Approaching: Sometimes we can't work something out directly ... but we **can** see what it should be as we get closer and closer!

Example: $(x^2 - 1)/(x - 1)$

Let's work it out for x=1:

$$(1^2 - 1)/(1 - 1) = (1 - 1)/(1 - 1) = 0/0$$

Now 0/0 is a difficulty! We don't really know the value of 0/0 (it is "indeterminate"), so we need another way of answering this.

So instead of trying to work it out for x=1 let's try **approaching** it closer and closer:

x		$(x^2 - 1)/(x - 1)$
0.5		1.50000
0.9		1.90000
0.99		1.99000
0.999		1.99900
0.9999		1.99990
0.99999		1.99999
...		...

Now we see that as x gets close to 1, then $(x^2-1)/(x-1)$ gets **close to 2**

We are now faced with an interesting situation:

- When x=1 we don't know the answer (it is **indeterminate**)
- But we can see that it is **going to be 2**

We want to give the answer "2" but can't, so instead mathematicians say exactly what is going on by using the special word "limit"

The **limit** of $(x^2-1)/(x-1)$ as x approaches 1 is **2**

DEFINITION Limit

Assume f is defined in a neighborhood of c and let c and L be real numbers. The function **f has limit L as x approaches c** if, given any positive number ε, there is a positive number δ such that for all x,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

We write

$$\lim_{x \to c} f(x) = L.$$

Computing Limits

For real numbers b and c, and positive integers n:

Basic Limits

1. $\lim_{x \to c} b = b$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$

Properties of Limits For functions f and g such that $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist,

1. $\lim_{x \to c} [bf(x)] = b \lim_{x \to c} f(x)$
2. $\lim_{x \to c} [f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$
3. $\lim_{x \to c} [f(x)g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$
4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$
5. $\lim_{x \to c} [f(x)]^n = \left[\lim_{x \to c} f(x)\right]^n$
6. $\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right)$

A. <u>"Plug-Ins"</u>

Using these basic limits and properties of limits, we can prove that the limit at c of the following kinds of functions can be evaluated by <u>direct substitution of c for x:</u>

Polynomial Function

Rational Function with c in its domain

Radical Function with c in its domain

Trigonometric Function with c in its domain

Examples:

$\lim_{x\to -3}(3x+2)=$ $\lim_{x\to 1}(3x^3-2x^2+4)=$ $\lim_{x\to -4}\frac{2}{x+2}=$

$\lim_{x\to 8}\sqrt{x+1}=$ $\lim_{x\to \pi/4}\tan x=$

B. <u>Rational Functions without c in their domain ("Single Holes")</u>

<u>Example 1</u> $\lim_{x\to 1}\frac{x^3-1}{x-1}=$ Factor.

Simplify.

Substitute c for x.

<u>Example 2</u> $\lim_{x\to 0}\frac{x^2-3x}{x}=$

C. <u>Functions with Radicals without c in their domain</u>

<u>Example 1</u> $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=$

Rationalize the numerator

using conjugates.

Simplify.

Substitute c for x.

Example 2 $\lim_{x\to 3}\frac{\sqrt{x+1}}{x-4}=$

Example 3

$$\lim_{x\to 1}.\frac{x^3-1}{x-1}=\lim_{x\to 1}.\frac{(x-1)(x^2+x+1)}{(x-1)}=\lim_{x\to 1}.(x^2+x+1)=3$$

Example 4

$$\lim_{x\to 0}.\frac{\sqrt{x+9}-3}{x}=\lim_{x\to 0}.\frac{\sqrt{x+9}-3}{x}.\frac{\sqrt{x+9}+3}{\sqrt{x+9}+3}=\lim_{x\to 0}.\frac{x+9-9}{x(\sqrt{x+9}+3)}=\frac{1}{6},$$

Squeeze Theorem

If $h(x)\le f(x)\le g(x)\forall x$ in an open interval containing c, except possibly at c itself, and if

$\lim_{x\to c}h(x)=L$ and $\lim_{x\to c}g(x)=L$, then $\lim_{x\to c}f(x)$ exists and = L. (see text p. 65)

D. Special Trigonometric Limits

(1) $\lim_{x\to 0}\frac{\sin x}{x}=1$ and (2) $\lim_{x\to 0}\frac{1-\cos x}{x}=0$

Examples of Use:

$\lim_{\theta\to 0}\frac{\cos\theta\tan\theta}{\theta}=$ $\lim_{x\to 0}\frac{\tan^2 x}{x}=$

Homework:

1- $\lim_{x\to 2}. \frac{x^4-2x^2-8}{x^2-4}$ 2 - $\lim_{x\to 0}. \frac{(1+x)^{\frac{3}{2}}-1}{x}$ 3- $\lim_{x\to 3}. \frac{\sqrt{3x}-3}{x-3}$

