1 Vector space

Definition 1.1. A vector space V over a field $\mathbb{K}$ is a set V with two operations called addition $+$ and multiplication $\cdot$ such that the following axioms are satisfied:

(1) (i) $u + v \in V$ for all $u, v \in V$. (Addition is closed)

(ii) $u + v = v + u$ for all $u, v \in V$. (Addition is commutative)

(iii) $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$. (Addition is associative)

(iv) There exists an element $0 \in V$, called the zero vector, such that $u + 0 = 0 + u = u$ for all $u \in V$.

(v) For all $u \in V$ there exists an element $-u \in V$, called the additive inverse of u, such that $u + (-u) = 0 = -u + u$.

(2) (i) $\alpha \cdot u \in V$ for all $u \in V$ and $\alpha \in \mathbb{K}$.

(ii) $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$ for all $u, v \in V$ and $\alpha \in \mathbb{K}$.

(iii) $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$. for all $u \in V$ and $\alpha, \beta \in \mathbb{K}$.

(iv) $(\alpha \beta) \cdot u = \alpha \cdot (\beta \cdot u)$ for all $u \in V$ and $\alpha, \beta \in \mathbb{K}$.

(v) For all $u \in V$ there exists an element $1 \in \mathbb{K}$, called the multiplicative identity of u, such that $1 \cdot u = u \cdot 1 = u$.

Example 1.2. Let $\mathbb{C}$ be the set of complex numbers. Define addition in $\mathbb{C}$ by

$$(a + bi) + (c + di) = (a + c) + (b + d)i \quad \text{for all } a, b, c, d \in \mathbb{R}; \tag{1}$$

and define scalar multiplication by

$$\alpha \cdot (a + bi) = \alpha a + \alpha bi \quad \text{for all scalars } \alpha \in \mathbb{R}; \text{ and for all } a, b \in \mathbb{R}. \tag{2}$$

Show that $(C; +; \cdot)$ is a vector space over R.

Solution : Let $u = a + bi;\ v = c + di;\ w = e + fi \in C$, where $a; b; c; d; e; f \in R$, we have

(1)

(i) The addition is closed :

$$\begin{aligned} u + v &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i \qquad \text{by (1).} \end{aligned}$$

Since $(a + c)$ and $(b + d)$ are real numbers then $u + v \in C$.

(ii) The addition is commutative:

$$\begin{aligned} u + v &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i \qquad \text{by (1),} \\ &= (c + a) + (d + b)i \qquad \text{because addition on } R \text{ is commutative,} \\ &= (c + di) + (a + bi) \qquad \text{by (1),} \\ &= v + u \end{aligned}$$

(iii) The addition is associative: we have to prove that $u + (v + w) = (u + v) + w$ for all $u; v; w \in C$.

The left hand side (L.H.S):

$$\begin{aligned} u + (v + w) &= u + [(c + di) + (e + fi)] \\ &= (a + bi) + [(c + e) + (d + f)i] \qquad \text{by (1),} \\ &= [a + (c + e)] + [b + (d + f)]i \qquad \text{by (1),} \\ &= [(a + c) + e] + [(b + d) + f]i \qquad \text{because addition on } R \text{ is associative.} \end{aligned}$$

The right hand side (R.H.S):

$$
\begin{aligned}
(u+v)+w &= [(a+bi)+(c+di)]+w \\
&= [(a+c)+(b+d)i]+(e+fi) \quad \text{by (1)}, \\
&= [(a+c)+e]+[(b+d)+f]i \quad \text{by (1)}.
\end{aligned}
$$

Then L.H.S = R.H.S

(iv) The additive identity : For all $u = a+bi \in C$, we have

$$
\begin{aligned}
(a+bi)+(0+0i) &= (a+0)+(b+0)i \quad \text{by (1)}, \\
&= a+bi \quad \text{because 0 is the additive identity in } R.
\end{aligned}
$$

Then the additive identity of C is $(0+0i)$.

(v) The additive inverse : For all $u = a+bi \in C$, we have

$$
\begin{aligned}
(a+bi)+\left(-a+(-b)i\right) &= \left(a+(-a)\right)+\left(b+(-b)\right)i \quad \text{by (1)}, \\
&= 0+0i \quad \text{because } (-a) \text{ is the additive inverse of } a \text{ in } R.
\end{aligned}
$$

Then the additive inverse of $a+bi \in C$ is $-a+(-b)i$.

(2) Let $u = a+bi;\, v = c+di \in C$ and $\alpha \in R$.

(i) We have to prove that $\alpha u \in C$.

$$
\begin{aligned}
\alpha u &= \alpha(a+bi) \\
&= \alpha a+\alpha bi
\end{aligned}
$$

Since $\alpha a;\ \alpha b \in R$, then $\alpha u \in C$.

(ii) We have to prove that $\alpha(u+v) = \alpha u+\alpha v$ for all $u; v \in C$ and $\alpha \in R$.

The left hand side (L.H.S) :

(u + v) = [(a + bi) + (c + di)]

= [(a + c) + (b + d)i] by (1)

= (a + c) + (b + d)i by (2)

= (a + c) + (b + d)i because multiplication distributes over addition in R.

The right hand side (R.H.S) :

u + v = (a + bi) + (c + di)

= (a + bi) + (c + di) by (2),

= (a + c) + (b + d)i by (1),

Then L.H.S = R.H.S

(iii) We have to prove that (+) u = u + u for all u 2 C and ; 2 R.

The L.H.S :

(+) u = (+) (a + bi)

= (+)a + (+)bi by (2),

= (a + a) + (b + b)i because multiplication distributes over addition in R.

The R.H.S :

u + u = (a + bi) + (a + bi)

= (a + bi) + (a + bi) by (2),

= (a + a) + (b + b)i by (1).

Then L.H.S= R.H.S

(iv) We have to prove that ( ) u = (u) for all u 2 C and ; 2 R. The L.H.S :

$$(\ \ ) \ u = (\ \ ) \ (a + bi)$$

$$= (\ \ )a + (\ \ )bi \qquad \text{by (2)},$$

$$= \ \ a + \ \ bi \qquad \text{because multiplication is associative in R.}$$

The R.H.S :

$$(\ \ u) = \ \ [\ \ (a + bi)]$$

$$= \ \ [\ a + \ \ bi] \qquad \text{by (2)},$$

$$= \ \ a + \ \ bi \qquad \text{by (2)}.$$

Then L.H.S= R.H.S

(v) The multiplicative identity : we have to show that 1 u = u for all u = a + bi 2 C. (Note that, 1 represents scalar from the eld R and NOT from the set C).

$$1 \ \ u = 1 \ \ (a + bi)$$

$$= 1a + 1bi \qquad \text{by (2)},$$

$$= a + bi$$

$$= u$$

We have proved that all axioms hold in C. Hence, (C; + ;) is a vector space over R.

