, if it is Homogeneous Equation.

2- Homogeneous Equations:

An ordinary differential equation is said to be a homogeneous differential equation if the following condition is satisfied

$$y' = f(zt, zy) = f(t, y) \quad \forall\, z \in \mathbb{R}$$

Let us come back to the previous example and see if it is homogenous

<u>EXAMPLE</u>: Check if the ODE is homogenous: $(x - t)dt + tdy = 0$

<u>Solution</u>: We need to show that $y' = f(zt, zy) = f(t, y)$

We know that $y' = \frac{y}{t} - 1 = f(t, y)$ So, $f(zt, zy) = \frac{zy}{zt} - 1 = \frac{y}{t} - 1 = f(t, y)$

Therefore, the ODE is homogenous.

<u>QUESTION</u>: How can the method for solving homogeneous equations transform the not separable ODE into separable.

<u>ANSWER</u>: The method for solving homogeneous equations follows from this fact: "The substitution $y = vt$ (and therefore $dy = vdt + tdv$) transforms a homogeneous equation into a separable one."

Let us come back again to the previous example and see how is it going to be separable?

EXAMPLE: Transfer the not separable ODE below into separable by using the method of homogenous DE:

$$(t-y)dt + tdy = 0$$

Solution: we know that $y' = \frac{y}{t} - 1 = f(t,y)$

Set $y = vt \quad \rightarrow y' = \frac{dy}{dt} = \frac{d(vt)}{dt} = v\frac{dt}{dt} + t\frac{dv}{dt} = v + tv' = f(t,vt)$

$$\rightarrow tv' = f(t,vt) - v \qquad \rightarrow v' = \frac{f(t,vt) - v}{t}$$

$$f(t,y) = \frac{y}{t} - 1 \quad \rightarrow f(t,vt) = \frac{vt}{t} - 1 = v - 1$$

$$\rightarrow v' = \frac{f(t,vt) - v}{t} = \frac{v - 1 - v}{t} = \frac{-1}{t}$$

Now, this new form of our ODE $v' = \frac{-1}{t}$ is separable, so we can use the separation of variables method to solve it as follows:

$v' = \frac{dv}{dt} = \frac{-1}{t} \quad \rightarrow \quad dv = -\frac{dt}{t} \quad \rightarrow \int dv = -\int \frac{dt}{t} \quad \rightarrow \quad v = -ln(t) + c$

But, $y = vt \quad \rightarrow y = (-ln(t) + c)$ t

To verify the solution:

$$y = (-\ln(t) + c)\text{t} \quad \rightarrow y' = (-\ln(t) + c) + t\left(-\frac{1}{t}\right)$$

$$= (-\ln(t) + c) - 1 = \frac{y}{t} - 1$$

$$\rightarrow \quad y' = \frac{y}{t} - 1$$ correct (☺👍)

Strategy: The approach of solving a first order DE using **the method of homogenous DE** is as follows:

Step 0: If required, check if the DE satisfies (**The homogeneous property**)

$$y' = f(zt, zy) = f(t, y)$$

Step 1: set that $y = vt$

Step 2: from $f(t, y)$, find $f(t, vt)$

Step 3: substitute $f(t, vt)$ in $v' = \frac{f(t,vt)-v}{t}$ to make the DE separable

Step 4: use the separation of variable method to solve for v

Step 5: substitute $v = \frac{y}{t}$ in the solution from step 4 to get the solution of the DE.

EXAMPLE: Verify that the ODE below is homogenous and find its solution:

$$(t^2 - y^2)dt + tydy = 0$$

Step 0: To verify that the ODE above is separable, we set that $y' = f(t, y)$ and we have to prove that $y' = f(zt, zy) = f(t, y)$

$$(t^2 - y^2)dt + tydy = 0 \;\rightarrow\; tydy = -(t^2 - y^2)dt \;\rightarrow\; \frac{dy}{dt} = \frac{y^2 - t^2}{ty}$$

$$f(t, y) = \frac{y^2 - t^2}{ty}$$

$$f(zt, zy) = \frac{(zy)^2 - (zt)^2}{zt.zy} = \frac{z^2(y^2 - t^2)}{z^2\, ty} = \frac{y^2 - t^2}{ty} = f(t, y)$$

The ODE is homogenous.

Step 1: set that $y = vt$

Step 2: from $f(t, y) = \frac{y^2-t^2}{ty}$, we find $f(t, vt) = \frac{(vt)^2-t^2}{t(vt)} == \frac{t^2(v^2-1)}{t^2 v} = \frac{v^2-1}{v}$

Step 3: substitute $f(t,vt) = \frac{v^2-1}{v}$ in $v' = \frac{f(t,vt)-v}{t}$ to make the DE separable

$$v' = \frac{\frac{v^2-1}{v} - v}{t} = \frac{\frac{v^2-1-v^2}{v}}{t} = \frac{-\frac{1}{v}}{t} = \frac{-1}{vt}$$

Step 4: use the separation of variable method to solve for v

$$v' = \frac{dv}{dt} = \frac{-1}{vt} \;\rightarrow\; vdv = -\frac{dt}{t} \;\rightarrow \int vdv = \int -\frac{dt}{t} \;\rightarrow\; \frac{v^2}{2} = -ln(t) + c$$

$$\rightarrow\; v^2 = 2(-ln(t) + c)$$

Step 5: substitute $v = \frac{y}{t}$ in the solution from step 4 to get the solution of the ODE.

$$v^2 = 2(-ln(t) + c) \rightarrow \left(\frac{y}{t}\right)^2 = 2(-ln(t) + c) \rightarrow\; y^2 = 2t^2(-ln(t) + c)$$

