Chapter Three: Second order Ordinary Differential Equations

The general form of second order ODEs is $y'' = f(t, y, y')$ and the general solution of this equation contains two constants:

i.e $y = y(x, c_1, c_2) \ldots\ldots (1)$

So, in order to find values for c_1, c_2, we need to impose two initial conditions:

$$y(t_0) = y_0, y'(t_0) = y_1 \ldots\ldots (2)$$

Where $t_0 \in$ Domain of y.

Definition: - the second order ODE (1), with the initial conditions (2) is called initial value problem (I.V.P.)

Chapter Three

Solutions of Second order Ordinary Differential Equations.

This chapter is divided into three parts

1- Reducing the order
- Type 1
- Type 2

2- Homogeneous linear equations with constant coefficients.

3- Nonhomogeneous linear equations with constant coefficients.

1- Reducing the order

For some types of second order ODEs, we can reduce the order from two to one by using a certain substitutions.

Which means, in order to find the general solutions for an equation of these types, we need to solve two O.D.E. of first order.

In this chapter, we will study two types of these equations.

Type One:

The general form for this type of equations: $y'' = f(t, y')$

Which means the D.E. depends only on t & y' and y does not appear in the equation

To solve this type of equation we use the following method

Solving method

Step 1: set $p = y' = \frac{dy}{dt} \Rightarrow p' = y'' = \frac{d^2y}{dt^2}$

And substitute p & p' in the general equation $p' = f(t, p)$ → O.D.E. of first order

Step 2: solve the last equation to get

$$p(t) = H(t) + c_1 \Rightarrow y'(t) = H(t) + c_1 \rightarrow \text{O.D.E. of first order}$$

Step 3: solve the last equation to get the general solution of the O.D.E.

$$y(t) = \int (H(t) + c_1)\, dt + c_2$$

EXAMPLE: find the general solution of the following equation

$$t\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$$

Solution: since in this equation y dose not appear, it is of type 1.

Step 1: set $p = y' = \frac{dy}{dt} \Rightarrow p' = y''$

So the equation becomes $tp' + p = 0$

Step 2: to solve the last equation we can use separation of variables method

$$t\frac{dp}{dt} = -p \Rightarrow \int \frac{dp}{p} = -\int \frac{dt}{t}$$

Thus $\ln(p) = -\ln(t) + c_1$

i.e. $e^{\ln p} = e^{\ln(\frac{1}{t})+c_1}$

$\therefore p = \frac{1}{t} \cdot k_1$, where $k_1 = e^{c_1}$

Step 3: $y' = \frac{k_1}{t} \Rightarrow \frac{dy}{dt} = \frac{k_1}{t}$

Again, we use separation of variables method to get the general solution

$$\int dy = k_1 \int \frac{dt}{t} \Rightarrow y = k_1 \ln(t) + k_2$$

To verify the solution

$$t\frac{d^2y}{dt^2} + \frac{dy}{dt} = t\left[\frac{-k_1}{t^2}\right] + \left[\frac{k_1}{t}\right] = 0$$

Correct (☺👍)

<u>EXAMPLE</u>: find the solution of I.V.P

$$y'' + y' = t, \quad y(0) = 0, \quad y'(0) = 1$$

<u>Solution</u>: it is clear that, this equation is of type 1?

Step 1: set $p = y' \Rightarrow p' = y''$

So, the equation can be rewritten as follows:

$p' = -p + t \rightarrow$ Linear equation with respect to p.