Example 1.3. Let $M_{2\ 2}(R) = \ {}^{n} \begin{matrix} a & b \\ c & d \end{matrix} \ j\ a; b; c; d\ 2\ R^{o}$ be the set of all two by two matrices with entries in R. For $A = \begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix}; B = \begin{matrix} b_1 & b_2 \\ b_3 & b_4 \end{matrix}\ 2\ M_{2\ 2}$ and

2 R, addition and scalar multiplication of matrices de ned by

$$A + B = \begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix} + \begin{matrix} b_1 & b_2 \\ b_3 & b_4 \end{matrix} = \begin{matrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{matrix} \qquad (3)$$

$$A = \begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix} = \begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix} : \qquad (4)$$

Prove that $(M_{2\ 2}; +;\ )$ is a vector space over R.

Solution: Let $A = \begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix}; B = \begin{matrix} b_1 & b_2 \\ b_3 & b_4 \end{matrix}; C = \begin{matrix} c_1 & c_2 \\ c_3 & c_4 \end{matrix}\ 2\ M_{2\ 2}$.

(1)

(i)

$$A + B = \begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix} + \begin{matrix} b_1 & b_2 \\ b_3 & b_4 \end{matrix} = \begin{matrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{matrix} \quad \text{by (3).}$$

Since $a_1; a_2; a_3; a_4; b_1; b_2; b_3; b_4$ are real numbers, then $a_1 + b_1; a_2 + b_2; a_3 + b_3; a_4 + b_4\ 2$ R. Hence, $A + B\ 2\ M_{2\ 2}(R)$.

(ii) We have to show that $A + B = B + A$ for all $A; B \in M_{2\times 2}$.

$$
\begin{aligned}
A + B &= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \\
&= \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{pmatrix} \quad \text{by (3),} \\
&= \begin{pmatrix} b_1 + a_1 & b_2 + a_2 \\ b_3 + a_3 & b_4 + a_4 \end{pmatrix} \quad \text{because addition on R is commutative} \\
&= \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} + \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad \text{by (3),} \\
&= B + A
\end{aligned}
$$

(iii) We have to show that $A + (B + C) = (A + B) + C$ for all $A; B; C \in M_{2\times 2}$.

The L.H.S:

$$
\begin{aligned}
A + (B + C) &= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \left[\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} + \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \right] \\
&= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} b_1 + c_1 & b_2 + c_2 \\ b_3 + c_3 & b_4 + c_4 \end{pmatrix} \quad \text{by (3),} \\
&= \begin{pmatrix} a_1 + (b_1 + c_1) & a_2 + (b_2 + c_2) \\ a_3 + (b_3 + c_3) & a_4 + (b_4 + c_4) \end{pmatrix} \quad \text{by (3),} \\
&= \begin{pmatrix} (a_1 + b_1) + c_1 & (a_2 + b_2) + c_2 \\ (a_3 + b_3) + c_3 & (a_4 + b_4) + c_4 \end{pmatrix} \quad \text{because addition on R is associative.}
\end{aligned}
$$

The R.H.S:

$$(A+B)+C = \left[\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\right] + \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

$$= \begin{pmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{pmatrix} + \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \quad \text{by (3)},$$

$$= \begin{pmatrix} (a_1+b_1)+c_1 & (a_2+b_2)+c_2 \\ (a_3+b_3)+c_3 & (a_4+b_4)+c_4 \end{pmatrix} \quad \text{by (3)}.$$

Then L.H.S= R.H.S

(iv) For all $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in M_{2\times 2}$, we have

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

Then the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the additive identity.

(v) For all $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in M_{2\times 2}$, we have $(-A) = \begin{pmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{pmatrix} \in M_{2\times 2}$, where

$$A + (-A) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the matrix $(-A)$ is the additive inverse for the matrix A.

(2)

(i) We have to show that $\lambda A \in M_{2\times 2}(R)$ for all $A \in M_{2\times 2}$ and $\lambda \in R$.

$$\lambda A = \lambda \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} \lambda a_1 & \lambda a_2 \\ \lambda a_3 & \lambda a_4 \end{pmatrix} \quad \text{by (4)}.$$

Since $\lambda; a_1; a_2; a_3; a_4$ are real numbers then $\lambda a_1; \lambda a_2; \lambda a_3; \lambda a_4 \in R$.

Hence, A 2 $M_{2\ 2}$(R).

(ii) We have to show that (A + B) = A + B for all A; B 2 $M_{2\ 2}$ and 2 R.

The L.H.S:

$$(A + B) = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$$

$$= \begin{matrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{matrix} \quad \text{by (3),}$$

$$= \begin{matrix} (a_1 + b_1) & (a_2 + b_2) \\ (a_3 + b_3) & (a_4 + b_4) \end{matrix} \quad \text{by (4),}$$

$$= \begin{matrix} a_1 + \ b_1 & a_2 + \ b_2 \\ a_3 + \ b_3 & a_4 + \ b_4 \end{matrix} \quad \text{because multiplication distributes over addition in R.}$$

The R.H.S:

$$A + \ B = \begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix} + \begin{matrix} b_1 & b_2 \\ b_3 & b_4 \end{matrix}$$

$$= \begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix} + \begin{matrix} b_1 & b_2 \\ b_3 & b_4 \end{matrix} \quad \text{by (4),}$$

$$= \begin{matrix} a_1 + \ b_1 & a_2 + \ b_2 \\ a_3 + \ b_3 & a_4 + \ b_4 \end{matrix} \quad \text{by (3).}$$

Then L.H.S= R.H.S

(iii) We have to show that (+) A = A + A for all A 2 $M_{2\ 2}$ and ; 2 R.

The L.H.S:

$$(+) A = (+) \begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix}$$

$$= \begin{matrix} (+)a_1 & (+)a_2 \\ (+)a_3 & (+)a_4 \end{matrix} \qquad \text{by (4),}$$

$$= \begin{matrix} a_1 + \ a_1 & a_2 + \ a_2 \\ a_3 + \ a_3 & a_4 + \ a_4 \end{matrix} \qquad \text{because multiplication distributes over addition in R.}$$

The R.H.S:

$$A + \quad A = \begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix} + \begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix}$$

$$= \begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix} + \begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix} \qquad \text{by (4),}$$

$$= \begin{matrix} a_1 + \ a_1 & a_2 + \ a_2 \\ a_3 + \ a_3 & a_4 + \ a_4 \end{matrix} \qquad \text{by (3).}$$

Then L.H.S = R.H.S

(iv) We have to show that ( ) A = (A) for all l A 2 $M_{2\ 2}$ and ; 2 R.