To verify the solution:

$$y^2 = 2t^2(-ln(t) + c) \quad \rightarrow 2yy' = 2t^2\left(\frac{-1}{t}\right) + (-ln(t) + c)4\text{t}$$

$$\rightarrow y' = \frac{-2t+(-ln(t)+c)4\text{t}}{2y} = \frac{2t(-1+2(-ln(t)+c))}{2y} = \frac{t(-1+\frac{y^2}{t^2})}{y} = \frac{t(\frac{-t^2+y^2}{t^2})}{y} = \frac{y^2-t^2}{ty}$$

correct (☺👍)

<u>EXAMPLE</u>: Verify that the ODE below is homogenous and find its solution:

$$(t^3 + y^3)dt - 2ty^2dy = 0$$

Step 0: To verify that the ODE above is separable, we set that $y' = f(t,y)$ and we have to prove that $y' = f(zt,zy) = f(t,y)$

$$(t^3 + y^3)dt - 2ty^2dy = 0 \;\rightarrow\; 2ty^2dy = (t^3 + y^3)dt \;\rightarrow\; \frac{dy}{dt} = \frac{(t^3 + y^3)}{2ty^2}$$

$$f(t,y) = \frac{(t^3+y^3)}{2ty^2}\;,\;\; f(zt,zy) = \frac{((zt)^3+(zy)^3)}{2zt(zy)^2} = \frac{z^3(t^3+y^3)}{z^3.2ty^2} = \frac{t^3+y^3}{2ty^2} = f(t,y)$$

The ODE is homogenous.

Step 1: set that $y = vt$

Step 2: from $f(t, y) = \frac{t^3+y^3}{2ty^2}$, we find $f(t, vt) = \frac{t^3+(vt)^3}{2t(vt)^2} = \frac{t^3(1+v^3)}{t^3(2v^2)} = \frac{1+v^3}{2v^2}$

Step 3: substitute $f(t, vt) = \frac{1+v^3}{2v^2}$ in $v' = \frac{f(t,vt)-v}{t}$ to make the DE separable

$$v' = \frac{\frac{1+v^3}{2v^2} - v}{t} = \frac{\frac{1+v^3-2v^3}{2v^2}}{t} = \frac{\frac{1-v^3}{2v^2}}{t}$$

Step 4: use the separation of variable method to solve for v

$$v' = \frac{dv}{dt} = \frac{\frac{1-v^3}{2v^2}}{t}$$

$$\to \frac{dv}{\frac{1-v^3}{2v^2}} = \frac{dt}{t} \quad \to \frac{2v^2 dv}{1-v^3} = \frac{dt}{t} \quad \to \int \frac{2v^2 dv}{1-v^3} = \int \frac{dt}{t}$$

$$\to \frac{-3}{-3}\int \frac{2v^2 dv}{1-v^3} = \int \frac{dt}{t} \qquad \to \frac{2}{-3}\int \frac{-3v^2 dv}{1-v^3} = \int \frac{dt}{t}$$

$$\to \frac{2}{-3} ln(1-v^3) = ln(t) + c$$

Step 5: substitute $v = \frac{y}{t}$ in the solution from step 4 to get the solution of the ODE.

The solution of the ODE is $\frac{2}{-3} ln(1-(\frac{y}{t})^3) = ln(t) + c$

To verify the solution:

$$\frac{2}{-3} ln(1-(\frac{y}{t})^3) = ln(t) + c \quad \to \quad \frac{2}{-3} ln(1-\frac{y^3}{t^3}) = ln(t) + c$$

$$\to \frac{2}{-3}\frac{-(3\frac{y^2}{t^3}y' - 3\frac{y^3}{t^4})}{1-\frac{y^3}{t^3}} = \frac{1}{t} \to \frac{2\frac{y^2}{t^3}y' - 2\frac{y^3}{t^4})}{1-\frac{y^3}{t^3}} = \frac{1}{t} \to \frac{\frac{2ty^2y'-2y^3}{t^4})}{\frac{t^3-y^3}{t^3}} = \frac{1}{t}$$

$$\to \frac{2ty^2y'-2y^3}{t^3-y^3} = 1 \quad \to 2ty^2y' - 2y^3 = t^3 - y^3 \quad \to 2ty^2y' = t^3 + y^3$$

$\to y' = \frac{(t^3+y^3)}{2ty^2}$ correct (☺👍)

Homework: verify that each of the following ODEs are homogenous and find the solution for each one:

1- $y' - \frac{y}{t} + 1 = 0$

2- $y^2 + (t^2 + ty)\frac{dy}{dt} = 0$

3- $(t^2 + y^2)dt + (t^2 - ty)dy = 0$ Hint: use long division

EXAMPLE: Check if the ODE below is homogenous and find its solution:

$$2tydt + (t^2 - 1)dy = 0$$

SOLUTION:

Step 0: To verify that the ODE above is separable, we set that $y' = f(t, y)$ and we have to prove that $y' = f(zt, zy) = f(t, y)$

$$2tydt + (t^2 - 1)dy = 0 \rightarrow (t^2 - 1)dy = -2tydt \rightarrow \frac{dy}{dt} = \frac{-2ty}{(t^2 - 1)}$$

$$f(t, y) = \frac{-2ty}{(t^2 - 1)}$$

$$f(zt, zy) = \frac{-2(zt)(zy)}{((zt)^2 - 1)} = \frac{-2z^2ty}{(z^2t^2 - 1)} \neq f(t, y)$$

The ODE is nonhomogeneous.