Step 2: use integration factor for to solve the last equation

Set $\mu = e^{-\int a(t)dt} = e^{-\int(-1)dt} = e^{t}$

Multiply the linear equation by μ, we get

$$e^{t}p' + e^{t}p = te^{t} \Rightarrow \frac{d}{dt}(e^{t}p) = te^{t}$$

$$\therefore \int d(e^{t}p) = \int te^{t}\,dt$$

It follows: $[e^{t}p = te^{t} - e^{t} + c_1]e^{-t}$

Thus $p = t - 1 + c_1e^{-t} \Rightarrow y' = t - 1 + c_1e^{-t}$

Step 3: use separation of variables method to solve the last equation

$$\int dy = \int [t - 1 + c_1e^{-t}]\,dt$$

$$\therefore y = \frac{t^2}{2} - t - c_1e^{-t} + c_2$$

Next, we aim to find c_1, c_2 based on the initial conditions

$$y(0) = \frac{0^2}{2} - 0 - c_1e^{-0} + c_2 = 0$$

$$\therefore c_1 = c_2$$

$$y'(0) = p(0) = 0 - 1 + c_1e^{-0} = 1$$

$$\therefore c_1 = 2 \Rightarrow c_2 = 2.$$

Thus the solution of the I.V.P is

$$y = \frac{t^2}{2} - t - 2e^{-t} + 2.$$

To verify the solution

$$y'' + y' = (1 - 2e^{-t}) + (t - 1 - 2e^{-t}) = t$$

Correct (☺👍)

Type Two:

The general form for this type of equations:

$$y'' = f(y, y')$$

Which means the D.E. depends only on y & y' and t does not appear in the D.E.

To solve this type of equation we use the following method

Solving method

Step 1: set $p = y' \Rightarrow \frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dp}{dt} = \frac{dp}{dy} \cdot \frac{dy}{dt} = p\frac{dp}{dy}$

$$\therefore y'' = p\frac{dp}{dy}$$

Thus the general equation becomes $p\frac{dp}{dy} = f(y, p) \rightarrow$ O.D.E. of first order

Step 2: solve the last equation to get

$$p = H(y) + c_1 \Rightarrow y' = H(y) + c_1 \rightarrow \text{O.D.E. of first order}$$

Step 3: solve the last equation, using separation of variables, to get

$$\int \frac{dy}{H(y) + c_1} = \int dt$$

$$G(y, c_1) = t + c_2$$

Which is the general solution of the D.E.

EXAMPLE: find the general solution of the following differential equation

$$y'' + y = 0$$

Solution: it is clear that, this equation is of type 2?

Step 1: set $p = y', \quad y'' = p\frac{dp}{dy}$

The D.E. becomes

$$p\frac{dp}{dy} + y = 0 \Rightarrow p\frac{dp}{dy} = -y$$

Step 2:

$$\therefore \int p\,dp = \int -y\,dy$$

$$2\left[\frac{p^2}{2} = -\frac{y^2}{2} + c\right] \Rightarrow p^2 = -y^2 + a^2 \quad , a^2 = 2c$$

So, $p = \sqrt{a^2 - y^2}$

Step 3: $y' = \sqrt{a^2 - y^2}$

$$\therefore \int \frac{dy}{\sqrt{a^2 - y^2}} = \int dt$$

$$\sin^{-1}\left(\frac{y}{a}\right) = t + b$$

Thus $\frac{y}{a} = \sin(t+b) \Rightarrow y = a\sin(t+b)$

The general solution of the D.E.

To verify the solution

$$y' = a\cos(t+b)\,, y'' = -a\sin(t+b)$$

$$y'' + y = a\sin(t+b) - a\sin(t+b) = 0$$

Correct (☺👍)

2- Linear second order DEs with constant coefficients.

This type of equations take the general form:

$$ay'' + by' + cy = f(t)$$

If $f(0) \equiv 0$, then the D.E. is called Homogeneous otherwise it's called nonhomogeneous.

Solution of Homogeneous equation

Theorem: let y_1, y_2 are two solutions of the Homogeneous equation

$$ay'' + by' + cy = 0$$

Such that $\{y_1, y_2\}$ are linearly independent

$$(if \quad c_1 y_1 + c_2 y_2 = 0, \ then \quad c_1 = c_2 = 0\,)$$

Then the general solution of the D.E. takes the form:

$$y_h = c_1 y_1 + c_2 y_2 \text{ , where } c_1, c_2 \text{ are constant.}$$

Note: $\{y_1, y_2\}$ is called the fundamental set of solutions to the Homogeneous equation.

In order to find $\{y_1, y_2\}$, we assume that

$$y = e^{\lambda t} \Rightarrow y' = \lambda e^{\lambda t} \Rightarrow y'' = \lambda^2 e^{\lambda t}$$

Substitute y and its derivatives in the homo. Equation, we get

$$a\,\lambda^2 e^{\lambda t} + b\,\lambda e^{\lambda t} + c\,e^{\lambda t} = 0$$

$$e^{\lambda t}[a\,\lambda^2 + b\,\lambda + c] = 0$$

Since $e^{\lambda t} \neq 0 \Rightarrow a\,\lambda^2 + b\,\lambda + c = 0$

Which means, to solve the D.E. we need to find the roots of the last polynomial of second order.

