The Introduction

Riemann sums and the definite integral

Partition of the interval $[a, b]$

تقسيم

فترات جزئية

[divides $[a,b]$ in to subinterval

s.t ; $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

$\Delta x_1 = x_1 - x_0$ $\qquad \Delta x_K = \frac{b-a}{n}$

$\Delta x_2 = x_2 - x_1$

$\vdots$

$\Delta x_n = x_n - x_{n-1}$

مساحة

$$A = \lim_{\max \Delta x_m \to 0} \sum_{K=1}^{n} f(x_K^*) \Delta x_K$$

اطول مقطع

اكبر

Def : A function f is said to be integrable ~~interval~~ on finite closed $[a,b]$ if the Limit

$$\lim_{\max \Delta x_K \to 0} \sum_{K=1}^{n} f(x_K^*) \Delta x_K \text{ exists and}$$

does not depend on the choice of Partitions or on the choice of the numbers x_K^* in the subinterval, when this is the case we, denote the

④ $y^2 = x$, $x = 2y$ about the y axis.

sol

$y^2 = 2y \Rightarrow y^2 - 2y = 0 \Rightarrow y(y-2) = 0$ اما $y = 0$, أو $y = 2$

$$V = \pi \int_0^2 (2y)^2 - (y^2)^2 \, dy = \pi \int_0^2 4y^2 - y^4 \, dy$$

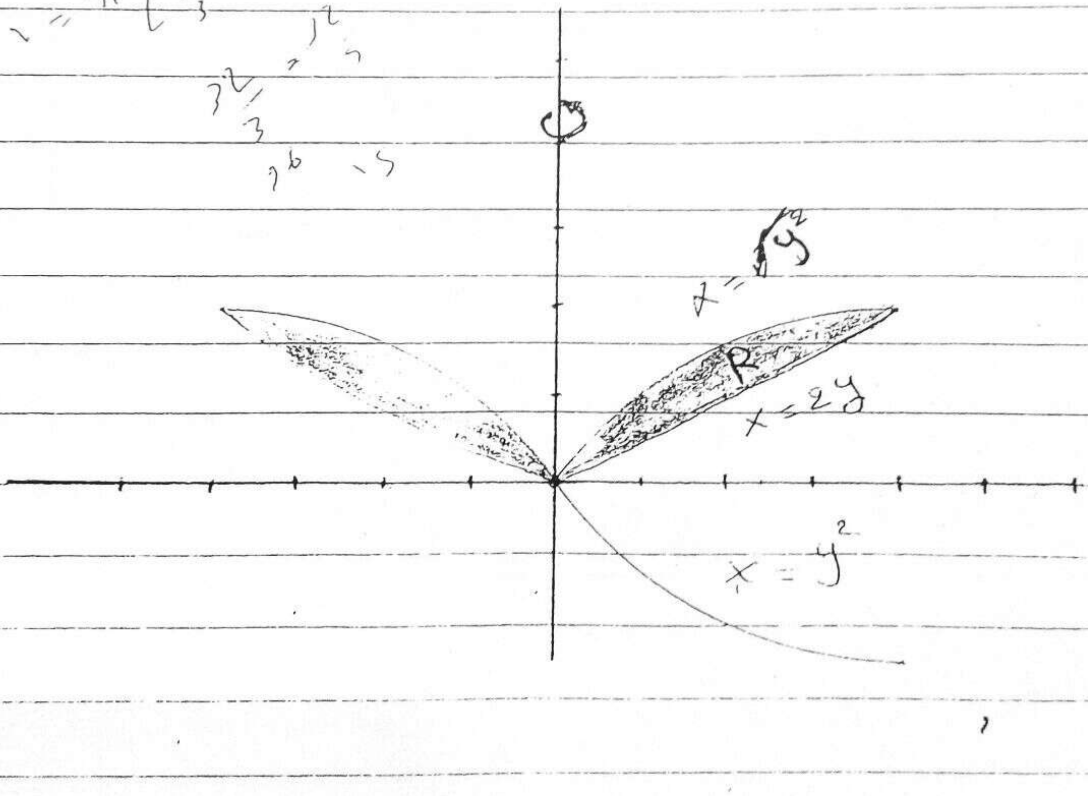
$y^2 = x$

$x = 2y$

$$V = \pi \left[\frac{4}{3} y^3 - \frac{y^5}{5} \Big|_0^2 \right] = \pi \left(\frac{32}{3} - \frac{32}{5} \right)$$

$y^2 = 2y$

$$V = \pi \left(\frac{64}{15} \right) = \frac{64}{15} \pi$$



الموضوع : التاريخ / / ٢٠

⑤ $y = e^{-x}$, $y = 1$, $x = 2$, about $y = 2$

$\because e^{-x} = 1 \Rightarrow x = 0$

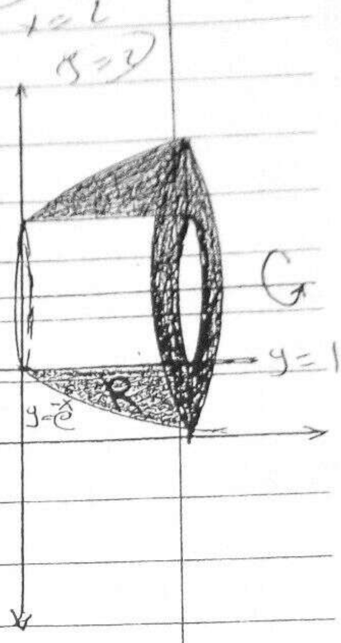
$$V = \pi \int_0^2 (e^{-x} - 2)^2 - (1-2)^2 \, dx$$

$$V = \pi \int_0^2 (e^{-2x} - 4e^{-x} + 4 - 1) \, dx$$

$$V = \pi \int_0^2 e^{-2x} - 4e^{-x} + 3 \, dx$$

$$V = \pi \left[\frac{-1}{2} e^{-2x} + 4e^{-x} + 3x \Big|_0^2 \right]$$

$$V = \pi \left[\frac{-1}{2} e^{-4} + 4e^{-2} + 6 \right]$$



(4) $y = x$, $y = \sqrt{x}$ about $x = 2$

Sol.. $y = \sqrt{x} \Rightarrow x = y^2$

$\therefore y = y^2 \Rightarrow y^2 - y = 0 \Rightarrow y(y-1) = 0$

$\therefore y = 0$ or $y = 1$

$$V = \pi \int_0^1 \left((y^2 - 2)^2 - (y - 2)^2\right) dy$$

$$V = \pi \int_0^1 (y^4 - 4y^2 \cancel{+4} - y^2 + 4y \cancel{-4})\, dy$$

$$V = \pi \int_0^1 (y^4 - 5y^2 + 4y)\, dy$$

$$V = \pi \left[\frac{y^5}{5} - 5\frac{y^3}{3} + 2y^2 \Big|_0^1 \right] = \pi \left[\frac{1}{5} - \frac{5}{3} + 2 \right]$$

$$V = \pi \left(\frac{3 - 25 + 30}{15} \right) = \pi \left(\frac{8}{15} \right) = \frac{8\pi}{15}$$