QUESTION: Is it possible to overcome this problem, and make the above ODE is solvable?

ANSWER: **YES**, if it is Exact Equation.

3- Exact Equation:

Consider the differential equation which takes the form

$$M(t,y)dt + N(t,y)dy = 0$$

we say that this differential equation is exact if it is satisfied this condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Solving Method: The approach of solving a first order DE using **the method of Exact DE** is as follows:

Step 0: If required, check if the DE satisfies (**The Exactness condition**)

First, the equation should be in the form:

$$M(t,y)dt + N(t,y)dy = 0$$

and check it satisfies the condition: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$

Step 1: Assume that the function $\emptyset = \emptyset(t,y)$ (the solution of the general equation)

such that $\frac{\partial \emptyset}{\partial t} = M(t,y)$ and $\frac{\partial \emptyset}{\partial y} = N(t,y)$ (the old one)

(Which means that $M(t,y)dt + N(t,y)dy = 0$ becomes

$$\frac{\partial \emptyset}{\partial t}dt + \frac{\partial \emptyset}{\partial y}dy = 0$$

$\rightarrow \quad \partial\emptyset = 0 \quad \rightarrow \emptyset = C \quad \forall t, y \in \mathbb{R}^2$)

Step 2: Integrate $M(t,y)$ with respect of t to get:

$$\emptyset(t,y) = \int_t M(t,y)dt + h(y)$$

Step 3: Calculate the new $\frac{\partial \emptyset}{\partial y}$ as following:

$$\frac{\partial \emptyset}{\partial y} = \frac{\partial}{\partial y}\left(\int_t M(t,y)dt + h(y)\right) = \frac{\partial}{\partial y}\int_t M(t,y)dt + h'(y)$$

Step 4: Compare the new $\frac{\partial \emptyset}{\partial y}$ with the old $\frac{\partial \emptyset}{\partial y} = N(t,y)$ and solve to get $h(y)$

Step 5: Substitute $h(y)$ in the equation from step 2:

$$\emptyset(t,y) = \int_t M(t,y)dt + h(y)$$

to get the solution of the DE.

Strategy: The steps of **the Exact DE method** are:

Step 0: Find $M(t,y)$ and $N(t,y)$ from $M(t,y)dt + N(t,y)dy = 0$

and verify that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$

Step 1: $A = \int M(t,y)dt$, with ignoring the integration constant, which should be a function of y .

Step 2: $B = \frac{\partial}{\partial y} A$

Step 3: $h(y) = \int (N(t,y) - B)dy$

Step 4: $k = A + h(y)$ is the solution where k is constant.

EXAMPLE: Verify that the ODE is Exact and find its solution using the Exact

equation method: $2tydt + (t^2 - 1)dy = 0$

Solution: **Step 0: Find $\mathbf{M(t,y)}$ and $\mathbf{N(t,y)}$ from $\mathbf{M(t,y)dt + N(t,y)dy = 0}$**

and verify that $\frac{\partial \mathbf{M}}{\partial \mathbf{y}} = \frac{\partial \mathbf{N}}{\partial \mathbf{t}}$

$M(t,y) = 2ty$ and $N(t,y) = (t^2 - 1)$

$\frac{\partial M}{\partial y} = 2t$ is equal to $\frac{\partial N}{\partial t} = 2t$ the ODE is Exact

Step 1: $A = \int M(t,y)dt$, $A = \int 2tydt = t^2y$, we integrate without the integration constant.

Step 2: $B = \frac{\partial}{\partial y}A$, $B = \frac{\partial}{\partial y}(t^2y) = t^2$

Step 3: $h(y) = \int (N(t,y) - B)dy$

$$h(y) = \int (t^2 - 1 - t^2)dy = -\int dy = -y + c_1$$

Step 4: $k = A + h(y)$ is the solution where k is constant.

$k = t^2y - y + c_1 \rightarrow t^2y - y = c$ where $c = k - c_1$

To verify the solution: $t^2y - y = c \rightarrow t^2y' + 2ty - yy' = 0 \rightarrow y' = \frac{dy}{dt} = \frac{2ty}{t^2-1}$

$\rightarrow 2tydt + (t^2 - 1)dy = 0$ correct (☺👍)

<u>EXAMPLE</u>: Verify that the ODE is Exact and find its solution using the Exact equation method: $(e^{2y} - y\,cos(ty))dt + (2te^{2y} - t\,cos(ty) + 2y)dy = 0$

<u>Solution</u>:

Step 0: **Find $\mathbf{M(t,y)}$ and $\mathbf{N(t,y)}$ from $\mathbf{M(t,y)dt + N(t,y)dy = 0}$**

and verify that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$

$M(t,y) = e^{2y} - y\,cos(ty)$ and $N(t,y) = 2te^{2y} - t\,cos(ty) + 2y$

$$\frac{\partial M}{\partial y} = 2e^{2y} - \left(y\left(-sin(ty)\right)t + cos(ty)\right) = 2e^{2y} + ty\,sin(ty) - cos(ty))$$

$$\frac{\partial N}{\partial t} = 2e^{2y} - (t\left(-sin(ty)\right)y + cos(ty)) = 2e^{2y} + ty\,sin(ty) - cos(ty))$$

Step 1: $A = \int M(t,y)dt$,

Here, we find the integral with ignoring the integration constant.