The L.H.S:

$$( ) A = ( ) \begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix}$$

$$= \begin{matrix} ( )a_1 & ( )a_2 \\ ( )a_3 & ( )a_4 \end{matrix} \qquad \text{by (4),}$$

$$= \begin{matrix} (a_1) & (a_2) \\ (a_3) & (a_4) \end{matrix} \qquad \text{because multiplication on R is associative.}$$

The R.H.S:

$$\alpha \odot (\beta \odot A) = \alpha \odot \left[\beta \odot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \right]$$

$$= \alpha \odot \begin{pmatrix} \beta a_1 & \beta a_2 \\ \beta a_3 & \beta a_4 \end{pmatrix} \quad \text{by (4)},$$

$$= \begin{pmatrix} \alpha(\beta a_1) & \alpha(\beta a_2) \\ \alpha(\beta a_3) & \alpha(\beta a_4) \end{pmatrix} \quad \text{by (4)}.$$

Then L.H.S= R.H.S

(v) For all $A \in M_{2\times 2}$, we have $1 \in \mathbb{R}$ such that

$$1 \odot A = 1 \odot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 1a_1 & 1a_2 \\ 1a_3 & 1a_4 \end{pmatrix} = A.$$

Then $1 \in \mathbb{R}$ is the multiplicative identity.

Example 1.4. Let $V = \{x \in \mathbb{R} \mid x > 0\}$. For $x, y \in V$ and $\alpha \in \mathbb{R}$, we define addition and scalar multiplication as following

$$x \oplus y = xy;$$

$$\alpha \odot x = x^{\alpha}.$$

Show that $(V, \oplus, \odot)$ is a vector space over $\mathbb{R}$.

Example 1.5. Is the set $V = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b > 0 \right\}$ with the usual addition and scalar multiplication of matrices define a vector space over $\mathbb{R}$?

Solution: Let $\alpha = -2 \in \mathbb{R}$, then $\alpha \begin{bmatrix} a \\ b \end{bmatrix} = -2 \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2a \\ -2b \end{bmatrix} \notin V$.

Since $a, b > 0$ then $-2a, -2b < 0$.

Proposition 1.6. Let V be a vector space over K, then we have

(1) The additive identity, 0 2 V, is unique.

(2) The additive inverse, (u) 2 V, for u 2 V is unique.

(3) For all u 2 V we have 0 u = 0.

(4) For all u 2 V we have (1) u = u.

(5) For all u; v; w 2 V, if u + v = u + w then v = w.

(6) For all u; v 2 V, the equation u + x = v has a unique solution x = v u 2 V.

(7) For all u 2 V, we have (u) = u.

2 Subspace

In this section we suppose that $(V; +; \cdot)$ is a vector space over $\mathbb{K}$.

Definition 2.1. A non-empty subset U of V is called a subspace of V if $(U; +; \cdot)$ is a vector space over $\mathbb{K}$.

Proposition 2.2. A non-empty subset U of a vector space V over $\mathbb{K}$ is a subspace of V if and only if the following conditions are satisfied:

(1) $0 \in U$.

(2) For all $u; v \in U$, we have $u + v \in U$.

(3) For all $u \in U$ and $\alpha \in \mathbb{K}$, we have $\alpha u \in U$.

Remark 2.3. Every vector space V has two subspaces namely V and $\{0\}$, which are called trivial subspaces. Any other subspace of V is called a proper subspace of V.

Example 2.4. Show that which of these sets are subspace of $\mathbb{R}^3$

(1) $U = \{(x; y; 0) \mid x; y \in \mathbb{R}\}$.

(2) $U = \{(x; y; 1) \mid x; y \in \mathbb{R}\}$.

.

Proposition 2.5. If W_1 and W_2 are subspaces of V, then $W_1 \cap W_2$ is a subspace of V.

Proof. We have to satisfy the three conditions in Proposition 2.2.

(1) Since W_1 and W_2 are subspaces of V, then $0 \in W_1$ and $0 \in W_2$.

Hence,

$$0 \in W_1 \cap W_2.$$

(2) Let $u, v \in W_1 \cap W_2$, then $u, v \in W_1$ and $u, v \in W_2$.

Since W_1 and W_2 are subspaces of V, then $u + v \in W_1$ and $u + v \in W_2$.

Hence,

$$u + v \in W_1 \cap W_2.$$

(3) Let $\alpha \in K$ and $u \in W_1 \cap W_2$, then $u \in W_1$ and $u \in W_2$.

Since W_1 and W_2 are subspaces of V then $\alpha u \in W_1$ and $\alpha u \in W_2$.

Hence,

$$\alpha u \in W_1 \cap W_2.$$

□

Example 2.6. Show that if W_1 and W_2 are subspaces of a vector space V, then $W_1 \cup W_2$ is NOT a subspace of V.

To prove this, we have $W_1 = \{(a, 0) \mid a \in \mathbb{R}\}$ and $W_2 = \{(0, b) \mid b \in \mathbb{R}\}$ are both subspaces of $\mathbb{R}^2$. But $W_1 \cup W_2$ is not a subspace of $\mathbb{R}^2$ because $(1, 0) \in W_1 \cup W_2$ and $(0, 1) \in W_1 \cup W_2$ while $(1, 0) + (0, 1) = (1, 1) \notin W_1 \cup W_2$.

Proposition 2.7. Let $W_1, W_2, \cdots, W_n$ are subspaces of a vector space V over a field K, then we have

(1) $W_1 \cap W_2 \cap \cdots \cap W_n$ is a subspace of V.

(2) $W_1 + W_2 + \cdots + W_n = \{w_1 + w_2 + \cdots + w_n \mid w_i \in W_i, i = 1, 2, \cdots, n\}$ is a subspace of V.

Proof. □

3 Linear Combinations and Span

De nition 3.1. Let $v_1; v_2; \quad ; v_n$ be vectors in a vector space V over K . A linear combination of these vectors is any expression of the form

$$_1v_1 + \quad _2v_2 + \quad + \quad _nv_n$$

for some scalars $_1; \quad _2; \quad ; \quad _n$ 2 K .