4- $\lim_{x\to a}. \frac{\sqrt{x^2+1}-\sqrt{a^2+1}}{x-a}$ 5- $\lim_{x\to 0}. \frac{1}{x}(\frac{1}{x+2}-\frac{1}{2})$ 6- $\lim_{x\to 0}. \frac{\sin 3x}{x}$

7- $\lim_{x\to 2}. \frac{(x^2-4)}{\sqrt{x-2}}$ 8- $\lim_{x\to 0}. \frac{\tan x}{x}$ 9- $\lim_{h\to 0}. \frac{\sqrt{x+h}-\sqrt{x}}{h}$

One -sided limits (right-hand limits and left-hand limits)

Definition: we say the limits of the function f(x) as x approaches a from the right equals L if given any $\in > 0$, $\exists\ a, \delta > 0$ such that for all x,

$a < x < a+\delta$ Implies $|f(x)-L| < \in$ denoted by $\lim_{x\to a^+} f(x) = L$

Definition: we say the limits of the function f(x) as x approaches a from the left equals L if given any $\in > 0$, $\exists\ a, \delta > 0$ such that for all x,

$a-\delta < x < a$ Implies $|f(x)-L| < \in$ denoted by $\lim_{x\to a^-} f(x) = L$

Note:

A function f(x) has a limit at a point a iff the right –hand and left- hand limits at a exist and are equal

$\lim_{x\to a} f(x) = L \leftrightarrow \lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x)$

Example: find the limit of the function

$$f(x) = \begin{Bmatrix} x+1, & x \geq 1 \\ x^2-1, & x < 1 \end{Bmatrix}$$

as 1. $x \to 5$ 2. $x \to -5$ 3. $x \to 1$

sol: 1. $\lim_{x\to 5} f(x) = \lim_{x\to 5} (x+1) = 6$

2. $\lim_{x\to -5} f(x) = \lim_{x\to -5} (x^2-1) = 24$

3. a/ $\lim_{x\to 1+} f(x) = \lim_{x\to 1+} (x+1) = 2$

b/ $\lim_{x\to 1-} f(x) = \lim_{x\to 1-} (x^2-1) = 0$

$\therefore \lim_{x\to 1+} f(x) \neq \lim_{x\to 1+} f(x)$

<u>Homework:</u> Find the limit of the following functions:

$$1 - f(x) = \begin{Bmatrix} x^2, & x \geq 0 \\ x^3, & -1 < x < 0 \\ x+1, & x \leq -1 \end{Bmatrix}$$

If 1. $x \to 3$ 2. $x \to -\frac{1}{2}$ 3. $x \to -5$ 4. $x \to 0$ 5. $x \to -1$

$$2 - f(x) = \begin{Bmatrix} x^2-1, & x \geq 1 \\ x+1, & 0 \leq x < 1 \\ x^3-1, & x < 0 \end{Bmatrix}$$

If 1. $x \to 1$ 2. $x \to \frac{1}{2}$ 3. $x \to -6$ 4. $x \to 0$ 5. $x \to 8$

3- $f(x) = \lim_{x\to 0}. \frac{|x|}{x}$

Limits at infinity (limits as $x \to \infty$ or $x \to -\infty$)

<u>Def:</u> We say the limits of the function f(x) as x approaches infinity is a real number L ($\lim_{x\to\infty} f(x) = L$) if given any $\in > 0$, $\exists\ M > 0$ such that for all x,

$M < x$ Implies $|f(x) - L| < \in$

<u>Def:</u> We say the limits of the function f(x) as x approaches negative infinity is a real number L ($\lim_{x\to-\infty} f(x) = L$) if given any $\in > 0$, $\exists\ N < 0$ such that for all x,

$x < N$ Implies $|f(x) - L| < \in$

<u>Theorem:</u> if f(x) =k for any no. k

1- $\lim_{x\to\infty} k = \lim_{x\to-\infty} k = k$ such that k is constant

2- if $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} g(x) = M$,then :

i) $\lim_{x\to\infty}\big(f(x)\mp g(x)\big)=\lim_{x\to\infty}f(x)\mp\lim_{x\to\infty}g(x)=L\mp M$

ii)

iii) $\lim_{x\to\infty}\big(f(x)\,.\,g(x)\big)=\lim_{x\to\infty}f(x)\,.\,\lim_{x\to\infty}g(x)=L\,.\,M$

iv) $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\frac{\lim_{x\to\infty}f(x)}{\lim_{x\to\infty}g(x)}=\frac{L}{M}$,M$\neq 0$

v) $\lim_{x\to\infty}\frac{\sin x}{x}=0$

vi) $\lim_{x\to\infty}\frac{\cos x}{x}=0$

vii) $\lim_{x\to\infty}\frac{1}{x}=0$

Ex: Find the limits of the following functions:

1. $\lim_{x\to\infty}\frac{x}{7x+4}=\lim_{x\to\infty}\frac{\frac{x}{x}}{\frac{7x}{x}+\frac{4}{x}}=\lim_{x\to\infty}\frac{1}{7+\frac{4}{x}}=\frac{1}{7}$

2. $\lim_{x\to\infty}\frac{2x^2+2x-3}{x^2+4x+4}=\lim_{x\to\infty}\frac{2+\frac{2}{x}-\frac{3}{x^2}}{1+\frac{4}{x}+\frac{4}{x^2}}=\frac{\lim_{x\to\infty}(2+\frac{2}{x}-\frac{3}{x^2})}{\lim_{x\to\infty}(1+\frac{4}{x}+\frac{4}{x^2})}=2$

H.W: Find the following limits:

1- $\lim_{x\to\infty}\sqrt{\frac{3x^4-x^2}{x^4-1}}$

2- $\lim_{x\to\infty}(2+\frac{\sin x}{x})$

3- $\lim_{x\to\infty}\frac{3x^2-5x^2+x}{x^4}$

Infinite limits

Def: We say the limits of the function f(x) as x approaches a $(a\in D_f)$ is infinity $(\lim_{x\to a} f(x)=\infty)$ if given any $M>0$, $\exists\ \delta>0$ such that for all x, such that $\forall\, x\in D_f$, $0<|x-a|<\delta$ implies f(x)>M

Def: We say the limits of the function f(x) as x approaches a $(a\in D_f)$ is -infinity $(\lim_{x\to a} f(x)=-\infty)$ if given any $M<0$, $\exists\ \delta>0$ such that for all x, s.t $\forall\, x\in D_f$, $0<|x-a|<\delta$ implies f(x)<M

Ex: Find the limits as follows:

1- $\lim_{x\to 2^+}\cdot \frac{1}{x^2-4}=\frac{1}{0}=\infty$

2. $\lim_{x\to\infty}\cdot \frac{2x^3+2x-1}{x^2-5x+2}=\lim_{x\to\infty}\cdot \frac{2+\frac{2}{x^2}-\frac{1}{x^3}}{\frac{1}{x}-\frac{5}{x^2}+\frac{2}{x^3}}=\frac{2}{0}=\infty$

3- $\lim_{x\to 0^+}\cdot \frac{1}{x}=\infty$

3- $\lim_{x\to 0^-}\cdot \frac{1}{x}=-\infty$

H.W: Find the following limits:

1- $\lim_{x\to\infty}\cdot \frac{x^2-x}{x+1}$ 2- $\lim_{x\to\infty}\cdot \frac{x^5}{3x^4+x^3-x+1}$

continuity

Def: We say the function y=f(x) is continuous at x=c iff all the three following conditions are meet:

1- f (c) is exists (c *is the domain of* f)
2- $\lim_{x\to c} f(x)$ is exists (f has a limit as x→ c)
3- $\lim_{x\to c} f(x)=f(c)$ (the limit equals the fun. value)

Ex: Let $f(x)=\begin{Bmatrix}\frac{\sin x}{x}, x\neq 0\\ 1, \; x=0\end{Bmatrix}$, is f cont. at x=0?

Sol:

1-f(0)=1 is exist

2- $\lim_{x\to 0} f(x)=\lim_{x\to a}\frac{\sin x}{x}=1 \; is\ exist\ as\ x\to 0$

$\therefore \lim_{x\to 0} f(x)=f(0)=1$

$\therefore f$ is cont. at 0

Discontinuity at a point

Def: If a function f is not continuous at a point c, we say that f is discontinuous at c, and c is called a point of discontinuity of f.

Ex: $f(x)=\begin{cases}\frac{1}{x+1}, & x\neq -1\\ 1, & x=-1\end{cases}$ is f cont. at x=-1

Sol:

1- f(-1)=1 is exist

2- $\lim_{x\to -1} f(x)=\lim_{x\to -1}\frac{1}{x+1}$ $not\ exist\ as\ x\to -1$

$\therefore f$ is not cont. at x=-1

Homework:

1- $f(x)=\begin{cases}\frac{X^2+X-6}{x}, & x\neq 2\\ \frac{5}{4}, & x=2\end{cases}$, is f cont. at x=2?

2- $f(x)=\begin{cases}x^2-1 & ,x\geq 0\\ x & ,\ x<0\end{cases}$, is f cont. at x=0?

3-1- $f(x)=\begin{cases}\frac{X^2-9}{x+3}, & x\neq -3\\ -6, & =-3\end{cases}$, is f cont. at x=-3?

4- $f(x)=\begin{cases}\frac{X^2-4}{x-2}, & x\neq 2\\ 4, & x=2\end{cases}$, is f cont. at x=2?

5- $f(x)=\begin{cases}|x+1|+2 & ,\ x\neq -1\\ 0, & x=-1\end{cases}$, is f cont. at x=-1?