Subgroups

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Spring term 2011, Lecture 9

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☞ Subgroups.
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Notation

Along with notation from previous lecture, other notations often used in algebra are:

Notation in Lecture 8	Additive notation	Multiplicative notation
$a * b$	$a + b$	ab
e	0	1
a'	$-a$	a^{-1}
$a * a * \cdots * a$ (n times)	na	a^n

Additive notation is used only for abelian groups.

Definition 1. The **order** $|G|$ of a group G is the cardinality of the set G.

Subgroups

A *subgroup* H of a group G is a group contained in G so that if h, $h' \in H$, then the product hh' in H is the same as the product hh' in G. The formal definition of subgroup, however, is more convenient to use.

Definition 2. (Thm 7.10, Sec. 7.3, p. 182)
A subset H of a group G is a **subgroup** if

① $1 \in H$;

② If a, $b \in H$, then $ab \in H$;

③ if $a \in H$, then $a^{-1} \in H$.

Theorem 1. *(Thm 7.11, Sec. 7.3, p. 182)*
If G is finite, then a non-empty $H \subset G$ is a subgroup if $a, b \in H \Rightarrow ab \in H$.
Proof. *In finite G, for any $a \in G$ there exists positive integer k such that $a^k = e$. Hence, for any $a \in H$,*

$$a^{-1} = a^{k-1} \in H \text{ and } a^k = e \in H$$

because $a, b \in H \Rightarrow ab \in H$.

If H is a subgroup of G, we write $H \leq G$; if H is a **proper** subgroup of G, that is, $H \neq G$, then we write $H < G$. G is the **improper** subgroup of G. The subgroup $\{1\}$ is the **trivial subgroup** of G. All other subgroups are **nontrivial**.

Definition 3. (Sec. 7.3, p. 183)
Center of G is the subset in G consisting of all elements which commute with every element in G:

$$Z(G) = \{a \in G \mid ag = ga \quad \forall g \in G\}$$

OBS! G is abelian $\Leftrightarrow Z(G) = G$

Theorem 2. *(Compare with Thm 7.12, Sec. 7.3, p. 183)*
The center $Z(G)$ is abelian subgroup of G.

Proof. Do this as an exercise. For detailed proof see the end of p. 183 in the book. □

Examples of subgroups

Example 1. For any $n \in \mathbb{Z}^+$, we have $(\mathbb{Z}_n,+) < (\mathbb{Z},+) < (\mathbb{Q},+) < (\mathbb{R},+) < (\mathbb{C},+)$.

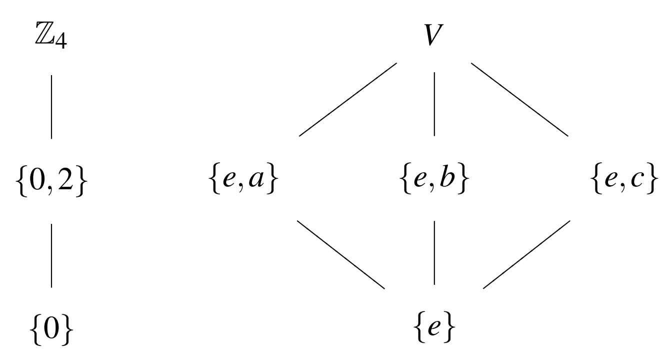
Example 2. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Then, for any $n \in \mathbb{Z}^+$, we have $(U_n,\cdot) < (U,\cdot) < (\mathbb{C}^*,\cdot)$.
Example 3. The set of cardinality 4 may carry exactly two different group structures. The first is $(\mathbb{Z}_4,+)$,

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

while the second is the **Klein 4-group** V (V abbreviates the original German term *Vierergruppe*):

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\mathbb{Z}_4$ has only one nontrivial proper subgroup $\{0,2\}$, while V has three nontrivial proper subgroups, $\{e,a\}$, $\{e,b\}$, and $\{e,c\}$. This is shown at the following *subgroup diagrams*.