8) $x - y = 1 \ , \ y = x^2 - 4x + 3 \ , \text{about } y = 3$

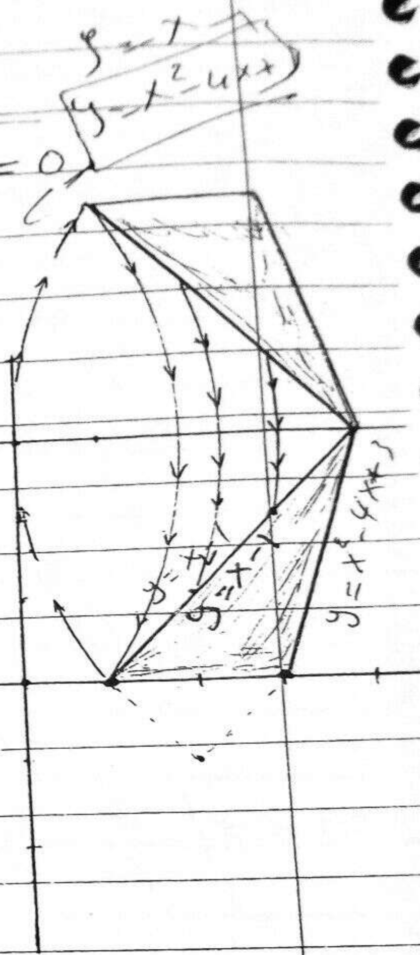
$\therefore x - y = 1 \Rightarrow y = x - 1$ $\because y = x - 1$ dx $y = x^2 - 4x + 3$

$\therefore x - 1 = x^2 - 4x + 3 \Rightarrow x^2 - 4x + 3 - x + 1 = 0$

$x^2 - 5x + 4 = 0$

$x^2 - 5x + 4 = 0 \Rightarrow (x - 4)(x + 1) = 0$

$x = 4$ or $x = 1$



$$V = \pi \int_1^4 (x^2 - 4x + 3 - 3)^2 - (x - 1 - 3)^2 \, dx$$

$$V = \pi \int_1^4 (x^2 - 4x)^2 - (x - 4)^2 \, dx$$

$$V = \pi \int_1^4 (x^4 \cancel{- 8x} + 16x^2 - x^2 \cancel{+ 8x} + 16) \, dx$$

$$V = \pi \int_1^4 (x^4 + 15x^2 + 16) \, dx = \pi \left[\frac{x^5}{5} + 5x^3 + 16x \right]_1^4$$

$$V = \pi \left[\left(\frac{1024}{5} + 320 + 64 \right) - \left(\frac{1}{5} + 5 + 16 \right) \right]$$

$$V = \pi \left[\left(\frac{1024 + 160 + 320}{5} \right) - \left(\frac{1 + 25 + 80}{5} \right) \right]$$

$$V = \pi \left(\frac{2944}{5} - \frac{106}{5} \right) = \frac{2838}{5} \pi$$

⑨ $y = x^3$, $y = \sqrt{x}$, about $x = 1$, $y = 1$

(a) about $x = 1$

sol

$\sqrt[3]{x^3} = \sqrt[3]{y} \Rightarrow x = \sqrt[3]{y}$, $x = y^2$

$(\sqrt[3]{y} = y^2)^3 \Rightarrow y = y^6 \Rightarrow y^6 - y = 0 \Rightarrow y(y^5 - 1) = 0$

$y = 0$ or $y = 1$

$y = y^6$

$y^6 - y =$

$$V = \pi \int_0^1 (y^2 - 1)^2 - (\sqrt[3]{y} - 1)^2 \, dy$$

$$V = \pi \int_0^1 (y^4 - 2y^2 + \cancel{1} - y^{\frac{2}{3}} + 2y^{\frac{1}{3}} - \cancel{1}) \, dy$$

$$V = \pi \left[\frac{y^5}{5} - \frac{2}{3} y^3 + \frac{3}{5} y^{\frac{5}{3}} + \frac{3}{2} y^{\frac{4}{3}} \Big|_0^1 \right]$$

$$V = \pi \left[\frac{1}{5} - \frac{2}{3} - \frac{3}{5} + \frac{3}{2} \right] = \pi \left(\frac{6 - 20 - 18 + 45}{30} \right)$$

$$V = \pi \left(\frac{13}{30} \right) = \frac{13\pi}{30}$$

① $\int \sin x \, dx = -\cos x + c$

② $\int \cos x \, dx = \sin x + c$

③ $\int \sec^2 x \, dx = \tan x + c$

④ $\int \csc^2 x \, dx = -\cot x + c$

⑤ $\int \sec x \tan x \, dx = \sec x + c$

⑥ $\int \csc x \cot x \, dx = -\csc x + c$

⑦ $\int \tan^2 x \, dx = (\sec^2 x - 1) + c$

⑧ $\int \cot^2 x \, dx = (\csc^2 x - 1) + c$

⑨ $\int \sec x \, dx = \ln|\sec x + \tan x| + c$

⑩ $\int \csc x \, dx = \ln|\csc x - \cot x| + c$

⑪ $\int \tan x \, dx = -\ln|\cos x| + c$

⑫ $\int \cot x \, dx = \ln|\sin x| + c$

⑬ $\int \sin^2 x \, dx = \int \frac{1}{2} - \frac{1}{2}\cos x \, dx = \frac{1}{2}x - \frac{1}{4}\sin x + c$

season

$\int \cos^2 x \, dx = \int \frac{1}{2} + \frac{1}{2}\cos x \, dx = \frac{1}{2}x + \frac{1}{4}\sin x + c$

C

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$\cos^2 x + \sin^2 x = 1$

$\sin 2x = 2\sin x \cdot \cos x$

$-\tan^2 x + \sec^2 x = 1$

$\csc^2 x - \cot^2 x = 1$

طريقة القشرة الدورانية الاسطوانية

volumes by cylindricl shells

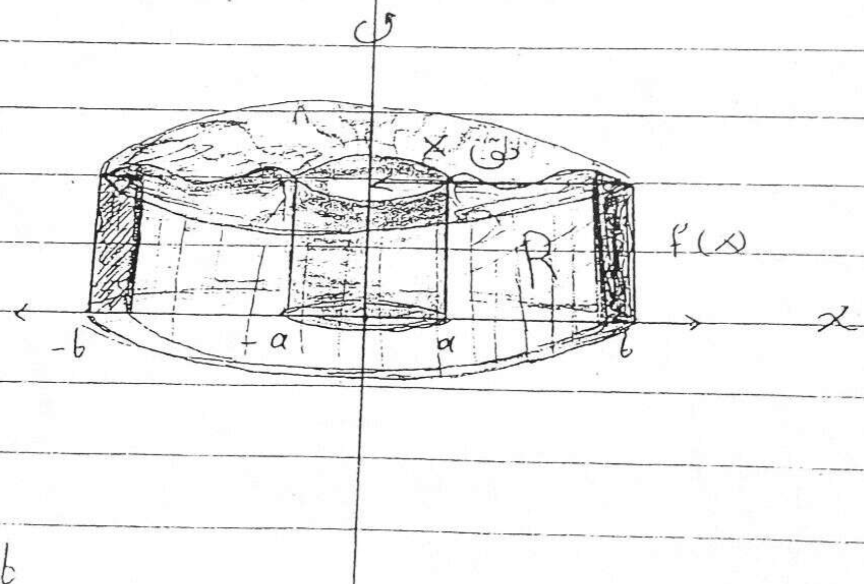
"الاسطوانة القشرية"

