1. Natural Number

A natural number is a number that occurs commonly and obviously in nature.

Definition 1.1. *The natural numbers, denoted as N, is the set of the positive whole numbers. We denote it as follows:*

$$N = \{0, 1, 2, ...\}$$

A possible second definition for N, that addresses the above criticism, is the following:

$$N = |We\ can\ write\ x\ as\ the\ sum\ 1 + 1 + ... + 1, for\ some\ number\ of 1's.\}$$

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Example 1.1. *Is* $5 \in N$*? Since* $5 = 1 + 1 + 1 + 1 + 1$ *then,* $5 \in N$.

Theorem 1.1. *For all natural numbers* $m, n,$ *and* p *we have*

1. $m + n \in N$ *(closure +).*
2. $(m + n) + p = m + (n + p)$ *(commutativity +).*
3. $m + n = n + m$ *(associativity +).*
4. $n + 0 = 0$ *(identity +).*
5. $nm \in N$ *(closure .).*
6. $(mn)p = m(np)$ *(commutativity .).*
7. $mn = nm$ *(associativity .).*
8. $m1 = m$ *(identity .).*

Example 1.2. *Let* n, m *and* b *be natural numbers. Then*

$$(mn)b = (mb)n.$$

Sol: Based on Theorem 1.1

$$(mn)b = m(nb)$$

then,

$$m(nb) = m(bn)$$

and,

$$m(bn) = (mb)n.$$

Then,

$$(mn)b = (mb)n.$$

Remark 1.1. *Each $n \in N$ has a successor $n+1$. For example, the successor of 5 is 6.*

Proposition 1.1. *The set N satisfies the following properties:*

N1 $0 \in N$;

N2 *if $n \in N$, then its successor $n+1 \in N$;*

N3 *0 is not the successor of any element in N;*

N4 *if n and m have the same successor, then $n = m$;*

N5 *suppose S is a subset of N satisfying: $1 \in N$ and if $n \in S$ then $n+1 \in S$, then $S = N$.*

Definition 1.2. *Natural number is either the value zero or the successor of some other natural numbers.*

Theorem 1.2. *Let m and n be natural numbers. There exactly one of the following three statement is true:*

1. $m > n$
2. $n > m$
3. $m = n$

1.1. Some result on Natural number (counting numbers)

1. Sum of all first n natural numbers is

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

2. Sum of square of all first n natural numbers is

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. Sum of all cube of all first n natural numbers is

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

4. Sum of first n odd natural numbers is

$$1 + 3 + 7 + \cdots = n^2.$$

5. Sum of first n even natural numbers is

$$2 + 4 + 6 + \cdots = n(n+1).$$

2. Mathematical Induction

2.1. The Principle of Mathematical Induction

Suppose we have some statement $P_{(n)}$ and we want to demonstrate that $P_{(n)}$ is true for all $n \in N$. Even if we can provide proofs for

$$P_{(1)}, P_{(2)}, \cdots P_{(k)}$$

where kis some large number, we have accomplished very little. However, there is a general method, the Principle of Mathematical Induction.
Induction is a defining difference between discrete and continuous mathematics.
Principle of Induction. In order to show that $\forall n, P_{(n)}$, holds, it suffices to establish the following two properties:

1. Base case: Show that $P_{(n)}$ holds.
2. Induction step: Assume that $P_{(n)}$ holds, and show that $P_{(n+1)}$ also holds.

2.2. Induction Examples

Example 2.1. *By using mathematical induction prove*

$$3^n - 1$$

a multiple of 2, $\forall n \in N$.

1. *Step one: for* $n = 1$
$$3^1 - 1 = 3 - 1 = 2$$
Since 2 is multiple of 2 then, $3^1 - 1$ *is true*
2. *Assume it is true for* $n = k$
$$3^k - 1$$
is true.
Now, prove that $3^{k+1} - 1$ *is multiple of 2*
$$3^{k+1} - 1 = 3^k 3 - 1 = 3^k(2 + 1) - 1 = 23^k + (3^k - 1)$$

Example 2.2. *Prove for* $n \geq 1$

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n$$

!= (n+1)!-1

Step one: for $n = 1$ *The left hand side is* $1 \times 1! = 1$. *The right hand side is* $2! - 1 + 1$. *They are equal.*

Assume it is true for $n = k$

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + k \times k! = (k+1)! - 1$$

is true.
Now, prove that for $n = k + 1$

$$\begin{aligned} 1 \times 1! + \cdots + k \times k! + (k+1) \times (k+1)! \quad &= (k+1)! - 1 + (k+1) \times (k+1)! \\ &= [(k+1)! + (k+1) \times (k+1)!] - 1 \\ &= (k+1)![1 + (k+1)] - 1 \\ &= (k+1)![k+2] - 1 \\ &= (k+2)! - 1 \end{aligned}$$

3. Integer number

For several reasons, it is convenient to extend the set N of natural numbers to the group Z of integers by throwing in the identity element 0 and an inverse n for each natural number n. One reason for doing this is to ensure that the difference m n of any two integers is meaningful. Thus Z is a set on which all three operations +, , and × are defined. (The notation Z comes from the German "Zahlen", meaning "numbers".)