Extra info on **Klein four-group**

(See more in Wikipedia article **Klein four-group**)

The Klein four-group is the smallest non-cyclic group. The only other group with four elements, up to isomorphism, is $\mathbb{Z}_4$, the cyclic group of order four (see also the list of small groups).

All non-identity elements of the Klein group have order 2. It is abelian, and isomorphic to the dihedral group of order (cardinality) 4. It is also isomorphic to the direct sum $\mathbb{Z}_2 \bigoplus \mathbb{Z}_2$.

In $2D$ it is the symmetry group of a rhombus and of a rectangle which are not squares, the four elements being the identity, the vertical reflection, the horizontal reflection, and a 180 degree rotation.

In $3D$ there are three different symmetry groups which are algebraically the Klein four-group V:

- one with three perpendicular 2-fold rotation axes: D_2
- one with a 2-fold rotation axis, and a perpendicular plane of reflection: $C_{2h} = D_{1d}$
- one with a 2-fold rotation axis in a plane of reflection (and hence also in a perpendicular plane of reflection):

$$C_{2v} = D_{1h}$$

The three elements of order 2 in the Klein four-group are interchangeable: the automorphism group is the group of permutations of the three elements. This essential symmetry can also be seen by its permutation representation on 4 points:

$$V = \{identity, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$$

In this representation, V is a normal subgroup of the alternating group A_4 (and also the symmetric group S_4) on 4 letters. In fact, it is the kernel of a surjective map from S_4 to S_3. According to Galois theory, the existence of the Klein four-group (and in particular, this representation of it) explains the existence of the formula for calculating the roots of quartic equations in terms of radicals, as established by Lodovico Ferrari: the map corresponds to the resolvent cubic, in terms of Lagrange resolvents.

Another example of the Klein four-group is the multiplicative group $\{1,3,5,7\}$ with the action being multiplication modulo 8.

In the construction of finite rings, eight of the eleven rings with four elements have the Klein four-group as their additive substructure.

Cyclic subgroups

Definition 4. (Sec. 7.3, p. 84)
If G is a group and $a \in G$, write

$$\langle a \rangle = \{ a^n : n \in \mathbb{Z} \}.$$

$\langle a \rangle$ is called the **cyclic subgroup** of G **generated** by a. A group G is called **cyclic** if there exists $a \in G$ with $G = \langle a \rangle$, in which case a is called a **generator** for G.

Obs! The fact that $\langle a \rangle$ is a subgroup of G is an easy exercise stated as **Theorem 7.13.** on page 184 in the book.

Can you prove it yourself NOW in 3 minutes?

Example 4. For any $n \in \mathbb{Z}^+$, U_n is a cyclic group with $\zeta = e^{2\pi i/n}$ as a generator, i.e., $U_n = \langle 1 \rangle$. Because$\mathbb{Z}_n$ is isomorphic to U_n, $\mathbb{Z}_n$ is also a cyclic group with 1 as a generator, i.e., $\mathbb{Z}_n = \langle 1 \rangle$. Check that $\mathbb{Z}_4 = \langle 3 \rangle$.

Example 5. V is *not* cyclic, because $\langle a \rangle$, $\langle b \rangle$, and $\langle c \rangle$ are proper subgroups.

Example 6. $(\mathbb{Z}, +) = \langle 1 \rangle$. For any $n \in \mathbb{Z}$, the cyclic subgroup generated by n, $\langle n \rangle$, consists of all multiples of n, and is denoted by $n\mathbb{Z}$. We have $n\mathbb{Z} = -n\mathbb{Z}$.

Properties of cyclic groups

Definition 5. (equivalent way to define order of element, Theorem 7.14. p. 184 in the book)
Let G be a group, and let $a \in G$. If $\langle a \rangle$ is finite, then the **order** of a is the order $|\langle a \rangle|$ of this cyclic subgroup. Otherwise, we say that a is of **infinite order**.