Let f be continuous and non negative on [a, b] and let R be the region that is bounded above by y = f(x) below by the x-axis and on the sides by the lines x = a, x = b

Then the volume V of the solid of revolution that generated by revoling the region R about y-axis is given by

$$V = 2\pi \int_a^b x\, f(x)\, dx$$

x - n (x, y)

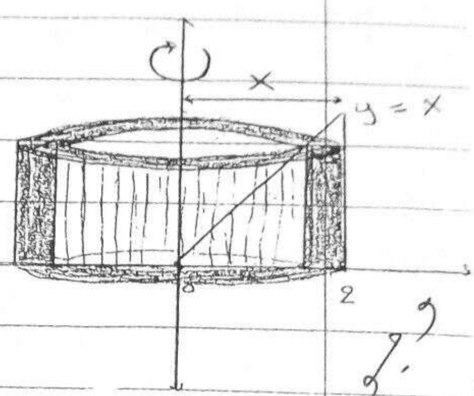


$$V = 2\pi \int_a^b y\, f(y)\, dy$$

if about x-axis

Ex: Use cylindrical shells to find the volume of the solid generated when the region enclosed between $y = x \ \forall x \in [0,2]$ is revolved about the ① y-axis ② x-axis

$$V = 2\pi \int_0^2 x \cdot x \, dx$$



$$V = 2\pi \int_0^2 x^2 \, dx$$

$$V = 2\pi \left[\frac{x^3}{3} \Big|_0^2 \right] = \frac{16\pi}{3}$$

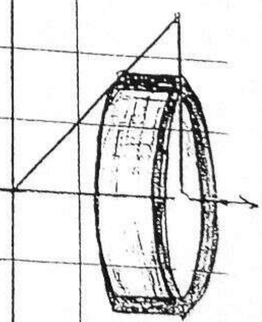
Other way

$$V = \pi \int_0^2 (2^2 - y^2) \, dy$$

$$= \pi \left[4y - \frac{y^3}{3} \Big|_0^2 \right] = \frac{16\pi}{3}$$

x-axis

$$V = 2\pi \int_0^2 y(2-y) \, dy = 2\pi \int_0^2 (2y - y^2) \, dy$$



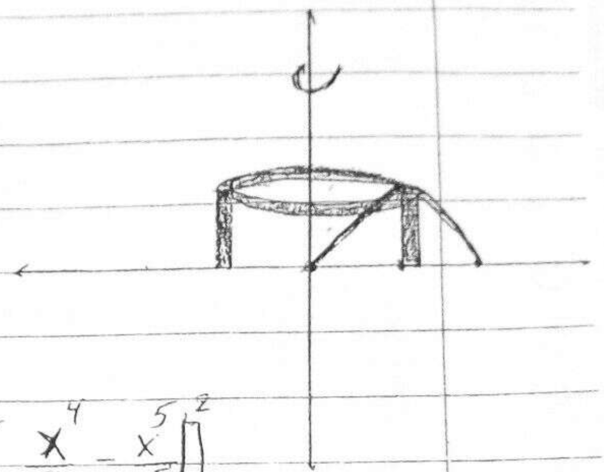
$$= 2\pi \left[y^2 - \frac{y^3}{3} \Big|_0^2 \right] = 2\pi \left[4 - \frac{8}{3} \right]$$

$$= \frac{8\pi}{3}$$

يمكن حله بطريقة القرص

Ex: find the volume of the solid generated by rotaing about the y-axis the region by $y = 2x^2 - x^3$, $y = 0$?

Sol:



$2x^2 - x^3 = 0$

$x^2(2 - x) = 0 \Rightarrow x = 0$ or $x = 2$

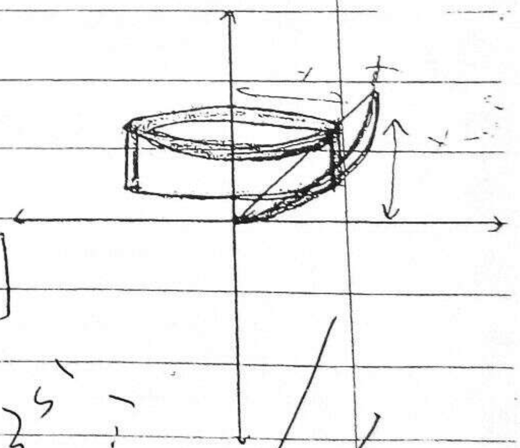
$$V = 2\pi \int_0^2 x(2x^2 - x^3)\,dx$$

$$V = 2\pi \int_0^2 (2x^3 - x^4)\,dx = 2\pi \left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2$$

$$V = 2\pi\left(8 - \frac{32}{5}\right) = 2\pi\left(\frac{40 - 32}{5}\right) = \frac{16\pi}{5}$$

Find the volume about y-axis, $y = x$, $y = x^2$

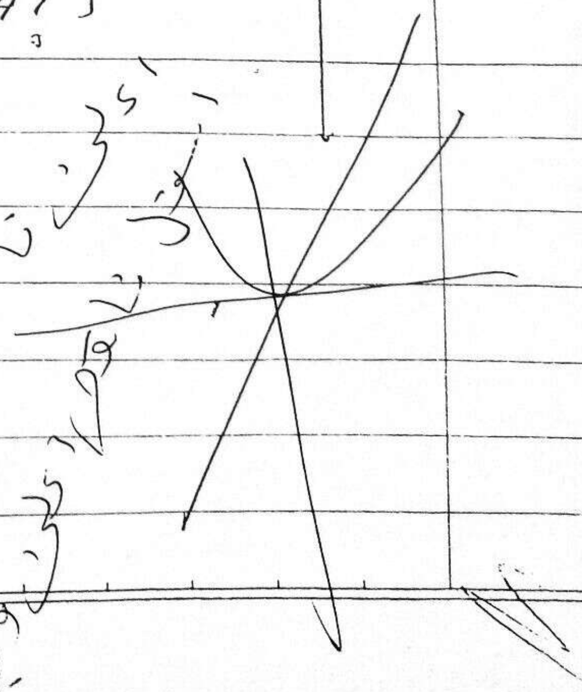
sol/ $x = x^2 \Rightarrow x - x^2 = 0 \Rightarrow x = 0$ or $x = 1$



$$V = 2\pi \int_0^1 x(x - x^2)\,dx$$

$$V = 2\pi \int_0^1 (x^2 - x^3)\,dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$V = 2\pi\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{\pi}{6}$$



Ex: use cylindrical shells to find the volume of the solid obtained by rotating about x axis is the region under the curve $y = \sqrt{x}$ from $x = 0$ to $x = 1$

① about x-axis

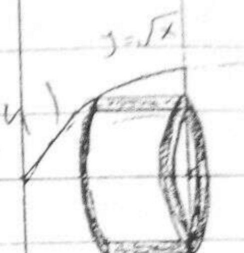
sol.