Example 3.2. Consider the vector space R^2. The vector v = (7; 13) is a linear combination of v_1 = (2; 1) and v_2 = (1; 5), where

$$v = 2v_1 + (\quad 3)v_2:$$

Example 3.3. Consider the vector space R^2. The vector v = (1; 3) is a linear combination of v_1 = (0; 1) , v_2 = (2; 1), v_3 = (1; 2) and v_4 = (0; 3) where

$$v = (\quad 2)v_1 + (0)v_2 + 1v_3 + (\frac{1}{3})v_4:$$

Sometimes we cannot write a vector v in a vector space V as a linear combination of $v_1; v_2; \quad ; v_n$ 2 V, as explained in this example.

Example 3.4. Let v_1 = (2; 5; 3); v_2 = (1; 1; 1), and v = (4; 2; 0). Because there exist no scalars $_1; \quad _2$ 2 K such that $v = \quad _1v_1 + \quad _2v_2$ then v is not a linear combination of v_1 and v_2.

De nition 3.5. Let V be a vector space over K , and let S = f $v_1; v_2; \quad ; v_n$ g be a subset of V. We say that S spans V, or S generates V, if every vector v in V can be written as a linear combination of vectors in S. That is, for all v 2 V, we have

$$v = \quad _1v_1 + \quad _2v_2 + \quad + \quad _nv_n$$

for some scalars $\alpha_1, \alpha_2, \cdots, \alpha_n \in K$.

Example 3.6. Show that the set $S = \{(1,0),(0,1)\}$ spans the vector space $\mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}\}$.

Solution: We have to show that for all $v = (a,b) \in \mathbb{R}^2$ there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $v = \alpha_1(1,0) + \alpha_2(0,1)$.

$$
\begin{aligned}
(a,b) &= \alpha_1(1,0) + \alpha_2(0,1) \\
&= (\alpha_1, 0) + (0, \alpha_2) \\
&= (\alpha_1, \alpha_2)
\end{aligned}
$$

Then $\alpha_1 = a$ and $\alpha_2 = b$. So, any vector $v = (a,b) \in \mathbb{R}^2$ can be written in the form $(a,b) = a(1,0) + b(0,1)$. Thus S spans $\mathbb{R}^2$.

Example 3.7. Let $S = \left\{ v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, and

$V = \left\{ v = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a,b,c,d \in \mathbb{R} \right\}$

(1) Does S spans V ?

(2) Define a vector space U such that S spans U.

(3) Find a set that spans V.

Solution: (1) If S spans V then for all $v \in V$, there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$
\begin{aligned}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \alpha_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \alpha_2 \end{bmatrix} \\
&= \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}
\end{aligned}
$$

So, $_1 = a$ and $_2 = d$. But if b or c is non-zero then v cannot be written as a linear combination of the vectors in S. Hence, S not spans V.

(2) From (1), we can see that if $U = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ then S spans U.

(3) The set that spans V is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

Example 3.8. Show that the set $S = \{(0,1,1),(1,0,1),(1,1,0)\}$ spans R^3 and write the vector $(2,4,8)$ as a linear combination of vectors in S.

Solution:

A vector in R^3 has the form $v = (x,y,z)$.

Hence we need to show that, for some scalars $\alpha_1, \alpha_2, \alpha_3 \in R$, every such v can be written as

$$\begin{aligned}(x,y,z) &= \alpha_1(0,1,1) + \alpha_2(1,0,1) + \alpha_3(1,1,0)\\ &= (\alpha_2 + \alpha_3, \alpha_1 + \alpha_3, \alpha_1 + \alpha_2)\end{aligned}$$

This give us system of equations

$$\begin{aligned}x &= \alpha_2 + \alpha_3\\ y &= \alpha_1 + \alpha_3\\ z &= \alpha_1 + \alpha_2\end{aligned}$$

This system of equations can be written in matrix form

$$\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

We can write it as $A\alpha = b$. Since $\det(A) = 2$ then this system has a solution.

Now, to write $(2,4,8)$ as a linear combination of vectors in S, we find that

$$A^{-1} = \begin{bmatrix}-0.5 & 0.5 & 0.5\\ 0.5 & -0.5 & 0.5\\ 0.5 & 0.5 & -0.5\end{bmatrix}$$

Then

$$\alpha = A^{-1}b$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} -0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

So, $\alpha_1 = 5;\ \alpha_2 = 3;\ \alpha_3 = -1$, and

$$(2;4;8) = 5(0;1;1) + 3(1;0;1) + (-1)(1;1;0).$$

4 Linear independence

Definition 4.1. Let V be a vector space over a field K. A subset $\{v_1; v_2; \cdots; v_n\}$ in V is linearly dependent over K if there exists scalars $\alpha_1; \alpha_2; \cdots; \alpha_n \in K$, (not all zero), such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0.$$

Definition 4.2. Let V be a vector space over a field K. A subset $\{v_1; v_2; \cdots; v_n\}$ in V is linearly independent over K if $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

Example 4.3. Show that the set $\{(1;0;1); (1;-1;1); (2;-1;2); (0;0;1)\}$ is linearly dependent over $\mathbb{R}$.

Solution: We have to show that there exists $\alpha_1; \alpha_2; \alpha_3; \alpha_4 \in \mathbb{R}$ not all zero such that

$$\alpha_1(1;0;1) + \alpha_2(1;-1;1) + \alpha_3(2;-1;2) + \alpha_4(0;0;1) = (0;0;0)$$

We have the following system of equations

$$
\begin{aligned}
{}_1 + {}_2 + 2\,{}_3 &= 0 \\
{}_2 \quad {}_3 &= 0 \\
{}_1 + {}_2 + 2\,{}_3 + {}_4 &= 0
\end{aligned}
$$

Put the rst equation in the last equation, we get ${}_4 = 0$.

From the second equation, we have ${}_2 = \quad {}_3$. Let ${}_2 = 1$ then ${}_3 = \quad 1$ and ${}_1 = 1$. Hence, $(1;0;1) + (1; \quad 1;1) + (\quad 1)(2; \quad 1;2) + (0)(0;0;1) = (0;0;0)$.

Example 4.4. Show that the set f $(1;0;1);(0;0;1)$g is linearly independent over R .

Solution:

$$
\begin{aligned}
{}_1(1;0;1) + {}_2(0;0;1) &= (0;0;0) \\
({}_1;0; {}_1) + (0;0; {}_2) &= (0;0;0) \\
({}_1;0; {}_1 + {}_2) &= (0;0;0)
\end{aligned}
$$

So, ${}_1 = 0$, ${}_1 + {}_2 = 0$ then ${}_2 = 0$. Then it is linearly independent over R.

Example 4.5. Show that the set S = f $i; i+1$g is linearly dependent over C, but it is linearly independent over R.