We have three cases:

1) $b^2 > 4ac$ in this case we have two different real roots i.e $\lambda_1 \neq \lambda_2$
2) $b^2 = 4ac$ in this case we have two equal real roots i.e $\lambda_1 = \lambda_2 = \lambda$

3) $b^2 < 4ac$ in this case we have two different complex roots such that λ_2 is the conjugate of λ_1 i.e $\lambda_1 = a + ib,\ \lambda_2 = \overline{\lambda_1} = a - ib$

Where $\lambda = \frac{-b \mp \sqrt{b^2-4ac}}{2a}$

For case one, the general solution of the Homogeneous D.E. take the form:

$$y_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

For case two, the general solution takes the form:

$$y_h = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$$

For case three,

$$y_h = c_1 e^{(a+ib)t} + c_2 e^{(a-ib)t}$$

$$= e^{at}\left[c_1 e^{ibt} + c_2 e^{-ibt}\right]$$

By using Euler's formula

$$y_h = e^{at}[c_1[\cos(bt) + i\sin(bt)] + c_2[\cos(bt) - i\sin(bt)]]$$

$$= e^{at}\,[\mathrm{A}\cos(bt) + B\sin(bt)]$$

Where $A = (c_1 + c_2), B = i(c_1 - c_2)$

<u>Solving method</u>

Step 1: let $y = e^{\lambda t} \Rightarrow y' = \lambda e^{\lambda t}, y'' = \lambda^2 e^{\lambda t}$

Step 2: solve the polynomial $a\,\lambda^2 + b\,\lambda + c = 0$

If $\lambda_1 \neq \lambda_2$, then $y_\lambda = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

If $\lambda_1 = \lambda_2 = \lambda$, then $y_\lambda = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$

If $\lambda_1 = a + ib,\ \lambda_2 = \overline{\lambda_1} = a - ib$, then $y_h = e^{at}[c_1\cos(bt) + c_2\sin(bt)]$

<u>EXAMPLE</u>: find the general solution of the following differential equation

$$y'' - 10y' + 21y = 0$$

<u>Solution</u>: set $y = e^{\lambda t} \Rightarrow y' = \lambda e^{\lambda t}, y'' = \lambda^2 e^{\lambda t}$

Plug y, y' and y'' in the D.E., we get

$$e^{\lambda t}[\,\lambda^2 - 10\,\lambda + 21] = 0$$

Since $e^{\lambda t} \neq 0 \Rightarrow\; \lambda^2 - 10\,\lambda + 21 = 0$

Thus $(\lambda - 3)(\lambda - 7) = 0 \Rightarrow \lambda_1 = 3,\; \lambda_2 = 7$

So, we have case one

∴ The general solution takes the form

$$y_h = c_1 e^{3t} + c_2 e^{7t}$$

Note: it's clear that each of $\{e^{3t}, e^{7t}\}$ satisfies the D.E.

EXAMPLE: solve the following I.V.P.

$$y'' - 9y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

Solution: let $y = e^{\lambda t} \Rightarrow y' = \lambda e^{\lambda t}, y'' = \lambda^2$

So, the D.E. becomes

$$e^{\lambda t}[\,\lambda^2 - 9] = 0$$

Since $e^{\lambda t} \neq 0 \Rightarrow\; \lambda^2 - 9 = 0 \Rightarrow \lambda = \mp 3$ (case 1)

Thus the general solution is

$$y_h = c_1 e^{-3t} + c_2 e^{3t} \Rightarrow y_\lambda' = -3c_1 e^{-3t} + 3c_2 e^{3t}$$

Now, to find the solution of the I.V.P.

$$y_h(0) = c_1 + c_2 = 1$$

$$y_h'(0) = -3c_1 + 3c_2 = 0 \Rightarrow 3c_1 = 3c_2 \Rightarrow c_1 = c_2$$

$$\therefore\; 2c_1 = 1 \Rightarrow c_1 = \frac{1}{2} = c_2$$

$$\therefore\; y_h = \frac{1}{2}e^{-3t} + \frac{1}{2}e^{3t}$$

EXAMPLE: solve the D.E. $y'' - 10y' + 25y = 0$

Solution: let $y = e^{\lambda t} \Rightarrow y' = \lambda e^{\lambda t}, y'' = \lambda^2 e^{\lambda t}$