$$V = 2\pi \int_0^1 y(1-y^2)\,dy$$

$$2\pi \int_a^b x f(x)\,dx \qquad 2\pi \int_a^b x(f(x) - \dots)$$

$$V = 2\pi \int_0^1 (y - y^3)\,dy$$

$$V = 2\pi \left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi}{2}$$

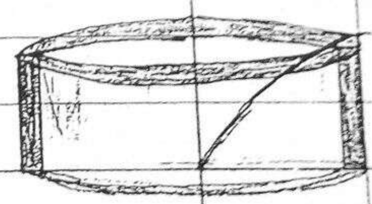


اذا لم تحدد الطريقة فيمكن حله بطريقة القرص

about y-axis $y = \sqrt{x}$, $x = 0$

$$V = 2\pi \int_0^1 x\sqrt{x}\,dx$$

$$V = 2\pi \int_0^1 x^{\frac{3}{2}}\,dx = 2\pi \left[\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1$$

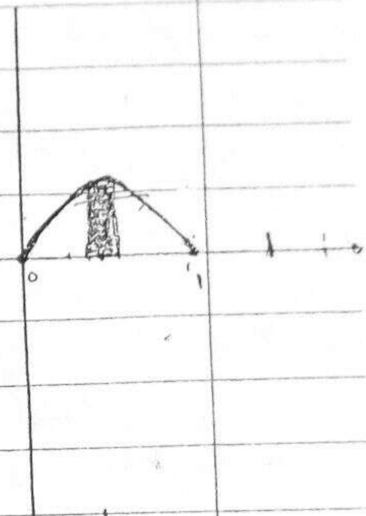


$$V = 2\pi\left(\frac{2}{5}\right) = \frac{4\pi}{5}$$

يمكن حله بطريقة الواشر

Ex; Rotation about a vertical axis find the volume of the solid obtained by rotation the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$

sol/ $x - x^2 = 0 \Rightarrow x = 0$ or $x = 1$



$$V = 2\pi \int_0^1 (2 - x)(x - x^2)\, dx$$

$$V = 2\pi \int_0^1 (2x - 2x^2 - x^2 + x^3)\, dx$$

$$V = 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\, dx = 2\pi \left[\frac{x^4}{4} - x^3 + x^2 \Big|_0^1 \right]$$

$$V = 2\pi \left(\frac{1}{4} - 1 + 1 \right) = \frac{\pi}{2}$$

طول المنحنى

length of plane curve

① let $y = f(x)$ on $[a,b]$ then

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\; dx$$

② let $x = f(y)$ on $[a,b]$ then

$$L = \int_a^b \sqrt{1 + [f'(y)]^2}\; dy$$

③ let $x = x(t)$, $y = y(t)$ on $[a,b]$ then

$$L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\; dt$$

Ex: find the arc length of curve $y = x^{\frac{3}{2}}$ from $(1,1)$ to $(2, 2\sqrt{2})$

soli

$$L = \int_1^2 \sqrt{1+\left[\tfrac{3}{2}x^{\frac{1}{2}}\right]^2}\, dx = \int_1^2 \sqrt{1+\tfrac{9}{4}x}\, dx$$

$$= \frac{4}{9}\int_1^2 \frac{9}{4}\left(1+\frac{9}{4}x\right)^{\frac{1}{2}} dx = \frac{4}{9}\,\frac{\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_1^2$$

$$= \frac{8}{27}\left[\left(1+\frac{9}{2}\right)^{\frac{3}{2}} - \left(1+\frac{9}{4}\right)^{\frac{3}{2}}\right] = \frac{8}{27}\left[\left(\frac{11}{2}\right)^{\frac{3}{2}} - \left(\frac{13}{4}\right)^{\frac{3}{2}}\right] \cong 2.09$$

$$y = x^{\frac{3}{2}} \Rightarrow x = y^{\frac{2}{3}}$$

$$L = \int_1^{2\sqrt{2}} \sqrt{1+\left[\tfrac{2}{3}y^{-\frac{1}{3}}\right]^2}\, dy = \int_1^{2\sqrt{2}} \sqrt{1+\tfrac{4}{9}y^{-\frac{2}{3}}}\, dy$$

$$= \int_1^{2\sqrt{2}} \sqrt{y^0 + \tfrac{4}{9}y^{-\frac{2}{3}}}\, dy = \int_1^{2\sqrt{2}} \sqrt{y^{\frac{2-2}{3}} + \tfrac{4}{9}y^{-\frac{2}{3}}}\, dy$$

$$= \int_1^{2\sqrt{2}} \sqrt{y^{-\frac{2}{3}}\left(y^{\frac{2}{3}}+\tfrac{4}{9}\right)}\, dy = \int_1^{2\sqrt{2}} y^{-\frac{1}{3}}\left(y^{\frac{2}{3}}+\frac{4}{9}\right)^{\frac{1}{2}} dy$$

$$= \frac{3}{2}\left[\frac{\left(y^{\frac{2}{3}}+\frac{4}{9}\right)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_1^{2\sqrt{2}}\right] = \left[\left((2\sqrt{2})^{\frac{2}{3}}+\frac{4}{9}\right)^{\frac{3}{2}} - \left(1+\frac{4}{9}\right)^{\frac{3}{2}}\right]$$

Ex: $x = a\cos t$, $y = a\sin t$, $a > 1$, $0 \leq t \leq 2\pi$

$$L = \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2(\underbrace{\sin^2 t + \cos^2 t}_{1})}\, dt = \int_0^{2\pi} a\, dt = at \Big|_0^{2\pi} = a2\pi$$

Ex: $x = t^2$, $y = t^3$ from $(\overset{a}{1}, \overset{b}{1})$ to $(\overset{4}{4}, 8)$

$$L = \int_1^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt = \int_1^2 \sqrt{4t^2 + 9t^4}\, dt$$

$$= \int_1^2 t\sqrt{4 + 9t^2}\, dt = \int_1^2 t(4 + 9t^2)^{\frac{1}{2}}\, dt$$

$$= \frac{1}{18}\left[\frac{(4+9t^2)^{\frac{3}{2}}}{\frac{3}{2}} \Bigg|_1^2 \right] = \frac{1}{27}\left[(4 + 9(2^2)) - (4 - 9) \right] = 7,6337$$

$1 = t^2 \Rightarrow t = \pm 1$
$4 = t^2 \Rightarrow t = \pm 2$
$1 = t^3 \Rightarrow t = 1$
$8 = t^3 \Rightarrow t = 2$

المساحة السطحية الدورانية

Def: if f is , non negative function on $[a,b]$. then surface area as of the surfaceof revolution that is genearted by rotating the portion of the curve $y=f(x)$ between $x=a$, $x=b$ about x-axis defined

$$S=\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$=\int_a^b 2\pi y\sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx$$