Solution: Since $(\quad 1+i)i + (1)(1+i) = 0$, so, S is linearly dependent over C.

Let $\quad (i) + \quad (1+i) = 0$, where $\; ; \quad 2$ R

Then

$$
\begin{aligned}
i + \quad + \quad i &= 0 + 0i \\
+ (\quad + \quad)i &= 0 + 0i
\end{aligned}
$$

So, = 0; + = 0 and then = 0. Hence, S is linearly independent over R.

Theorem 4.6. If $A = (a_{ij})$ 2 $M_{n\ n}(K)$, and C_j = f $a_{1j}; a_{2j};\ ; a_{nj}$ g , j = 1; 2; ; n are the n columns of A then f $C_1; C_2;\ ; C_n$ g is linearly dependent over K if and only if det A = 0.

Corollary 4.7. The n rows of a matrix A 2 $M_{n\ n}(K)$ are linearly dependent over K if and only if det A = 0.

5 Basis and dimension

De nition 5.1. Let V be a vector space over K . A subset S = f $v_1; v_2;\ ; v_n$ g is called a basis for V if

(i) V is spanned by S, that is, for every v 2 V there exists scalars $_1;\ _2;\ ;\ _n$ 2 K such that $v = \ _1v_1 + \ _2v_2 + \ + \ _nv_n$.

(ii) The set S is linearly independent over K .

Example 5.2. Show that the set S = f (1; 0; 0); (0; 1; 0); (0; 0; 1)g is a basis for the vector space R^3.

Solution: (i) we have to show that S spans R^3. That is, f or all v = (x; y; z) 2 R^3, we have to nd scalars $_1;\ _2;\ _3$ 2 R^3 such that $v = \ _1v_1 + \ _2v_2 + \ _3v_3$

$$(x; y; z) = \ _1(1; 0; 0) + \ _2(0; 1; 0) + \ _3(0; 0; 1)$$

$$(x; y; z) = (\ _1;\ _2;\ _3)$$

So, (x; y; z) = x(1; 0; 0) + y(0; 1; 0) + z(0; 0; 1) and, hence, R^3 is generated by S.

(ii) To show that S is linearly independent, Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since $\det(A) \neq 0$ then S is linearly independent.

Finally, we get S is a basis for $\mathbb{R}^3$.

Example 5.3. Let $e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1)$. Then $B = \{e_1, e_2, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$. This basis called the standard basis for $\mathbb{R}^n$.

Theorem 5.4. Let V be a vector space over a field K, and $S = \{v_1, v_2, \ldots, v_n\}$ be a basis of V containing n vectors.Then any subset containing more than n vectors in V is linearly dependent.

Definition 5.5. Let V be a vector space with a basis $S = \{v_1, v_2, \ldots, v_n\}$ has n vectors. Then, we say n is the dimension of V and we write $\dim(V) = n$.

Theorem 5.6. Any vector space V has a basis. All bases for V are of the same dimension.

Example 5.7. The following vector spaces over $\mathbb{R}$ have dimensions :

(1) $\dim(\mathbb{R}^n) = n$.

(2) $\dim \mathbb{R} = 1$.

(3) $\dim \mathbb{C} = 2$.

(4) $\dim M_{n,n}(\mathbb{R}) = n^2$.

Theorem 5.8. Let V be a vector space such that $\dim(V) = n$. Let $S = \{v_1, v_2, \ldots, v_n\}$ be a subset of V. Then we have

(1) If S spans V, then S is also linearly independent hence a basis for V.

(2) If S is linearly independent, then S also spans V hence is a basis for V.

Example 5.9. Show that S is not a basis for $\mathbb{R}^3$ where $S = \{(6;4;1);(3;-5;1);(8;13;6);(0;6;9)\}$.

Solution: Since $\dim(\mathbb{R}^3) = 3$, then any basis for $\mathbb{R}^3$ must have 3 vectors, while here S has four.

Example 5.10. Show that $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis for $M_{2;2}(\mathbb{R})$.

Solution: Since S has four vectors and $\dim(M_{2;2}(\mathbb{R})) = 4$ then, by Theorem 5.8, we have to show that either S spans V or S is linearly independent.

6 Dot and cross products

Definition 6.1. Let $v = (a_1;a_2;\cdots;a_n)$ be a vector in a vector space V. The length (or norm or magnitude) of v is

$$\|v\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

Example 6.2. Suppose that the vector $v = (2;-1;4;1)$, then the length of v is

$$\|v\| = \sqrt{2^2 + (-1)^2 + 4^2 + 1^2} = \sqrt{22}.$$

Definition 6.3. Let $u = (a_1;a_2;\cdots;a_n)$ and $v = (b_1;b_2;\cdots;b_n)$ are vectors in a vector space V. The dot product of u and v is defined by

$$u \cdot v = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

.

Definition 6.4. The angle θ between two vectors u and v is determined by the formula

$$u \cdot v = \|u\|\|v\| \cos\theta$$

Example 6.5. Let $u = (1; 3; 0)$ and $v = (-2; 1; 5)$. The dot product of u and v is

$$u \cdot v = 1(-2) + 3(1) + 0(5) = 1;$$

and the angle between them is

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{1}{\sqrt{10}\sqrt{30}}$$

So,

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{10}\sqrt{30}}\right):$$

Some properties of the dot product : Let $u; v$ and w are vectors in a vector space V over K. The dot product has the following properties:

(1) $v \cdot v = \|v\|^2$

(2) $u \cdot v = v \cdot u$

(3) $u \cdot (v + w) = u \cdot v + u \cdot w$

(4) $(\lambda u) \cdot v = u \cdot (\lambda v) = \lambda(u \cdot v)$, where $\lambda \in K$.

(5) If $u \cdot v > 0$ then the angle formed by the vectors $(0 < \theta < 90)$.

(6) If $u \cdot v < 0$ then the angle formed by the vectors, $(90 < \theta \leq 180)$.

(7) If $u \cdot v = 0$ then the angle formed by the vectors is 90 degrees.

Definition 6.6. Let u and v are vectors in a vector space V. If

$$u \cdot v = 0$$

then we say that u and v are orthogonal.

Definition 6.7. A subset $S = \{v_1; v_2; \cdots; v_n\}$ of a vector space V form an orthogonal set if all vectors in S are orthogonal to each other, $v_i \cdot v_j = 0$ for $i \neq j$. In addition, if all vectors in an orthogonal set S has length one, $\|v_i\| = 1$, then S is called an orthonormal set.

Theorem 6.8. Any orthogonal set is linearly independent.