So, the D.E. becomes

$$e^{\lambda t}[\lambda^2 - 10\lambda + 25] = 0$$

Since $e^{\lambda t} \neq 0 \Rightarrow \lambda^2 - 10\lambda + 25 = 0 \Rightarrow (\lambda - 5)(\lambda - 5) = 0 \Rightarrow \lambda_1 = \lambda_2 = 5$

So, we have case two

Thus the general solution is $y_h = c_1 e^{5t} + c_2 t e^{5t}$

EXAMPLE: find the solution of the I.V.P.

$$y'' + 16y = 0, \qquad y(0) = 1, \qquad y'(0) = 2$$

Solution: let $y = e^{\lambda t} \Rightarrow y' = \lambda e^{\lambda t}, y'' = \lambda^2 e^{\lambda t}$

Plug y, y' and y'' in the D.E., we get $e^{\lambda t}[\lambda^2 + 16] = 0$

Since $e^{\lambda t} \neq 0 \Rightarrow \lambda^2 - (i4)^2 = 0 \Rightarrow \lambda_1 = 4i,\ \lambda_2 = -4i$ (case 3)

$$\therefore y_h = e^{0t}[c_1 \cos(4t) + c_2 \sin(4t)]$$

So, $y_h' = -4c_1 \sin(4t) + 4c_2 \cos(4t)$

$$y_h(0) = c_1 \cos(0) = c_1 = 1$$

$$y_h'(0) = 4c_2 = 2 \Rightarrow c_2 = \frac{1}{2}$$

$$\therefore y_h = \cos(4t) + \frac{1}{2}\sin(4t)$$

To verify

$$y'' = -16(1)\cos(4t) - 16\left(\frac{1}{2}\right)\sin(4t)$$

$$16\,y = 16(1)\cos(4t) + 16\left(\frac{1}{2}\right)\sin(4t)$$

$$\therefore y'' + 16y = 0$$

$$y(0) = \cos(0) = 1$$

$$y'(0) = \frac{4}{2}\cos(0) = 2$$

Correct (☺👍)

3- Solution of Non-homogeneous equation.

In order to find the general solution of the nonhomogeneous equation:

$$ay'' + by' + cy = f(t), \qquad f(t) \not\equiv 0$$

We need to find first the homogeneous solution y_λ, then the particular solution y_p

So, $y = y_h + y_p$

In fact, the form of y_p depends on the form of $f(t)$ and we will study three forms

Form 1:- If f is a polynomial

i.e. $f(t) = a_0 t^2 + b_0 t + c_0$

Then, we assume that $y(t) = at^2 + bt + c \Rightarrow y'(t) = 2at + b, \;\; y'' = 2a$

If we plug y, y' and y'' in the D.E., we get 3×3 system of linear equation, by solving this system, we can get the values of a, b and c.

EXAMPLE: find the solution of the following I.V.P.

$$y'' - 10y' + 21y = t \, , y(0) = 0 \, , y'(0) = 0$$

Solution:

Step 1: find y_h

$y_h = c_1 e^{3t} + c_2 e^{7t}$ [H.W.]

Step 2: since $f(t) = t$ is a polynomial, we assume that

$$y_p = at^2 + bt + c \Rightarrow y_p' = 2at + b, \;\; y_p'' = 2a$$

Substitute y_p, y_p' and y_p'' in the D.E., we get

$$2a - 10(2at + b) + 21(at^2 + bt + c) = t$$

$$2a - 20at - 10b + 21at^2 + 21bt + 21c = t$$

Thus, we can get the following linear system in terms of a, b, c:

$$\begin{aligned} 2a - 10b + 21c &= 0 \\ -20a + 21b &= 1 \\ 21a &= 0 \end{aligned}$$