$x=g(y)$, $y=c$ $y=d$ about y-axis

$$S=\int_c^d 2\pi g(y)\sqrt{1+[g'(y)]^2}\,dy$$

$\sqrt{y}=x$

$$=\int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$y=x^2$

$$S\ 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

season

Ex: find the area of the surface that is generated rotating portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = \int_0^1 2\pi x^3 \sqrt{1 + (3x^2)^2}\, dx = \int_0^1 2\pi x^3 \sqrt{(1 + 9x^4)}\, dx$$

$$= \frac{\pi}{16} \, \frac{(1 + 9x^4)^{\frac{3}{2}} - 1}{\frac{3}{2}} = 3.56$$

Ex $y = x^2$, $x = 1$, $x = 2$ about y-axis $x = \sqrt{y}$

$$S = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dy = 2\pi \int_1^4 \sqrt{y} \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy$$

$$= 2\pi \int_1^4 \sqrt{y} \sqrt{1 + \frac{1}{4y}}\, dy = 2\pi \int_1^4 \sqrt{y\left(1 + \frac{1}{4y}\right)}\, dy$$

$$= 2\pi \int_1^4 \sqrt{y + \frac{1}{4}}\, dy = 2\pi \left[\frac{\left(y + \frac{1}{4}\right)^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$$

$$= 2\pi \, \frac{2}{3} \left[\left(4 + \frac{1}{4}\right)^{\frac{3}{2}} - \left(1 + \frac{1}{4}\right)^{\frac{3}{2}}\right]$$

$$= \frac{4\pi}{3} \left[\left(\frac{17}{4}\right)^{\frac{3}{2}} - \left(\frac{5}{4}\right)^{\frac{3}{2}}\right] = 2\pi[8.76 - 1.39]$$

$$= 46.28$$

Area of Sarfce of revolution

let $x(t)$, $y(t)$, $x'(t)$, $y'(t)$ are continuous functions and $a \leq t \leq b$ then the area surface generated by this curve about x-axis

$$S = \int_a^b 2\pi\, y(t) \sqrt{[x'(t)]^2 + (y'(t))^2}\, dt$$

about y-axis $\qquad \int_a^b 2\pi\, y\, t.$

$$S = \int_a^b 2\pi\, x(t) \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

Ex: $x = t \qquad y = 2t^2 \qquad 0 \leq t \leq 1$ about y-axis

$$S = \int_0^1 2\pi x(t) \sqrt{1 + (4t)^2}\, dt$$

$$= \int_0^1 2\pi t \sqrt{1 + 16t^2}\, dt = \frac{\pi}{16} \frac{(1+16t^2)^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^1$$

$$= \frac{\pi}{16} \frac{(1+16)^{\frac{3}{2}}}{\frac{3}{2}} = \frac{\pi}{24}(1+16)^{\frac{3}{2}} = 9.17$$

$2\pi\, y(t) \sqrt{x'(t)^2 + y'(t)^2}$

$dt = \frac{\pi}{16}$

Ex: $x = a\cos t$, $y = a\sin t$ $0 \le t \le \pi$ (x-axis

$$S = \int_0^{\pi} 2\pi\, a\sin t \sqrt{(-a\sin t)^2 + (a\cos t)^2}\; dt$$

$$= \int_0^{\pi} 2\pi a\sin t \sqrt{a^2\sin^2 t + a^2\cos^2 t}\; dt$$

$$= \int_0^{\pi} 2\pi a\sin t\, a\sqrt{\underbrace{\sin^2 t + \cos^2 t}_{1}}\; dt$$

$$= 2a^2\pi \int_0^{\pi} \sin t = 2a^2\pi\left(-\cos t\Big|_0^{\pi}\right]$$

$$= 2a^2\pi(-(-1)-(-1)) = 2a^2\pi(1+1) = 4a^2\pi$$

$2a^2\pi[-\cos t]$

$-(-1-(-1))$

$2a^2\pi(1+1)$

$4a^2\pi$

about y-axis

$$S = \int_0^{\pi} 2\pi\, a\cos t\sqrt{(a\sin t)^2 + (a\cos t)^2}\; dt$$

$$= 2a^2\pi \int_0^{\pi} \cos t\; dt = 2a^2\pi\left(\sin t\Big|_0^{\pi}\right]$$

$$= 2a^2\pi(\sin\pi - \sin 0] = 0$$

moments and centers of mass

العزوم و سنتر الكتلة

Def I let ρ (CRant) density then mass is define

$$M = \rho A$$

$$M = \rho \int_a^b f(x)\, dx$$ when $f(x)$ is continuous on $[a,b]$

$$M = \int_a^b \rho f(x)\, dx$$

$$M_y = \rho \int_a^b x f(x)\, dx$$

$$M_x = \rho \int_a^b \frac{1}{2}[f(x)]^2\, dx$$

$$\bar{x} = \frac{M_y}{M} = \frac{\rho \int_a^b x f(x)\, dx}{\rho \int_a^b f(x)\, dx} = \frac{1}{A}\int_a^b x f(x)\, dx$$

$$\bar{y} = \frac{M_x}{M} = \frac{\rho \int_a^b \frac{1}{2}[f(x)]^2\, dx}{\rho \int_a^b f(x)\, dx} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\, dx$$

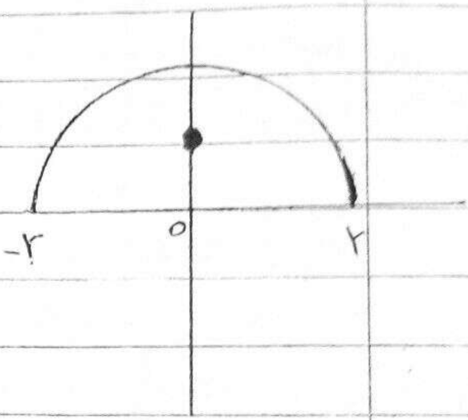
$\therefore$ center $= (\bar{x}, \bar{y})$

Ex: find the center of the

sol:

$$M = \rho A = \rho \, \frac{1}{2} \pi r^2$$ نصف مساحة الدائرة

$$\bar{x} = \frac{M_y}{M} = 0$$ لانه واقعة على (y)



$$\bar{y} = \frac{M_x}{M} = \frac{1}{A} \int_a^b \frac{1}{2} \left[f(x)\right]^2 dx$$

$$= \frac{1}{\frac{1}{2}\pi r^2} \int_{-r}^{r} \frac{1}{2} \left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \frac{1}{\cancel{\frac{1}{2}}\pi r^2} \, \cancel{\frac{1}{2}} \, 2 \int_0^r r^2 - x^2 \, dx$$

$$= \frac{2}{\pi r^2} \left[r^2 x - \frac{x^3}{3} \Big|_0^r \right] = \frac{2}{\pi r^2} \left[r^3 - \frac{r^3}{3} \right]$$