$$A = \int (e^{2y} - y\,cos(ty))dt = te^{2y} - sin(ty)$$

Step 2: $B = \frac{\partial}{\partial y} A$,

$$B = \frac{\partial}{\partial y}(te^{2y} - sin(ty)) = 2te^{2y} - tcos(ty)$$

Step 3: $h(y) = \int (N(t,y) - B)dy$

$$h(y) = \int (2te^{2y} - t\,cos(ty) + 2y - (2te^{2y} - tcos(ty)))dy$$

$$= \int 2ydy = y^2 + c_1$$

Step 4: $k = A + h(y)$ is the solution where k is constant.

$$k = te^{2y} - sin(ty) + y^2 + c_1 \;\rightarrow\; te^{2y} - sin(ty) + y^2 = c$$

where $c = k - c_1$

To verify the solution:

$$te^{2y} - sin(ty) + y^2 = c$$

$$\rightarrow\; 2te^{2y}y' + e^{2y} - (tcos(ty)y' + ycos(ty) + 2yy' = 0$$

$$\rightarrow\; y'(2te^{2y} - t\,cos(ty) + 2y) = -e^{2y} + y\,cos(ty)$$

$$\rightarrow\; y' = \frac{dy}{dt} = \frac{-e^{2y} + y\,cos(ty)}{(2te^{2y} - t\,cos(ty) + 2y)}$$

$$\rightarrow\; (e^{2y} - y\,cos(ty))dt + (2te^{2y} - t\,cos(ty) + 2y)dy = 0$$

correct (☺👍)

4- Solving the first-order Linear ODE by Integrating Factor method:

We continue our quest for solutions of first-order differential equations by next examining linear equations.

Def: A first-order differential equation of the form

$$y' = a(t)y + b(t)$$

is said to be a linear equation in the variable y where a: $\mathbb{R} \to \mathbb{R}$ and b: $\mathbb{R} \to \mathbb{R}$.

Solving Method: The approach of solving a first order linear ODE using **the integrating factor** is as follows:

Step 0: Make sure that the equation is in the standard form:

$$y' = a(t)y + b(t)$$

Step 1: set $\mu = e^{-\int a(t)dt}$

This is called the **Integrating Factor**, and no constant need be used in evaluating the indefinite integral $\int a(t)dt$

Step 2: Multiply both sides of the equation in step 0 by μ, to get

$$\mu\, y' = \mu\, (a(t)y + b(t))$$

$$\to \quad e^{-\int a(t)dt}\, \frac{dy}{dt} = e^{-\int a(t)dt} a(t)y + e^{-\int a(t)dt} b(t))$$

$$\to \quad e^{-\int a(t)dt}\, \frac{dy}{dt} - e^{-\int a(t)dt} a(t)y = e^{-\int a(t)dt} b(t))$$

$$\to \quad \frac{d}{dt}(e^{-\int a(t)dt}\, y) = e^{-\int a(t)dt} b(t)$$

$$\to \quad d(e^{-\int a(t)dt}\, y) = e^{-\int a(t)dt} b(t)dt$$

Step 3: Integrate both sides of the equation above

$$\to \int d(e^{-\int a(t)dt}\, y) = \int e^{-\int a(t)dt} b(t)dt$$

$$\rightarrow e^{-\int a(t)dt}\, y = \int e^{-\int a(t)dt} b(t)dt$$

$$\rightarrow\; y = e^{\int a(t)dt} \int e^{-\int a(t)dt} b(t)dt$$

Strategy Summary: The steps of solving a first order linear DE using **the integrating factor** are:

Step 0: The equation should be in the form: $y' = a(t)y + b(t)$

Step 1: set $\mu = e^{-\int a(t)dt}$ (integrate without constant)

Step 2: The solution of the equation is

$$y = e^{\int a(t)dt} \int e^{-\int a(t)dt} b(t)dt$$

Step 3: Only for IVP, use the initial condition to find the value of the constant

EXAMPLE: Solve the differential equation $\frac{dy}{dt} - 3y = 0$

Solution:

Step 0: The equation should be in the form: $y' = a(t)y + b(t)$

$\frac{dy}{dx} - 3y = 0 \rightarrow y' = 3y \qquad \rightarrow a(t) = 3, b(t) = 0$

Step 1: set $\mu = e^{-\int a(t)dt}$ (integrate without constant)

$$\mu = e^{-\int a(t)dt} = e^{-\int 3dt} = e^{-3t} \quad \text{the Integrating Factor}$$

Step 2: The solution of the equation is $y = e^{\int a(t)dt} \int e^{-\int a(t)dt} b(t)dt$

$$y = e^{3t} \int e^{-3t}\, 0\, dt = e^{3t} \int 0\, dt = e^{3t}(0 + c) = ce^{3t}$$

To verify the solution:

$$y = ce^{3t} \rightarrow \quad y' = 3ce^{3t} \rightarrow y' = 3y$$ correct (☺👍)

EXAMPLE: Solve the IVP $\frac{dy}{dt} - 2y = 6 \;, y(0) = 9$

Solution:

Step 0: The equation should be in the form: $y' = a(t)y + b(t)$

$\frac{dy}{dx} - 2y = 6 \;\rightarrow y' = 2y + 6 \qquad \rightarrow a(t) = 2, b(t) = 6$

Step 1: set $\mu = e^{-\int a(t)dt}$ (integrate without constant)

$$\mu = e^{-\int a(t)dt} = e^{-\int 2dt} = e^{-2t} \quad \text{the Integrating Factor}$$

Step 2: The solution of the equation is $y = e^{\int a(t)dt} \int e^{-\int a(t)dt} b(t)dt$

$$y = e^{2t} \int e^{-2t}\, 6\, dt = 6e^{2t} \int\; e^{-2t}\, dt = 6e^{2t}\left(\frac{e^{-2t}}{-2} + c\right) = -3 + 6ce^{2t}$$