By eq. (3) $a = 0$,

By eq. (2) $b = \frac{1}{21}$,

By eq. (1), $21c = \frac{10}{21} \Rightarrow c_1 = \frac{10}{(21)^2}$

Thus $y_p = \frac{1}{21}t + \frac{10}{(21)^2}$

Step 3: the general solution is: $y = y_h + y_p$

$$y = c_1e^{3t} + c_2e^{7t} + \frac{1}{21}t + \frac{10}{(21)^2} \ldots\ldots (i)$$

Now, to find the solution of the I.V.P., we need to use the I.C.S. to find c_1, c_2.

$$y' = 3c_1e^{3t} + 7c_2e^{7t} + \frac{1}{21}$$

$$\therefore y'(0) = 3c_1 + 7c_2 + \frac{1}{21} = 0 \quad \ldots\ldots (1)$$

$$y(0) = c_1 + c_2 + \frac{10}{(21)^2} = 0 \quad \ldots\ldots (2)$$

$$y'(0) - 3y(0) = 4c_2 + \frac{1}{21} - \frac{30}{(21)^2} = 0$$

$$\therefore c_2 = \frac{9}{4(21)^2}$$

By eq. (2) we have

$$c_1 = \frac{-10}{(21)^2} - c_2 = \frac{-10}{(21)^2} - \frac{9}{4(21)^2} = \frac{-40-9}{4(21)^2}$$

$$c_1 = \frac{-49}{4(21)^2}$$

Thus the solution of the I.V.P. is

$$y = \left(\frac{-49}{4(21)^2}\right)e^{3t} + \left(\frac{9}{4(21)^2}\right)e^{7t} + \frac{t}{21} + \frac{10}{(21)^2} \ldots\ldots (ii)$$

To verify:-

$$y'' = 9c_1e^{3t} + 49c_2e^{7t}$$
$$-10y' = -30c_1e^{3t} - 70c_2e^{7t} - \frac{10}{21}$$
$$21y = 21c_1e^{3t} + 21c_2e^{7t} + t + \frac{10}{21}$$

$+$ ______________________________

So, $y'' - 10y' + 21y = t \Rightarrow y$ in (i) satisfies the D.E.

To verify that y in (ii) satisfy the BCs.

$$y(0) = \left(\frac{-49}{4(21)^2}\right) + \left(\frac{9}{4(21)^2}\right) + \frac{10}{(21)^2} = \frac{-49 + 9 + 40}{4(21)^2} = 0$$

$$y'(0) = 3\left(\frac{-49}{4(21)^2}\right) + 7\left(\frac{9}{4(21)^2}\right) + \frac{1}{21} = \frac{-147 + 63 + 84}{4(21)^2} = \frac{0}{4(21)^2} = 0$$

Correct (☺👍)

Form 2:- If $f(t)$ is a trigonometric function, namely:

$$f(t) = a\cos(\alpha t) + b\sin(\alpha t)$$

Then $y_p = A\cos(\alpha t) + B\sin(\alpha t)$

$y_p' = -A\alpha\sin(\alpha t) + B\alpha\cos(\alpha t)$

$y_p'' = -A\alpha^2\cos(\alpha t) - B\alpha^2\sin(\alpha t)$

Plug y, y' and y'' in the nonhomogeneous equation, we get a 2×2 linear system, by solving this system, we can get the values of A and B.

EXAMPLE: find the general solution of the D.E. $y'' - 9y = \cos(2t)$

Solution:

Step 1: $y_h = c_1e^{3t} + c_2e^{-3t}$ [H.W.]

Step 2: $y_p = A\cos(2t) + B\sin(2t)$

$\Rightarrow y_p' = -2A\sin(2t) + 2B\cos(2t)$

$y_p'' = -4A\cos(2t) - 4B\sin(2t)$

So, the D.E. becomes

$$-4A\cos(2t) - 4B\sin(2t) - 9A\cos(2t) - 9B\sin(2t) = \cos(2t)$$

Thus, $-4A - 9A = 1, -4B - 9B = 0 \Rightarrow -13B = 0 \Rightarrow B = 0$

And $-13A = 1 \Rightarrow A = {}^{-1}/_{13}$

$$\therefore\ y_p = \frac{-1}{13}\cos(2t)$$

Step 3: the general solution of the D.E. becomes

$y = y_h + y_p$

$\quad = c_1 e^{3t} + c_2 e^{-3t} - \frac{1}{13}\cos(2t)$

To verify

$$y' = -3c_1 e^{-3t} + 3c_2 e^{3t} + \frac{2}{13}\sin(2t)$$

$$y'' = 9c_1 e^{-3t} + 9c_2 e^{3t} + \frac{4}{13}\cos(2t)$$

$$-9y = -9c_1 e^{-3t} - 9c_2 e^{3t} + \frac{9}{13}\cos(2t)$$

$+$ ____________________________________

So, $y'' - 9y = \left(\frac{4}{13} + \frac{9}{13}\right)\cos(2t) = \cos(2t)$

Correct (☺👍)