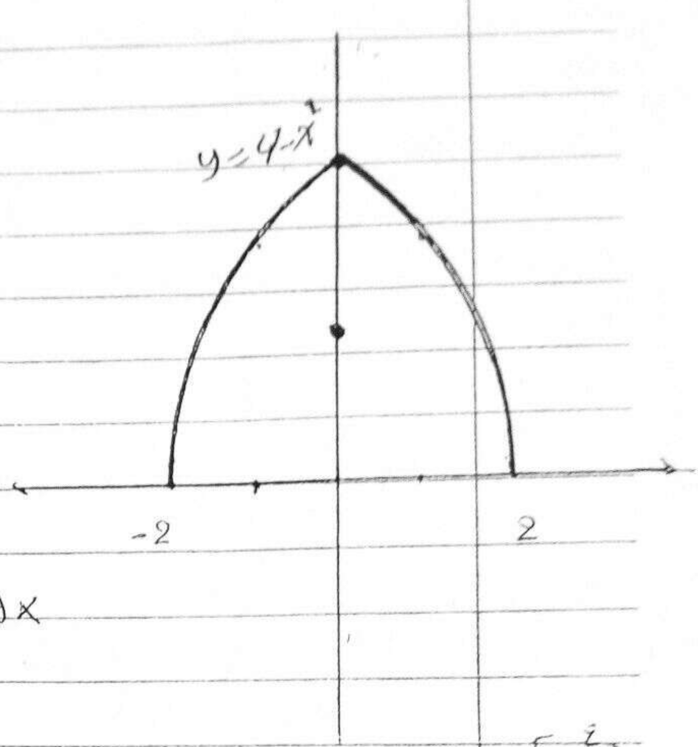
$$= \frac{2}{\pi \cancel{r^2}} \cdot \frac{2 r^{\cancel{3}}}{3} = \frac{4r}{3\pi}$$

The center is $\left(0, \frac{4r}{3\pi}\right)$

Ex: find the center
$y = 4 - x^2$ and $y = 0$

$4 - x^2 = 0 \Rightarrow x = \pm 2$

$M = \rho A = 2\int_0^2 (4 - x^2)\,dx$



$= 2\left[4x - \frac{x^3}{3}\Big|_0^2\right] = \frac{32}{3}$

$\bar{X} = \frac{M_y}{M} = 0$

$\bar{y} = \frac{M_x}{M} = \frac{1}{A}\int_{-2}^{2} \frac{1}{2}(4 - x^2)^2\,dx$

$= \frac{1}{\frac{32}{3}} \cdot \frac{1}{2} \cdot 2\int_0^2 (16 - 8x^2 + x^4)\,dx = \frac{3}{32}\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\Big|_0^2\right]$

$= \frac{3}{32}\left[32 - \frac{64}{3} + \frac{32}{5}\right] = \frac{3}{32}\left[\frac{256}{15}\right]$

$= \frac{256}{160}$

center $= (0, 1.6)$

ملاحظة يمكن ايجاد M_x من خلال القانون التالي

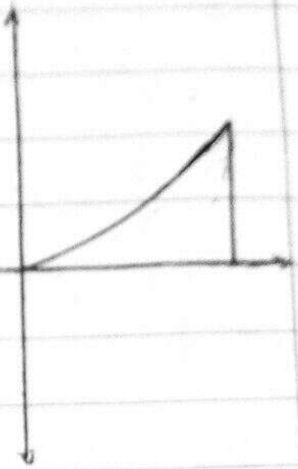
$M_x = \int_{f(a)=c}^{f(b)=d} \rho\, y\, f(y)\, dy$

season

Ex: find the center of mass of the curve
$y = x^2$, $y = 0$ and $x = 1$, $y = 0$

$P = 1$

$$M = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$



$$M_y = \int_0^1 x\,x^2\,dx = \int_0^1 x^3\,dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$$

$$M_x = \int_0^1 y(1 - \sqrt{y})\,dy = \int_0^1 (y - y^{\frac{3}{2}})\,dy$$

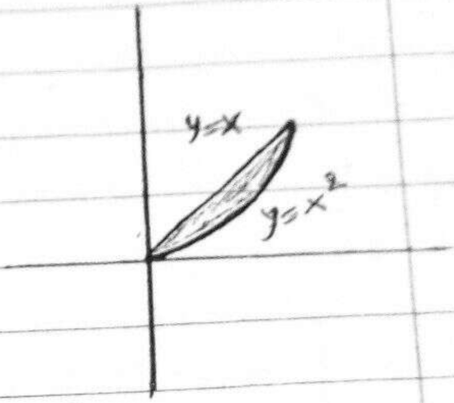
$$= \frac{y^2}{2} - \frac{y^{\frac{5}{2}}}{\frac{5}{2}}\Big|_0^1 = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}$$

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{1}{10}}{\frac{1}{3}} = \frac{3}{10}$$

$\therefore$ the center is $\left(\frac{3}{4}, \frac{3}{10}\right)$

Ex: find the center of curves $y = x^2$, $y = x$



$$M = \int_0^1 (x - x^2)\, dx = \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$M_y = \int_0^1 x(x - x^2)\, dx =$$

$$= \int_0^1 (x^2 - x^3)\, dx = \frac{x^3}{3} - \frac{x^4}{4} \Big|_0^1 = \frac{1}{12}$$

$$M_x = \int_0^1 \frac{1}{2} [f(x)]^2 = \frac{1}{2} \int_0^1 (x^2 - x^4)\, dx$$

$$\frac{1}{2} \left[\frac{x^3}{3} - \frac{x^5}{5} \Big|_0^1 \right] = \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{1}{15}$$

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{1}{15}}{\frac{1}{6}} = \frac{6}{15} = \frac{2}{5}$$

the center is $\left(\frac{1}{2} , \frac{2}{5} \right)$

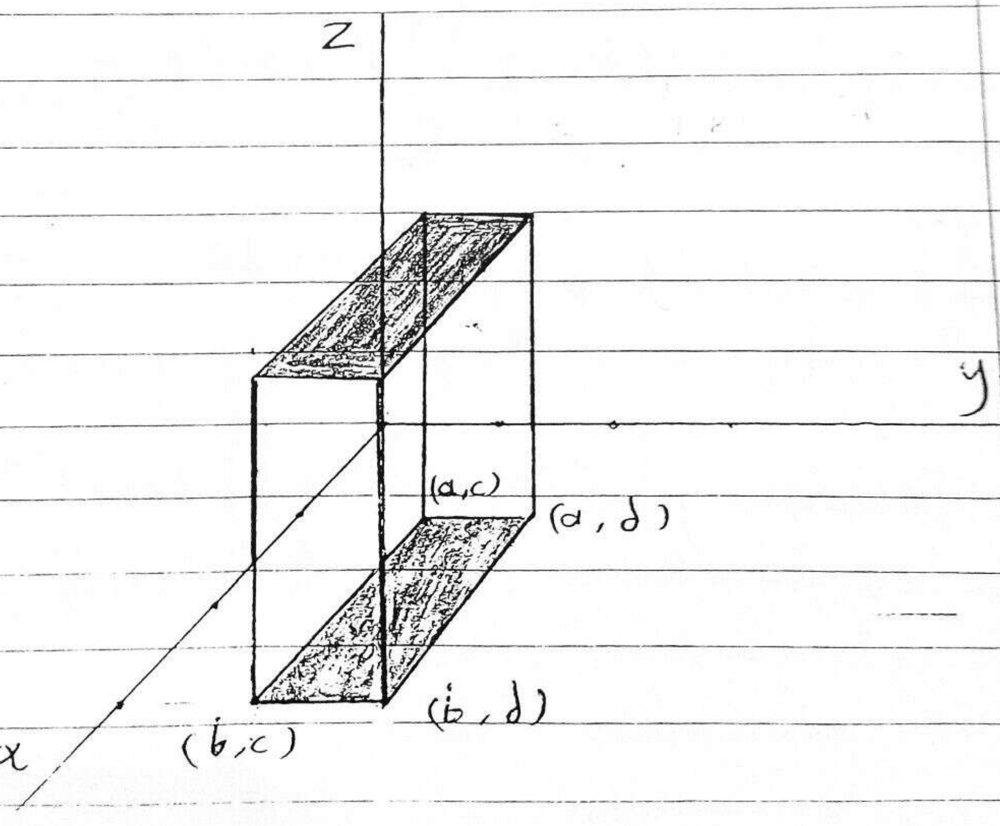
Double integrals   التكامل الثنائي

let f a function of two variables that is continous on the rgion R in the xy - Plane then