Gram-Schmidt process: If $B = \{v_1; v_2; \cdots; v_n\}$ is a basis for a vector space V. Then we can define an orthogonal basis $W = \{w_1; w_2; \cdots; w_n\}$ for V by using the following steps:

$$
\begin{aligned}
w_1 &= v_1 \\
w_2 &= v_2 - \frac{w_1 \cdot v_2}{w_1 \cdot w_1} w_1 \\
w_3 &= v_3 - \frac{w_1 \cdot v_3}{w_1 \cdot w_1} w_1 - \frac{w_2 \cdot v_3}{w_2 \cdot w_2} w_2 \\
&\vdots \\
w_n &= v_n - \frac{w_1 \cdot v_n}{w_1 \cdot w_1} w_1 - \frac{w_2 \cdot v_n}{w_2 \cdot w_2} w_2 - \cdots - \frac{w_{n-1} \cdot v_n}{w_{n-1} \cdot w_{n-1}} w_{n-1}
\end{aligned}
$$

In addition, the set

$$\left\{ \frac{w_1}{\|w_1\|}; \frac{w_2}{\|w_2\|}; \cdots; \frac{w_n}{\|w_n\|} \right\}$$

is an orthonormal basis for V.

Example 6.9. Let $S = \{v_1 = (1;1;0); v_2 = (1;1;1); v_3 = (3;1;1)\}$ be a basis for R^3. We will use Gram-Schmidt process to find orthogonal and orthonormal bases for R^3.

$$w_1 = v_1 = (1; 1; 0)$$

$$w_2 = v_2 \quad \frac{w_1 \quad v_2}{w_1 \quad w_1} w_1$$

$$= (1; 1; 1) \quad \frac{(1; 1; 0) \quad (1; 1; 1)}{(1; 1; 0) \quad (1; 1; 0)} (1; 1; 0)$$

$$= (1; 1; 1) \quad \frac{1 + 1 + 0}{1 + 1 + 0} (1; 1; 0)$$

$$= (0; 0; 1)$$

$$w_3 = v_3 \quad \frac{w_1 \quad v_3}{w_1 \quad w_1} w_1 \quad \frac{w_2 \quad v_3}{w_2 \quad w_2} w_2$$

$$= (3; 1; 1) \quad \frac{(1; 1; 0) \quad (3; 1; 1)}{(1; 1; 0) \quad (1; 1; 0)} (1; 1; 0) \quad \frac{(0; 0; 1) \quad (3; 1; 1)}{(0; 0; 1) \quad (0; 0; 1)} (0; 0; 1)$$

$$= (3; 1; 1) \quad \frac{4}{2}(1; 1; 0) \quad \frac{1}{1}(0; 0; 1)$$

$$= (3; 1; 1) \quad (2; 2; 0) \quad (0; 0; 1)$$

$$= (1; \quad 1; 0)$$

Then $W = f\, w_1; w_2; w_3 g = f\,(1; 1; 0); (0; 0; 1); (1; \quad 1; 0) g$ is an orthogonal basis for R^3.

Since $kw_1k = {}^{p}\overline{2}; kw_2k = 1; kw_3k = {}^{p}\overline{2}$ then the set

$$U = {}^{n} \frac{w_1}{kw_1k}; \frac{w_2}{kw_2k}; \frac{w_3}{kw_3k} {}^{o} = {}^{n} p\frac{1}{\overline{2}}(1; 1; 0); (0; 0; 1); p\frac{1}{\overline{2}}(1; \quad 1; 0)^{o}$$

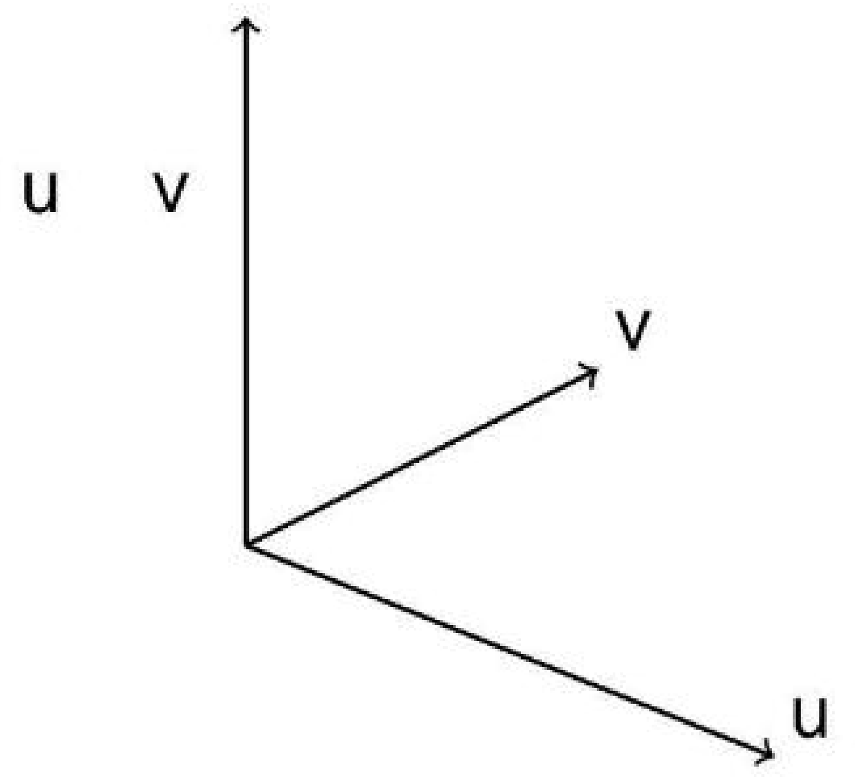
is an orthonormal basis for R^3.

Definition 6.10. Let $u = (a_1, a_2, a_3), v = (b_1, b_2, b_3) \in \mathbb{R}^3$ then we define the cross product of u and v as following

$$u \times v = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i(a_2b_3 - b_2a_3) - j(a_1b_3 - b_1a_3) + k(a_1b_2 - b_1a_2).$$

That is, $u \times v = (a_2b_3 - b_2a_3, a_3b_1 - a_1b_3, a_1b_2 - b_1a_2)$.