Step 3: Only for IVP, use the initial condition to find the value of the constant

$$y = -3 + 6ce^{2t} \rightarrow \quad y(0) = -3 + 6c = 9 \;\rightarrow c = 2$$

$$\rightarrow y = -3 + 6ce^{2t} \quad \rightarrow \quad y = -3 + 12e^{2t}$$

To verify the solution:

$$y = -3 + 12e^{2t} \rightarrow \quad y' = 24e^{2t} \rightarrow y' = 2(12e^{2t} - 3 + 3) = 2y + 6$$

correct (☺👍)

EXAMPLE: Solve $t\frac{dy}{dt} - 4y = t^6 e^t$

Solution:

Step 0: The equation should be in the form: $y' = a(t)y + b(t)$

$t\frac{dy}{dt} - 4y = t^6 e^t \;\rightarrow y' = \frac{4}{t}y + t^5 e^t \qquad \rightarrow a(t) = \frac{4}{t}, b(t) = t^5 e^t$

Step 1: set $\mu = e^{-\int a(t)dt}$ (integrate without constant)

$$\mu = e^{-\int a(t)dt} = e^{-\int \frac{4}{t}dt} = e^{-4ln(t)} = e^{ln(t^{-4})} = t^{-4} \quad \text{the Integrating Factor}$$

Step 2: The solution of the equation is $y = e^{\int a(t)dt} \int e^{-\int a(t)dt} b(t)dt$

$y = t^4 \int t^{-4}\, t^5 e^t\, dt = t^4 \int t\, e^t\, dt$

We should use the integration by parts to solve $\int t\, e^t\, dt$ as follows:

Let $u = t\, , dv = e^t\, dt \;\; \to du = dt\, , v = \int\; e^t\, dt = e^t$

$\to \int t\, e^t\, dt = uv - \int v\, du = te^t - \int e^t dt = te^t - e^t + c$

$$\to y = t^4 \int t\, e^t\, dt = t^4(te^t - e^t + c)$$

To verify the solution:

$$y = t^4(te^t - e^t + c) \to \quad y' = t^4(te^t + e^t - e^t) + 4t^3(te^t - e^t + c)$$

$$\to \quad y' = t^4(te^t) + 4t^3(te^t - e^t + c) = t^5 e^t + \frac{4}{t} y$$

correct (☺👍)

5- Nonlinear Equations can be transformed to linear by Substitutions:

We usually solve a differential equation by recognizing it as a certain kind of equation (say, separable, linear, or exact) and then carrying out a procedure, consisting of equation specific mathematical steps, that yields a solution of the equation. Sometimes a well-chosen substitution allows us to solve an equation. Sometimes it allows us to simplify an equation before we resort to numerical or qualitative techniques.

Next, we will study a type of differential equation, which take the general form:

$$g'(y)y' = a(t)g(y) + b(t)$$

Solving Method: The approach of solving this type of equations is as follows:

Step 0: Make sure that the equation is in the standard form

$$g'(y)y' = a(t)g(y) + b(t)$$

Step 1: Set $v = g(y)$, which leads to $\frac{dv}{dt} = g'(y)y'$

Step 2: Substitute v , v' in the general equation, we get a linear equation with respect to new dependent variable v

$v' = a(t)v + b(t)$,

Step 3: Solve the last linear equation using integrating factor method to get v

Step 4: Use $v = g(y)$, to get the solution in terms of y .

EXAMPLE: Solve $e^y y' + e^y = cos(t)$

Solution: The good choice is when we let $\boxed{\boldsymbol{v = e^y}}$

$$v = e^y \;\; \rightarrow ln(v) = ln(e^y) \;\; \rightarrow \;\; ln(v) = y$$
$$\rightarrow \; v' = (e^y)y'$$

So, $e^y y' + e^y = cos(t)$ becomes $v' + v = cos(t) \;\; \rightarrow v' = cos(t) - v$

Now, this equation is linear with respect to v, so we can solve it using **the integrating factor method.**

<u>HW:</u> Finish the solution of the example above using the integrating factor method.

<u>EXAMPLE</u>: Solve $\quad cos(y)y' = sin(y)t + 5e^{\frac{t^2}{2}}$

<u>Solution</u>: The good choice is when we let $\boxed{\boldsymbol{v = sin(y)}}$

$$v = sin(y) \ \rightarrow v' = cos(y)y' \ \ and \ \ y = sin^{-1}(v)$$

$$cos(y)y' = sin(y)t + 5e^{\frac{t^2}{2}} \ \text{ becomes } \ v' = vt + 5e^{\frac{t^2}{2}}$$

Now, this equation is linear with respect to v, so we can solve it using **the integrating factor method.**

<u>HW:</u> Finish the solution of the example above using the integrating factor method.