$\iint_R f(x,y)\, dA$ which is called

s.t $z = f(x,y)$, $R = [x \in (a,b), y \in (c,d)]$

$R = \{(x,y) \in R^2 / a \leq x \leq b, c \leq y \leq d\}$



properties of double inegrals:

① $$\iint_R c\, f(x,y)\, dA = c \iint_R f(x,y)\, dA$$

② $$\iint_R [f(x,y) \mp g(x,y)]\, dA$$

$$= \iint_R f(x,y)\, dA \mp \iint_R g(x,y)\, dA$$

③ if $f(x,y) \geq g(x,y) \Rightarrow \iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA$

④ if $R = R_1 \cup R_2 \Rightarrow \iint_R f(x,y)\, dA =$

$$\iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$$

⑤ Theorem: $\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx$

Ex1: Evaluate the double integral

$$\iint_R (x-3y^2)\, dA \quad \text{when } R=\left\{(x,y)\in \mathbb{R}^2 / \begin{matrix} 0\le x\le 2 \\ 1\le y\le 2 \end{matrix}\right\}$$

sol
$$\int_0^2\int_1^2 (x-3y^2)\, dy\, dx = \int_0^2 xy - y^3 \Big|_1^2 \, dx$$

$$\int_0^2 2x - 8\, dx = \left[x^2 - 8x\Big|_0^2\right] = -12$$

$$\iint_R y\sin(xy)\, dA \quad \text{when } R=\left[\overset{c\ d}{(1,2)} \times \overset{a\ b}{(0,\pi)}\right]$$

sol
$$\int_0^\pi\int_1^2 y\sin(xy)\, dx\, dy = \int_0^\pi -\cos xy \Big|_1^2 \, dy$$

$$= \int_0^\pi -\cos 2y + \cos y\, dy = \left[-\frac{1}{2}\sin 2y + \sin y\Big|_0^\pi\right]$$

$$-\frac{1}{2}(0+1) - \left(-\frac{1}{2}(0)+0\right) = -\frac{1}{2}$$

volume under asurface

if f is afanction of two variables that is continuous and non negative on aregion R in the xy-plane then the volume of the solid enclosed between the surface $z = f(x,y)$ and the region R is defined by

$$V = \iint_R f(x,y)\,dA$$

Note : $A(y) = \int_a^b f(x,y)\,dx$

$$V = \int_a^b A(x)\,dx \rightarrow A(x) \int_c^d f(x,y)\,dy$$

$$V = \int_c^d A(y)\,dy$$

Ex③ use the double integral find the volume of the solid that is bounded about bf the plane $z = \boxed{4 - x - y}$ and betwo by the rectangle $R = \boxed{[0,1]}_x \times \boxed{[0,2]}_y$

sol

$$V = \iint_R f(x,y)\, dA = \int_0^2 \int_0^1 4 - x - y \; dx\, dy \qquad dx\, dy$$

$V = \iint R(x,y)\, dA$

$$= \int_0^2 4x - \frac{x^2}{2} - yx \Big|_0^1 dy = \int_0^1 4y - \frac{1}{2} - y \; dy$$

$$= \left[4y - \frac{1}{2}y - \frac{y^2}{2} \Big|_0^2 \right] = 8 - 1 - 2 = 8 - 3 = 5$$

Ex④ $\iint_R (\sin x \cos y)\, dA \quad R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$

sol, $\int_0^{\frac{\pi}{2}} \sin x\, dx \cdot \int_0^{\frac{\pi}{2}} \cos y\, dy$

يمكن ان نجرى التكامل
بالنسبة لـ x و y اذا
كانت الدالة حاصل ضرب
دالة لـ x في دالة لـ y

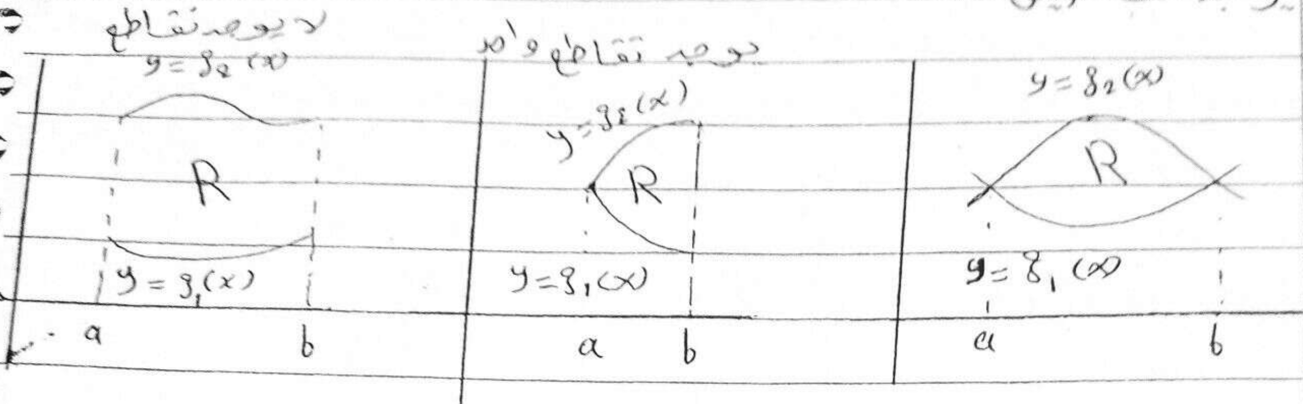
$$= -\cos x \Big|_0^{\frac{\pi}{2}} \cdot \sin y \Big|_0^{\frac{\pi}{2}}$$

$$= \left(-\cos \frac{\pi}{2} + \cos 0\right) \cdot \left(\sin \frac{\pi}{2} + \sin 0\right)$$

$$= (0 + 1) \cdot (1) = 1$$

Double Integrals over General Regions

التكامل الثنائي على منطقة عامة.