Form 3:- If $f(t)$ is an exponential function

i.e. $f(t) = \alpha e^{at}$, where α, a are constants.

For this form, we have three cases.

1. $y_p = Ae^{at}$ if $a \neq \lambda_1, a \neq \lambda_2, \lambda_1 \neq \lambda_2$
2. $y_p = Ate^{at}$ if $a \neq \lambda_1, a = \lambda_2, \lambda_1 \neq \lambda_2$ or $a = \lambda_1, a \neq \lambda_2, \lambda_1 \neq \lambda_2$
3. $y_p = At^2e^{at}$ if $a = \lambda_1 = \lambda_2$

EXAMPLE: find the general solution of the I.V.P.

$$y'' - 7y' + 6y = e^t \,, y(0) = 0 \,, y'(0) = 0$$

Solution: step one it's easy to show that $\lambda_1 = 1,\ \lambda_2 = 6$

Step 1: $y_h = c_1e^t + c_2e^{6t}$

Step 2: since $\lambda_1 \neq \lambda_2 \neq a$, we assume that (case one)

$$y_p = Ae^{2t} \Rightarrow y_p' = 2Ae^{2t},\ \ y_p'' = 2Ae^{2t}$$

Plug y_p, y_p' and y_p'' in the D.E., we get

$$4Ae^{2t} - 14Ae^{2t} + 6Ae^{2t} = e^{2t}$$

$$\therefore 4A - 14A + 6 = 1 \Rightarrow -4A = 1 \Rightarrow A = {}^{-1}\!/_{4}$$

Step 3: thus, the general solution is: $y = y_h + y_p$

$$y = c_1e^t + c_2e^{6t} - \frac{1}{4}e^{6t}$$

Step 4: $y(0) = c_1 + c_2 - \frac{1}{4} = 0 \Rightarrow c_1 + c_2 = \frac{1}{4} \quad \ldots\ldots(1)$

$$y'(t) = c_1e^t + 6c_2e^{6t} - \frac{1}{2}e^{2t}$$

$$y'(0) = c_1 + 6c_2 - \frac{1}{2} = 0 \Rightarrow c_1 + 6c_2 = \frac{1}{2} \qquad \ldots\ldots(2)$$

By (1) & (2), we have

$$L_2 - L_1 \Rightarrow 5c_2 = \frac{1}{4} \Rightarrow c_2 = \frac{1}{20}$$

Plug c_2 in L_1, we get

$$c_1 = \frac{1}{4} - c_2 = \frac{1}{4} - \frac{1}{20} = \frac{4}{20}$$

So, the solution of the I.V.P. is

$$y = \left(\frac{4}{20}\right)e^t + \left(\frac{1}{20}\right)e^{6t} - \frac{1}{4}e^{2t}$$

To verify

$$y'' = c_1e^t + 36c_2e^{6t} - e^{2t}$$

$$-7y' = -7c_1e^t - 42c_2e^{6t} + \frac{7}{2}e^{2t}$$

$$6y = 6c_1e^t + 6c_2e^{6t} - \frac{3}{2}e^{2t}$$

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So, $y'' - 7y' + 6y = \left(-1 + \frac{7}{2} - \frac{3}{2}\right)e^{2t} = (-1+2)e^{2t} = e^{2t}$

Correct (☺👍)

To verify the solution of I.V.P.:

$$y(0) = \left(\frac{4}{20}\right) + \left(\frac{1}{20}\right) - \frac{1}{4} = \frac{5-5}{20} = 0$$

$$y'(0) = \left(\frac{4}{20}\right) + \left(\frac{6}{20}\right) - \frac{1}{2} = \frac{10-10}{20} = 0$$