<u>NOTE:</u> Some differential equations may take different forms from the two examples above, So There are no general rules for finding good substitutions, see the following example:

<u>EXAMPLE</u>: Solve $\quad y' = -e^y - 1$

<u>Solution</u>: The good choice is when we let $\boxed{\boldsymbol{v = e^{-y}}}$

$$v = e^{-y} \ \rightarrow ln(v) = ln(e^{-y}) \ \rightarrow \ ln(v) = -y$$

$$\rightarrow \ y = -ln(v) \qquad \rightarrow \ y' = -\frac{v'}{v}$$

$$y' = -e^y - 1 \ \text{ becomes } \ -\frac{v'}{v} = -\frac{1}{v} - 1 \ \ \rightarrow v' = 1 + v$$

Now, this equation is linear with respect to v, so we can solve it using **the integrating factor method.**

Step 0: The equation should be in the form: $y' = a(t)y + b(t)$

$v' = 1 + v \quad \rightarrow a(t) = 1, b(t) = 1$

Step 1: set $\mu = e^{-\int a(t)dt}$ (integrate without constant)

$$\mu = e^{-\int 1dt} = e^{-\int dt} = e^{-t} \text{ the Integrating Factor}$$

Step 2: The solution of the equation is $v = e^{\int a(t)dt} \int e^{-\int a(t)dt} b(t)dt$

$$v = e^{t} \int e^{-t}(1)\, dt = e^{t} \int e^{-t}\, dt = e^{t}(-e^{-t} + c) = -1+ce^{t}$$

$v = -1+ce^{t}$ but $v = e^{-y}$

$$v = -1+ce^{t} \rightarrow e^{-y} = -1+ce^{t}$$

$$\rightarrow \ln(e^{-y}) = ln(-1+ce^{t}) \quad \rightarrow -y = ln(-1+ce^{t})$$

$$\rightarrow y = -ln(-1+ce^{t})$$

To verify the solution:

$$y = -ln(-1+ce^{t}) \rightarrow \quad y' = \frac{-ce^{t}}{(-1+ce^{t})} = \frac{-(ce^{t}-1)-1}{e^{-y}} = \frac{-e^{-y}-1}{e^{-y}} = \frac{-e^{-y}}{e^{-y}} - \frac{1}{e^{-y}}$$

$\rightarrow y' = -1 - e^{y}$ correct (☺👍)

6-Bernoulli's Equation:

Def: The differential equation $y' = a(t)y + y^{n}b(t)$

where n is any real number ($n \neq 0$ and $n \neq 1$), is called Bernoulli's equation.

Note: For n = 0 and n = 1, the equation above is linear.

Strategy: To solve Bernoulli's equation:

1- Do the substitution $z = y^{1-n}$ because this substitution reduces any equation of the form above to a linear equation with respect to *z*.

2- Solve the linear equation in terms of *z* to find *z* using the integrating factor method.

3- Use $z = y^{1-n}$ to make the solution in terms of y.

<u>EXAMPLE</u>: Solve the IVP $t\frac{dy}{dt} + y = t^2y^2,\ y(1) = 1$

<u>Solution</u>: $t\frac{dy}{dt} + y = t^2y^2 \quad \rightarrow \quad ty' + y = t^2y^2 \quad \rightarrow \quad y' = -\frac{1}{t}y + ty^2$

This equation is Bernoulli's equation with n=2

Let $z = y^{1-n} = y^{1-2} = y^{-1} \quad \rightarrow y = z^{-1} \quad \rightarrow y' = -z^{-2}z'$

$y' = -\frac{1}{t}y + ty^2$ becomes $-z^{-2}z' = -\frac{1}{t}z^{-1} + t(z^{-1})^2$

$$\rightarrow \quad -z^2(-z^{-2}z') = -z^2(-\frac{1}{t}z^{-1} + tz^{-2})$$

$$\rightarrow z' = \frac{1}{t}z - t$$

Now, this equation is linear with respect to z, so we can solve it using **the integrating factor method.**

Step 0: The equation should be in the form: $y' = a(t)y + b(t)$

$z' = \frac{1}{t}z - t \qquad \rightarrow a(t) = \frac{1}{t}, b(t) = -t$

Step 1: set $\mu = e^{-\int a(t)dt}$ (integrate without constant)

$$\mu = e^{-\int \frac{1}{t}dt} = e^{-ln(t)} = e^{ln(t^{-1})} = t^{-1} \quad \text{the Integrating Factor}$$

Step 2: The solution of the equation is $z = e^{\int a(t)dt} \int e^{-\int a(t)dt} b(t)dt$

$$z = t\int t^{-1}(-t)\,dt = t\int -dt = t(-t + c) = -t^2 + ct$$

$z = -t^2 + ct$ but $y = z^{-1} \rightarrow z = y^{-1}$

$$z = -t^2 + ct \rightarrow \quad y^{-1} = -t^2 + ct \quad \rightarrow y = \frac{1}{-t^2 + ct}$$

Step 3: Only for IVP, use the initial condition to find the value of the constant

$$y = \frac{1}{-t^2 + ct} \rightarrow \quad y(1) = \frac{1}{-1^2 + c(1)} = 1 \rightarrow c = 2$$

$$\to y = \frac{1}{-t^2 + 2t}$$

To verify the solution:

$$y = \frac{1}{-t^2+2t} \to \quad y' = \frac{-(-2t+2)}{(t^2+2t)^2} = \frac{-\frac{1}{t}(-t^2+2t)+t}{\frac{1}{y^2}} = \frac{-\frac{1}{t}\frac{1}{y}+t}{\frac{1}{y^2}}$$

$$\to \ y' = \left(-\frac{1}{t}\frac{1}{y} + t\right)y^2 = -\frac{1}{t}y + ty^2$$ correct (☺👍)

EXAMPLE: Solve $\frac{dy}{dt} - \frac{y}{3t} = e^t y^4$

Solution: $\frac{dy}{dt} - \frac{y}{3t} = e^t y^4 \quad \to \quad y' = \frac{y}{3t} + e^t y^4$