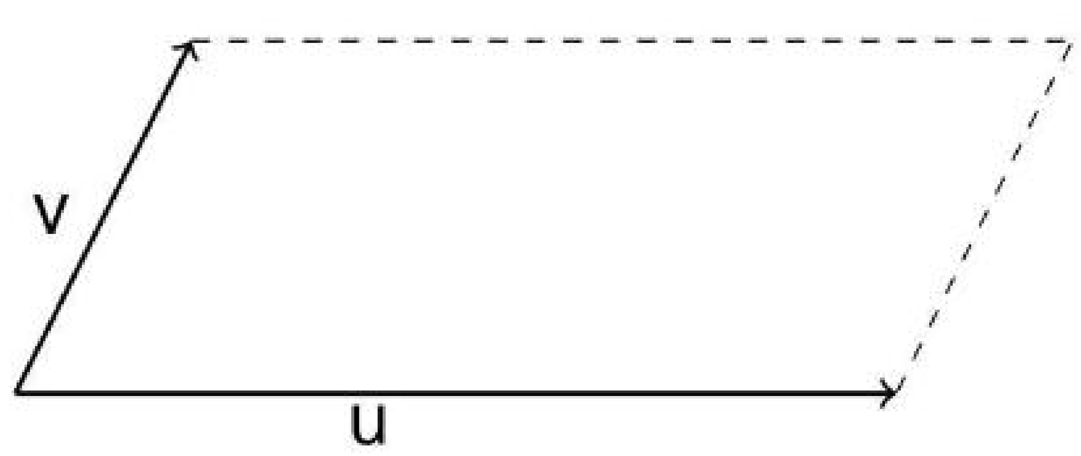
Geometrically, the cross product of vectors u and v represents a vector that is orthogonal to both of u and v.



Definition 6.11. The angle θ between two vectors u and v is determined by the formula

$$\|u \times v\| = \|u\| \|v\| \sin\theta.$$

Note that, the length of $u \times v$ represents the area of the parallelogram that spanned by u and v.



Example 6.12. Find the area of the parallelogram that spanned by the vectors $u = (1; 3; 2)$ and $v = (-2; 1; 0)$.

Solution :

$$u \times v = (-2; -4; 7)$$

$$\|u \times v\| = \sqrt{4 + 16 + 49} = \sqrt{69}$$

7 Eigenvalues and eigenvectors

Definition 7.1. Let A be an $n \times n$ matrix. If there is a number $\lambda \in \mathbb{C}$ and a vector $x \neq 0$ such that $Ax = \lambda x$, then we say that λ is an eigenvalue for A, and x is called an eigenvector for A with eigenvalue λ.

Example 7.2. If

$$A = \begin{pmatrix} 1 & 3 \\ 6 & -2 \end{pmatrix}; \quad \text{and} \quad x = \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

then

$$Ax = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4x :$$

So, $\lambda = 4$ is an eigenvalue of A, and x is an eigenvector for A with this eigenvalue.

We can write the equation $Ax = \lambda x$ as a linear system. Since $\lambda x = \lambda I x$, (where $I = I_n$ is the identity matrix), we have that

$$Ax = \lambda x \iff Ax - \lambda x = 0 \iff (A - \lambda I)x = 0$$

This linear system has a non-trivial solution $x \neq 0$ if and only if

$$\det(A - \lambda I) = 0; \quad \text{(why?)}:$$

Definition 7.3. The characteristic equation of a square matrix A is the equation

$$\det(A - \lambda I) = 0:$$

Theorem 7.4. The eigenvalues of a square matrix A are the solutions of the characteristic equation

$$\det(A \quad I) = 0.$$

How to nd the eigenvalues and the eigenvectors:

To nd the eigenvalues of a matrix A, we have to nd the solution of the characteristic equation det(A I) = 0, then to nd the eigenvectors for A with eigen value we have to solve the linear system (A I)x = 0, as explained in this example.

Example 7.5. Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}.$$

Solution: We have to nd A I.

$$A \quad I = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \quad & 3 \\ 3 & 6 \quad \end{pmatrix}$$

Now, we have to find the solution to the characteristic equation $\det(A - \lambda I) = 0$.

$$\begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = (2-\lambda)(-6-\lambda) - 3 \cdot 3 = \lambda^2 + 4\lambda - 21 = 0$$

Then

$$\lambda^2 + 4\lambda - 21 = (\lambda + 7)(\lambda - 3) = 0$$

So, the eigenvalues of A are

$$\lambda_1 = -7 \quad \text{and} \quad \lambda_2 = 3$$

To find the eigenvector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ for $\lambda_1 = -7$, we have to solve the following system

$$(A - \lambda_1 I)x = 0$$

$$\begin{pmatrix} 2-(-7) & 3 \\ 3 & -6-(-7) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using Gauss elimination, ($R_1 \to \frac{1}{9}R_1$; $R_2 \to -3R_1 + R_2$), we get

$$\begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We have only one equation with two variables $x_1 + \frac{1}{3}x_2 = 0$, then $x_1 = \frac{-1}{3}x_2$.

Assume $x_2 = c_1$, gives us $x = \begin{pmatrix} -\frac{1}{3}c_1 \\ c_1 \end{pmatrix} = c_1 \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}$, where $c_1 \in \mathbb{R}$.

Similarly, we can show that the eigenvector for $\lambda_2 = 3$ is $x = c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, where $c_2 \in \mathbb{R}$.

8 Linear transformation on vector spaces

De nition 8.1. Let V and W are vector spaces over a eld K . A linear transformation T from V into W is a mapping T : V ! W such that

(i) T(u + v) = T(u) + T(v)

(ii) T(u) = T(u)

for all u; v 2 V and 2 K . If T : V ! V then we say that T is a linear transformation on V.

Example 8.2. Show that T : R^3 ! R^2 de ned by T(a_1; a_2; a_3) = (a_1 + a_2; a_2 a_3) is a linear transformation.

Solution:

(i) Let u = (a_1; a_2; a_3); v = (b_1; b_2; b_3) 2 R^3. Then

u + v = (a_1 + b_1; a_2 + b_2; a_3 + b_3), and

$$\begin{aligned} T(u + v) &= T(a_1 + b_1; a_2 + b_2; a_3 + b_3) \\ &= (a_1 + b_1 + a_2 + b_2; a_2 + b_2 \quad a_3 \quad b_3) \\ &= (a_1 + a_2 + b_1 + b_2; a_2 \quad a_3 + b_2 \quad b_3) \\ &= (a_1 + a_2; a_2 \quad a_3) + (b_1 + b_2; b_2 \quad b_3) \\ &= T(u) + T(v) \end{aligned}$$

(ii) Let 2 K , then u = (a_1; a_2; a_3).

$$\begin{aligned} T(u) &= T(a_1; a_2; a_3) \\ &= (a_1 + a_2; a_2 \quad a_3) \\ &= (a_1 + a_2; a_2 \quad a_3) \\ &= T(u) \end{aligned}$$

Then T is a linear transformation.

Example 8.3. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(a_1, a_2, a_3) = (a_1 - 1, a_2)$. Is T a linear transformation?