<u>EXAMPLE</u>: find the general solution of the I.V.P.

$$y'' - 9y = 2e^{3t}$$

<u>Solution</u>: it's easy to show that $\lambda_1 = -3,\ \lambda_2 = 3 = a$

Thus $y_h = c_1e^{-3t} + c_2e^{3t}$

While, we assume that $y_p = Ate^{3t}$

$$\Rightarrow y_p' = A(3te^{3t} + e^{3t})$$

$$y_p'' = A(9te^{3t} + 3e^{3t} + 3e^{3t}) = A(9te^{3t} + 6e^{3t})$$

Plug y_p, y_p' and y_p'' in the D.E., we get

$$9Ate^{3t} + 6Ae^{3t} - 9Ate^{3t} = 2e^{3t}$$

$$\therefore 4A - 14A + 6 = 1 \Rightarrow -4A = 1 \Rightarrow A = {}^{-1}\!/_{4}$$

So, $6A = 2 \Rightarrow A = {}^{2}\!/_{6} = {}^{1}\!/_{3}$

$$\therefore y_p = \frac{t}{3}e^{3t}$$

The general solution is: $y = y_h + y_p$

Thus $y = c_1e^{-3t} + c_2e^{3t} + \frac{t}{3}e^{3t}$

To verify:

$$y' = -3c_1e^{-3t} + 3c_2e^{3t} + te^{3t} + \frac{e^{3t}}{3}$$

$$y'' = 9c_1e^{-3t} + 9c_2e^{3t} + 3te^{3t} + e^{3t} + e^{3t}$$

$$-9y = -9c_1e^{-3t} - 9c_2e^{3t} - 3te^{3t}$$

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$$\therefore y'' - 9y = 2e^{3t}$$

Correct (☺👍)

EXAMPLE: solve the following D.E.

$$y'' - 2y' + y = 3e^t$$

Solution: step one, it's easy to show that $\lambda_1 = \lambda_2 = a = 1$ (case 3)

And $y_h = c_1e^t + c_2te^t$ (check)

So, we assume that $y_p = At^2e^t$

$y_p' = A(t^2e^t + 2te^t)$

$y_p'' = A(t^2e^t + 2te^t + 2te^t + 2e^t)$

Plug y_p, y_p' and y_p'' in the D.E., we get

$$Ae^t[(t^2 + 4t + 2) - 2(t^2 + 2t) + t^2] = 3e^t$$

$$\therefore 2A = 3 \Rightarrow A = {}^3/_2$$

So, $y_p = \frac{3}{2}t^2e^t$

Thus, the general solution takes the form $y = y_h + y_p$

$$y = c_1e^t + c_2te^t + \frac{3}{2}t^2e^t$$

To verify:-

$$y' = c_1e^t + c_2(te^t + e^t) + \frac{3}{2}(t^2e^t + 2te^t)$$

$$y'' = c_1e^t + c_2(te^t + 2e^t) + \frac{3}{2}(t^2e^t + 4te^t + 2e^t)$$

$$-2y' = -2c_1e^t - 2c_2(te^t + e^t) - 3(t^2e^t + 2te^t)$$

$$y = c_1e^t + c_2te^t + \frac{3}{2}t^2e^t$$

$+$ ______________________________________

$$\therefore y'' - 2y' + y = 3e^t$$

Correct (☺👍)

Remark: one may think about the case, where $f(t)$ is combination of two or three functions of different types such as $f(t) = \cos(\alpha t) + e^{\beta t}$

Or $f(t) = at^2 + \sin(\alpha t) + be^{\beta t}$

To find the particular solution, in this case, we need to assume that

$$y_p = y_{p_1} + y_{p_2} + y_{p_3}$$

Where y_{p_1} takes a polynomial form

y_{p_2} is a trigonometric function

y_{p_3} is of exponential form.

EXAMPLE: $y'' - 5y' + 6y = e^t + 3\sin(4t) + (5t^2 + 1)$ is clear that

$y_h = c_1 e^{3t} + c_2 e^{2t}$ (check)

To find the particular solution, we assume that

$$y_p = Ae^t + \alpha\sin(4t) + B\cos(4t) + (at^2 + bt + c)$$

And then we continue the solution as previous examples.