This equation is Bernoulli's equation with n=4

Let $z = y^{1-\mathrm{n}} = y^{1-4} = y^{-3} \quad \to y = z^{-\frac{1}{3}} \to y' = -\frac{1}{3}z^{-\frac{4}{3}}\mathrm{z}'$

$y' = \frac{y}{3t} + e^t y^4$ becomes $-\frac{1}{3}z^{-\frac{4}{3}}\mathrm{z}' = \frac{z^{-\frac{1}{3}}}{3t} + e^t(z^{-\frac{1}{3}})^4$

$$\to \quad -3z^{\frac{4}{3}}(-\frac{1}{3}z^{-\frac{4}{3}}\mathrm{z}') = -3z^{\frac{4}{3}}(\frac{z^{-\frac{1}{3}}}{3t} + e^t z^{-\frac{4}{3}})$$

$$\to \mathrm{z}' = -\frac{1}{t}\mathrm{z} - 3e^t$$

Now, this equation is linear with respect to z, so we can solve it using **the integrating factor method.**

Step 0: The equation should be in the form: $y' = a(t)y + b(t)$

$\mathrm{z}' = -\frac{1}{t}\mathrm{z} - 3e^t \qquad \to a(t) = -\frac{1}{t}, b(t) = -3e^t$

Step 1: set $\mu = e^{-\int a(t)dt}$ (integrate without constant)

$$\mu = e^{-\int -\frac{1}{t}dt} = e^{ln(t)} = t \quad \text{the Integrating Factor}$$

Step 2: The solution of the equation is $z = e^{\int a(t)dt} \int e^{-\int a(t)dt} b(t)dt$

$$z = t^{-1} \int t(-3e^t)\, dt = -3t^{-1} \int te^t dt$$

We should use the integration by parts to solve $\int -3t\, e^t\, dt$ as follows:

Let $u = t\,, dv = e^t\, dt \;\; \to du = dt\,, v = \int\, e^t\, dt = e^t$

$$\to \int t\, e^t\, dt = uv - \int v\, du = te^t + \int e^t dt = te^t - e^t + c$$

$$\to\; z = -3t^{-1} \int te^t dt = -3t^{-1}(te^t - e^t + c)$$

$$z = -3t^{-1}(te^t - e^t + c) \quad \text{but } z = y^{-3}$$

$$\to\; y^{-3} = -3t^{-1}(te^t - e^t + c)$$

$$\to\; y^3 = \frac{-t}{3(te^t - e^t + c)}$$

To verify the solution:

$$y^3 = \frac{-t}{3(te^t-e^t+c)} \to \quad 3y^2y' = \frac{3(te^t-e^t+c)+t(3(te^t+e^t-e^t))}{(3(te^t-e^t+c))^2} = \frac{-3(te^t-e^t+c)+3t^2e^t}{(3(te^t-e^t+c))^2}$$

$$\to \quad 3y^2y' = \frac{\frac{t}{y^3}+3t^2e^t}{(\frac{-t}{y^3})^2} = \frac{3\frac{t^2}{y^3}(\frac{1}{3t}+e^ty^3)}{\frac{t^2}{y^6}} = \frac{3(\frac{1}{3t}+e^ty^3)}{\frac{1}{y^3}} = 3y^3(\frac{1}{3t} + e^ty^3)$$

$$\to\; y' = \frac{y}{3t} + e^ty^4 \qquad \text{correct (☺👍)}$$

<u>EXAMPLE</u>: Solve $\quad y' + \frac{1}{10}y - cos(t)y^2 = 0$

<u>Solution</u>: $\quad y' + \frac{1}{10}y - cos(t)y^2 = 0 \quad \to \quad y' = -\frac{1}{10}y + cos(t)y^2$

This equation is Bernoulli's equation with n=2

Let $\quad z = y^{1-n} = y^{1-2} = y^{-1} \quad \to y = z^{-1} \to y' = -z^{-2}z'$

$y' = -\frac{1}{10}y + cos(t)y^2$ becomes $-z^{-2}z' = -\frac{1}{10}z^{-1} + cos(t)(z^{-1})^2$

$$\rightarrow \; -z^2(-z^{-2}z') = -z^2(-\frac{1}{10}z^{-1} + cos(t)z^{-2}) \rightarrow z' = \frac{1}{10}z - cos(t)$$

Now, this equation is linear with respect to z, so we can solve it using **the integrating factor method.**

Step 0: The equation should be in the form: $y' = a(t)y + b(t)$

$z' = \frac{1}{10}z - cos(t) \qquad \rightarrow a(t) = \frac{1}{10}, b(t) = -cos(t)$

Step 1: set $\mu = e^{-\int a(t)dt}$ (integrate without constant)

$$\mu = e^{-\int \frac{1}{10}dt} = e^{-\frac{t}{10}} \text{ the Integrating Factor}$$

Step 2: The solution of the equation is $z = e^{\int a(t)dt} \int e^{-\int a(t)dt} b(t)dt$

$z = e^{\frac{t}{10}} \int e^{-\frac{t}{10}}(cos(t))\, dt$

We should use the integration by parts twice to solve $\int e^{-\frac{t}{10}} cos(t)\, dt$ as follows:

First, we let $u = cos(t), dv = e^{-\frac{t}{10}}\, dt$

$$\rightarrow du = -sin(t)dt\,, v = \int e^{-\frac{t}{10}}\, dt = -10e^{-\frac{t}{10}}$$

$$\rightarrow \int e^{-\frac{t}{10}} cos(t)\, dt = uv - \int v\, du = -10cos(t)e^{-\frac{t}{10}} - \int -10e^{-\frac{t}{10}}(-sin(t))dt$$