Solution: Let $u = (a_1, a_2, a_3)$ and $v = (b_1, b_2, b_3) \in \mathbb{R}^3$. Then

$$u + v = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

$$\begin{aligned} T(u+v) &= T(a_1 + b_1, a_2 + b_2, a_3 + b_3) \\ &= (a_1 + b_1 - 1, a_2 + b_2) \end{aligned}$$

On the other hand,

$$\begin{aligned} T(u) + T(v) &= T(a_1, a_2, a_3) + T(b_1, b_2, b_3) \\ &= (a_1 - 1, a_2) + (b_1 - 1, b_2) \\ &= (a_1 + b_1 - 2, a_2 + b_2) \end{aligned}$$

So, $T(u + v) \neq T(u) + T(v)$, and hence, T is NOT a linear transformation.

Example 8.4. Let $M \in M_{m,m}(K)$ and $N \in M_{n,n}(K)$. Define
$T : M_{m,n}(K) \to M_{m,n}(K)$ by $T(A) = MAN$ for all $A \in M_{m,n}(K)$.
Show that T is a linear transformation.

Solution: Let $A, B \in M_{m,n}(K)$ and $\lambda \in K$.

(i)

$$T(A + B) = M(A + B)N$$
$$= MAN + MBN$$
$$= T(A) + T(B)$$

(ii) $T(\ A) = M(\ A)N = \ (MAN) = \ T(A)$

Then T is a linear transformation.

9 Examples

Example 9.1. Find the eigenvalues and the eigenvectors for the matrix

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Solution: To find the eigenvalues of A we have to solve $\det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 3 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(-1-\lambda) = 0$$

Then $\lambda_1 = 1;\ \lambda_2 = 2$ and $\lambda_3 = -1$ are the eigenvalues for A.

Now, to find the eigenvectors for A, we have to solve the system $(A - \lambda I)x = 0$.

When $\lambda_1 = 1$ then

$$(A - \lambda_1 I)x = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, we have

$$3x_2 = 0$$
$$x_2 + x_3 = 0$$
$$2x_3 = 0$$

Then $x_2 = 0$ and $x_3 = 0$, since x_1 is not exists in the above equations so we can assume that $x_1 = 1$. Hence, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

When $\lambda_2 = 2$ then

$$(A - \lambda_2 I)x = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, we have

$$3x_3 = 0$$
$$x_1 - 3x_2 = 0$$

Then $x_3 = 0$ and $x_1 + 3x_2 = 0$. Let $x_1 = 1$ then $x_2 = -\frac{1}{3}$, and hence, $x = \begin{bmatrix} 1 \\ \frac{1}{3} \\ 0 \end{bmatrix}$

When $\lambda_3 = 1$ then

$$(A - \lambda_3 I)x = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, we have

$$2x_1 - 3x_2 = 0$$
$$3x_2 + x_3 = 0$$

Let $x_1 = 1$ then $x_2 = \frac{2}{3}$, and $x_3 = -2$. Hence, $x = \begin{bmatrix} 1 \\ \frac{2}{3} \\ -2 \end{bmatrix}$.

Example 9.2. Find the eigenvalues and the eigenvectors for the matrix

$$A = \begin{bmatrix} 5 & 0 \\ 2 & 3 \end{bmatrix}.$$

Solution: $A - \lambda I = \begin{bmatrix} 5 & 0 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5-\lambda & 0 \\ 2 & 3-\lambda \end{bmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 0 \\ 2 & 3-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda) = 0.$

Then either $5-\lambda = 0$ or $3-\lambda = 0$. So, the eigenvalues of A are

$$\lambda_1 = 5 \quad \text{and} \quad \lambda_2 = 3.$$

To find the eigenvectors of A, we have to solve the system $(A - \lambda I)x = 0$.

When $\lambda_1 = 5$ then we have to solve the system $(A - 5I)x = 0$.

$(A - 5I)x = \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$

Then $2x_1 - 2x_2 = 0$. Let $x_1 = 1$ then $x_2 = 1$. Hence, $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

When $_2 = 3$ then we have to solve the system $(A \quad 3I)x = 0$.

$$(A \quad 3I)x = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

Then $x_1 = 0$. Let $x_2 = 1$. Hence, $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Example 9.3. Let $B = \{(1,2,-1),(4,1,0),(3,5,7)\}$ be a basis for R^3. Find an orthonormal basis for R^3 by using Gram-Schmidt method.

Solution: $v_1 = (1,2,-1)$, $v_2 = (4,1,0)$, $v_3 = (3,5,7)$.

$$
\begin{aligned}
w_1 &= v_1 = (1,2,-1) \\
w_2 &= v_2 - \frac{w_1 \cdot v_2}{w_1 \cdot w_1} w_1 \\
&= (4,1,0) - \frac{(1,2,-1)\cdot(4,1,0)}{(1,2,-1)\cdot(1,2,-1)}(1,2,-1) \\
&= (4,1,0) - \frac{4+2+0}{1+4+1}(1,2,-1) \\
&= (4,1,0) - (1,2,-1) \\
&= (3,-1,1)
\end{aligned}
$$

$$
\begin{aligned}
w_3 &= v_3 - \frac{w_1 \cdot v_3}{w_1 \cdot w_1} w_1 - \frac{w_2 \cdot v_3}{w_2 \cdot w_2} w_2 \\
&= (3,5,7) - \frac{(1,2,-1)\cdot(3,5,7)}{(1,2,-1)\cdot(1,2,-1)}(1,2,-1) - \frac{(3,-1,1)\cdot(3,5,7)}{(3,-1,1)\cdot(3,-1,1)}(3,-1,1) \\
&= (3,5,7) - \frac{6}{6}(1,2,-1) - \frac{11}{11}(3,-1,1) \\
&= (3,5,7) - (1,2,-1) - (3,-1,1) \\
&= (-1,4,7)
\end{aligned}
$$

Then $W = \{w_1, w_2, w_3\} = \{(1,2,-1),(3,-1,1),(-1,4,7)\}$ is an orthogonal basis for R^3.

$\|w_1\| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$, $\|w_2\| = \sqrt{3^2 + (-1)^2 + 1^2} = \sqrt{11}$

$\|w_3\| = \sqrt{(-1)^2 + 4^2 + 7^2} = \sqrt{66}$

Then $U = \left\{\frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}, \frac{w_3}{\|w_3\|}\right\} = \left\{\frac{1}{\sqrt{6}}(1,2,-1), \frac{1}{\sqrt{11}}(3,-1,1), \frac{1}{\sqrt{66}}(-1,4,7)\right\}$ is an orthonormal basis for R^3.