Definition 3.1. *An integer numbers is a whole number that can be positive, negative, or zero. The set of integers, denoted Z, is formally defined as follows:*

$$..., -2, -1, 0, 1, 2, ...$$

Theorem 3.1. *Let* $a \in Z$ *then,*

1. $a + 0 = a$
2. $a - 0 = a$
3. $a.0 = 0.a = 0$
4. $\frac{a}{0}$ *is not defined.*
5. $a + a = 2a$
6. $a - a = 0$
7. $a \times a = a^2$
8. $\frac{a}{a} = 1$ *(except for* $a = 0$*, which is not defined, see rule 4)*
9. $a + (-a) = 0$

10. $a - (-a) = 2a$

11. $a \times (-a) = -a^2$

12. $\dfrac{a}{-a} = -1$ *(except for $a = 0$, which is not defined, see rule 4)*

Definition 3.2. *Let $a, b \in Z$,*

1. $a < b$ *if* $b - a \in Z$.
2. $a \leqslant b$ *if* $a - b \in Z$.

Theorem 3.2. *Let $a, b, c \in Z$ then,*

1. *If $a < b$ and $b < c$ then, $a < c$.*
2. *If $a < b$ $c > 0$ then, $ac < bc$.*

Example 3.1. *By using the integers number's properties prove if $a < b$ and $b < c$ then, $ac < bc$.*

Corollary 3.1. *Let a and b be an integer numbers then,*

1. *if* $a + c = b + c \to a = b$
2. $(-a)b = -(ab)$
3. $(-a)(-b) = ab$
4. *if* $a.b = a.c$ *and* $c \neq 0 \to a = b$

Proof. 1.

$$a + c = b + c$$

$$a + c - c = b + c - c$$

Based on Theorem 3.1(6)

$$a + 0 = b + 0$$

Based on Theorem 3.1(1)

$$a = b$$

2. Based on Theorem 3.1(3)

$$a.0 = 0$$

Based on Theorem 3.1(6)

$$0.a = 0$$

Based on Theorem 3.1(9)

$$(b + (-b))a = -(ab) + (ab)$$

$$ba + (-b)a = (ab) - (ab)$$

Based on Corollary 3.1(1) then,

$$(-b)a = -(ab).$$

3. Based on Corollary 3.1(2) then,

$$(-a)(-b) = -(-ab) = - \cdot -(ab) = ab$$

4. if $a.b = a.c$ and $c \neq 0 \rightarrow a = b$

5. $a \cdot b = 0 \rightarrow$ either $a = 0$ or $b = 0$

□

Theorem 3.3. *Let a and b be an integer numbers then,*

1. *if* $a \leqslant b \rightarrow -b \leqslant -a$

2. *if* $a \leqslant b$ $c \leqslant 0$ $bc \leqslant ac$

3. $0 \leqslant a$ *and* $0 \leqslant b \rightarrow 0 \leqslant ab$

4. $0 \leqslant a^2 \bigvee a$.

Proof. 1. Based on Corollary 3.1(1)

$$a + -b \leqslant b + -b.$$

Based on Theorem 3.1(6)

$$a + -b \leqslant 0.$$

$$-a + a + -b \leqslant -a + 0.$$

$$-b \leqslant -a.$$

2. Based on Theorem 3.3(1) $0 \leqslant -c$ and based on Corollary 3.1(1) then,

$$a \cdot (-c) \leqslant b \cdot (-c)$$

$$-(ac) \leqslant -(bc)$$

Based on Theorem 3.1(1)

$$-a \leqslant -b$$

3. Homework.

4. Homework.

□

3.1. Well-ordering principle

In mathematics, the well-ordering principle states that every non-empty set of positive integers contains a least element. In other words, the set of positive integers is well-ordered.

1. *Every nonempty subset S of the positive integers has a least element.*
2. *The set of positive integers does not contain any infinite strictly decreasing sequences.*

Theorem 3.4. *There are no positive integers strictly between 0 and 1.*

Proof. Let S be the set of integers x such that $0 < x < 1$ Suppose S is nonempty; let n be its smallest element. Multiplying both sides of $n < 1$ by n gives $n^2 < n$ The square of a positive integer is a positive integer, so n^2 is an integer such that $0 < n < n^2 < 1$. This is a contradiction of the minimality of n. Hence S is empty.

□