We should use the integration by parts again to solve $\int -10e^{-\frac{t}{10}}(-sin(t))dt$

Now, we let $u = -sin(t), dv = -10e^{-\frac{t}{10}}\, dt$

$$\rightarrow du = -cos(t)dt\,, v = \int -10e^{-\frac{t}{10}}\, dt = 100e^{-\frac{t}{10}}$$

$$\int -10e^{-\frac{t}{10}}(-sin(t))dt = uv - \int v\, du = -\,100sin(t)e^{-\frac{t}{10}} + 100\int e^{-\frac{t}{10}} \cos(t)dt$$

$$\rightarrow \quad \int e^{-\frac{t}{10}} cos(t)\, dt = -10cos(t)e^{-\frac{t}{10}} - \int -10e^{-\frac{t}{10}}(-sin(t))dt$$

$$= -10cos(t)e^{-\frac{t}{10}} - (-100sin(t)e^{-\frac{t}{10}} + 100\int e^{-\frac{t}{10}}\cos(t)\,dt)$$

$$= -10cos(t)e^{-\frac{t}{10}} + 100sin(t)e^{-\frac{t}{10}} - 100\int e^{-\frac{t}{10}}\cos(t)dt)$$

$$\rightarrow \quad 101\int e^{-\frac{t}{10}}cos(t)dt = -10cos(t)e^{-\frac{t}{10}} + 100sin(t)e^{-\frac{t}{10}}$$

$$\rightarrow \quad \int e^{-\frac{t}{10}}cos(t)dt = \frac{1}{101}(-10cos(t)e^{-\frac{t}{10}} + 100sin(t)e^{-\frac{t}{10}})$$

$$\rightarrow z = e^{\frac{t}{10}}\int e^{-\frac{t}{10}}\big(cos(t)\big)dt = e^{\frac{t}{10}}(\frac{1}{101}\Big(-10cos(t)e^{-\frac{t}{10}} + 100sin(t)e^{-\frac{t}{10}}\Big) + c$$

$$= \frac{10}{101}\big(10sin(t) - cos(t)\big) + ce^{\frac{t}{10}}$$

$z = \frac{10}{101}\big(10sin(t) - cos(t)\big) + ce^{\frac{t}{10}}$ but $y = z^{-1} \rightarrow z = y^{-1}$

$$z = \frac{10}{101}\big(10sin(t) - cos(t)\big) + ce^{\frac{t}{10}} \rightarrow \quad y^{-1} = \frac{10}{101}\big(10sin(t) - cos(t)\big) + ce^{\frac{t}{10}}$$

$$\rightarrow y = \frac{1}{\frac{10}{101}\big(10sin(t) - cos(t)\big) + ce^{\frac{t}{10}}}$$

To verify the solution:

$$y = \frac{1}{\frac{10}{101}\big(10sin(t) - cos(t)\big) + ce^{\frac{t}{10}}} \rightarrow \quad y' = \frac{-\frac{10}{101}\big(10cos(t) + sin(t)\big) - \frac{1}{10}ce^{\frac{t}{10}}}{(\frac{10}{101}\big(10sin(t) - cos(t)\big) + ce^{\frac{t}{10}})^2}$$

$$\rightarrow \quad y' = \frac{-\frac{10}{101}\big(10cos(t) + sin(t)\big) - \frac{1}{10}ce^{\frac{t}{10}}}{\frac{1}{y^2}} = \frac{-\frac{1}{10}\Big(\frac{10}{101}\big(10sin(t) - cos(t)\big) + ce^{\frac{t}{10}}\Big) + \frac{101}{101}cos(t)}{\frac{1}{y^2}}$$

$$\rightarrow \quad y' = \frac{-\frac{1}{10}\Big(\frac{1}{y}\Big) + cos(t)}{\frac{1}{y^2}} = (-\frac{1}{10}\Big(\frac{1}{y}\Big) + cos(t))y^2$$

$$\rightarrow \quad y' = -\frac{1}{10}y + cos(t)y^2$$ correct (☺👍)

Exercises

1- Determine if either of the following equations are separable.

$$y' = \cos(t+y) + \cos(t-y), \qquad y' = \cos(t+y) + \sin(t-y).$$

2- Solve the following homogeneous equations

(i) $y' = \frac{y}{t}(\frac{y}{t} + 1)$,

(ii) $y' = \frac{t^2 - 3y^2}{ty}$,

(iii) $y' = \frac{3t - 4y}{3t + 4y}$,

(iv) $ty' = y + \tan(\frac{y}{t})$,

(v) $y' = \frac{y}{t}[\frac{1}{\ln(\frac{y}{t})} - 1]$.

3- Determine which of the following differential equations are exact

$$(t^2 + ty)dt + tydy = 0,$$
$$(2y + y^2)dt - tdy = 0,$$
$$t^2y^3dt + t^3y^2dy = 0,$$
$$(e^t + y)dt + (2y + t + ye^y)dy = 0.$$

4- Show that the following equations are exact and then solve them

$$\frac{xdx}{(x^2+y^2)^{3/2}} + \frac{ydy}{(x^2+y^2)^{3/2}},$$

$$\frac{dy}{dx} = \frac{y + 6x^2}{x(2 - \ln(x))},$$

$$tdt + ydy = 0.$$

, 5- Solve the IVP,

$$y' + y = \cos t, \qquad y(0) = 1.$$

$$ty' + 2y = e^t, \qquad t > 0,$$

$$y' - \frac{3}{t}y = 4y^{-5}, \qquad y(1) = 2.$$