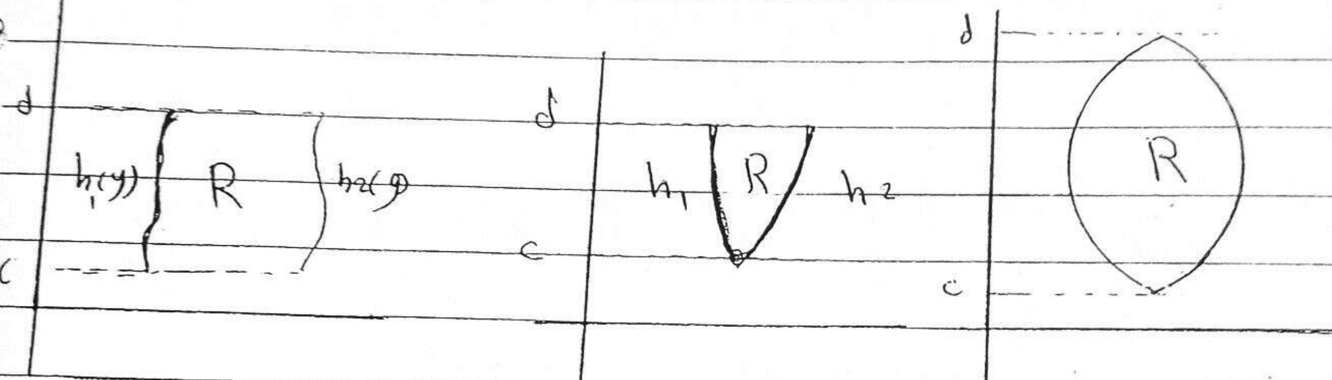


$y = g_1(x)$

$y = g_2(x)$

$x = a$

$x = b$



$x = h_1(y)$

$x = h_2(y)$

$y = c$

$y = d$

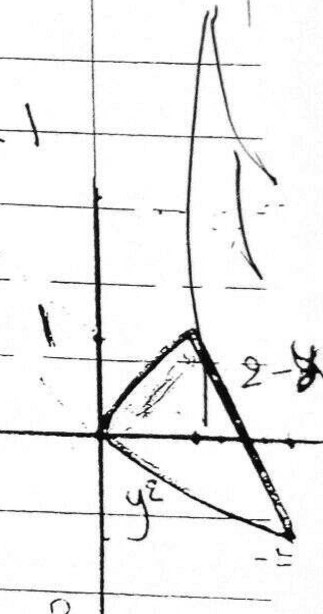
find $\iint_R x\,dA$ where $x = y^2$, $x = 2 - y$

$y^2 + y - 2 = 0 \Rightarrow (y+2)(y-1) = 0 \Rightarrow -2 \le y \le 1$

$$\int_{-2}^{1}\int_{y^2}^{2-y} x\,dx\,dy = \int_{-2}^{1} \frac{x^2}{2}\Big|_{y^2}^{2-y} dy = \int_{-2}^{1} \frac{(2-y)^2}{2} - \frac{y^4}{2}\,dy$$

$$\int_{-2}^{1} \frac{4-4y-y^2}{2} - \frac{y^4}{2}\,dy = 2y - y^2 - \frac{y^3}{3}\Big|_{-2}^{1}$$

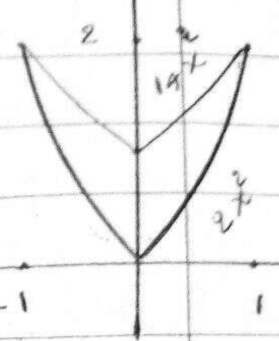
$$= \frac{2}{3} - \frac{16}{3} = \frac{-14}{3}$$



Ex: Evaluate $\iint x + 2y \, dA$ where R is the region bounded by the Para $y = 2x^2$ and $y = 1 + x^2$?

sol

$2x^2 - x^2 = 1 \Rightarrow x^2(2-1) = 1 \Rightarrow x^2 = 1 \therefore x = \pm 1$



$$\iint_R x + 2y \, dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} x - 2y \, dy \, dx$$

$$= \int_{-1}^{1} xy - y^2 \Big|_{2x^2}^{1+x^2} dx = \int_{-1}^{1} x + x^3 + (1+x^2)^2 - x(2x^2) - (2x^2)^2 \, dx$$

$$= \int_{-1}^{1} x + x^3 + 1 + 2x^2 + x^4 \quad 2x^3 - 4x^4 \, dx$$

$$= \int_{-1}^{1} -3x^4 - x^3 + 2x^2 + 1 \, dx = \frac{-3}{5}x^5 - \frac{x^4}{4} - \frac{2}{3}x^3 + x \Big|_{-1}^{1}$$

$$\left(\frac{-3}{5} - \cancel{\frac{1}{4}} - \frac{2}{3} + \cancel{1}\right) - \left(\frac{3}{5} - \cancel{\frac{1}{4}} + \frac{2}{3} - \cancel{1}\right)$$

$$\frac{-6}{5} - \frac{4}{3} = \frac{-18-20}{15} = \frac{-38}{15}$$

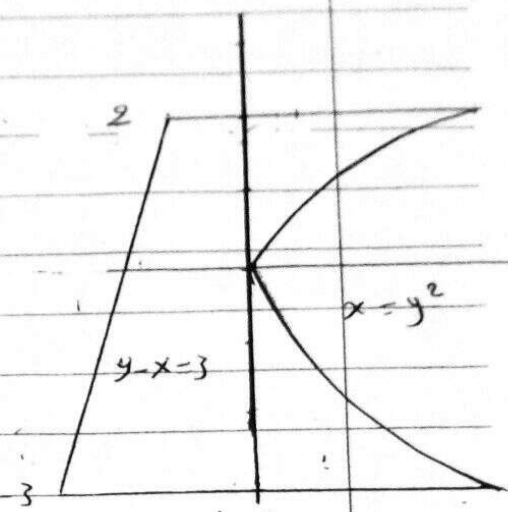
Using Double Integral to find Area

$$\iint_R dA = \text{Area of } R$$

Ex: find the area of the plan region bounded by the graphs of $x=y^2$, $y=-3$, $y-x=3$, $y=2$?

sol:

$$\iint_R dA = \int_{-3}^{2}\int_{y-3}^{y^2} dx\,dy$$



$$\int_{-3}^{2} x \Big|_{y-3}^{y^2} dy = \int_{-3}^{2} y^2 - y + 3 \, dy$$

$$\frac{y^3}{3} - \frac{y^2}{2} + 3y \Big|_{-3}^{2} = \left(\frac{8}{3} - 2 + 6\right) - \left(-9 - \frac{9}{2} - 9\right) = \frac{175}{6}$$

Moments and center mass العزوم ومركز الكتلة

$\rho(x,y)$ كثافة

$$M = \iint_R \rho(x,y)\,dA \quad , \quad M_y = \iint_R x\,\rho(x,y)\,dA$$

$$M_x = \iint_R y\,\rho(x,y)\,dA$$

$$\bar{x} = \frac{M_y}{M} \quad , \quad \bar{y} = \frac{M_x}{M}$$

$\rightarrow$ center of mass $(\bar{x}, \bar{y})$

Rana

نسخي ٣ Zoera

الموضوع : التكامل المتقدم

التاريخ / / ٢٠١

The Introduction

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② volumes of slicing, Disk, and washer
شريحة

③ volumes of cylindrical shells
الاسطواني القواقع

④ Length of A plane curve
طول المستوي منحني

⑤ Double integrals

⑥ Applications of integration

المصادر

① مخلصات شوم في التفاضل و التكامل المتقدم

② Calculus 4E

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