Theorem 3. *Every cyclic group is abelian.*

Theorem 4. *(Thm 7.16, Sec. 7.3, p. 185)*
Any subgroup H of a cyclic group $G = \langle a \rangle$ is cyclic (and more precisely $H = \langle a^k \rangle$ where $k = min\{k > 0 \mid a^k \in H\}$)

Proof. Let $k = min\{k > 0 \mid a^k \in H\}$. Any m such that $a^m \in H$ can be written by division algorithm in $\mathbb{Z}$ as $m = qk + r, \quad 0 \leq r < k$. Thus $r = m - kq$ and hence $a^r = a^m(a^k)^{-q} \in H$ and therefore $r = 0$ by choice of k as minimal. So, $a^m = (a^k)^q \in \langle a^k \rangle$ and hence $H = \langle a^k \rangle$. □

Corollary 1. *The subgroups of $(\mathbb{Z}, +)$ are $(n\mathbb{Z}, +)$ for $n \in \mathbb{Z}$.*

The structure and generators of cyclic groups and subgroups

Theorem 5 (The structure of cyclic groups, Thm 7.18, Sec. 7.4, p. 193)**.** *Every infinite cyclic group is isomorphic to the group* $(\mathbb{Z},+)$ *and every finite cycle group of order* m *is isomorphic to the group* $(\mathbb{Z}_m,+_m)$*.*

Proof. If $G=\langle a\rangle$ is a cyclic group, then $f(k)=a^k$ defines isomorphism in both cases. For more details see p. 193 in the book. □

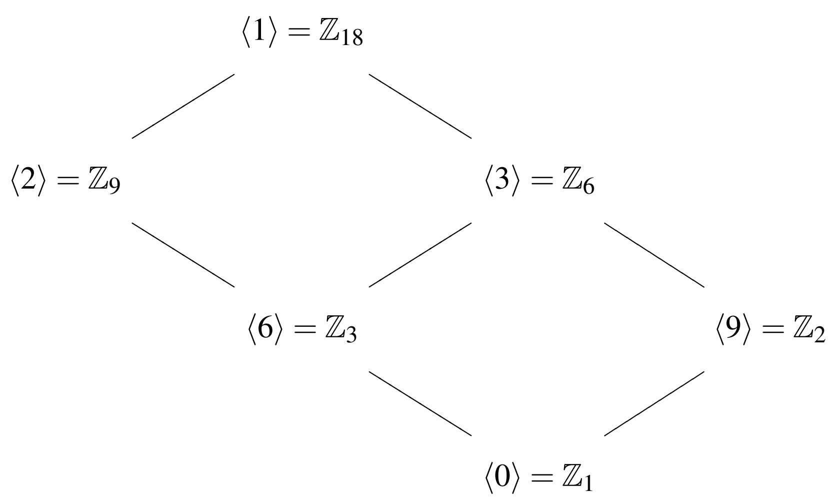
Let $r\in\mathbb{Z}^+$ and $s\in\mathbb{Z}^+$. Let $H=\langle r,s\rangle$ denotes the smallest subgroup in $(\mathbb{Z},+)$ containing both r and s. H is a subgroup of $(\mathbb{Z},+)$. One can prove that $H=\{\,nr+ms\colon\ n,m\in\mathbb{Z}^+\,\}$. By Corollary 1, H has a generator $d\in\mathbb{Z}\setminus\{0\}$, that can be chosen to be positive.

Definition 6. The positive generator d of the cyclic group $H=\{\,nr+ms\colon\ n,m\in\mathbb{Z}^+\,\}$ is called the **greatest common divisor** of r and s.

Definition 7. Two positive integers r and s are **relatively prime** if their greatest common divisor is 1.

Theorem 6. *Let* $G=\langle a\rangle$ *and* $|G|=n$*. Let* $b=a^s\in G$*. Let* d *be the greatest common divisor of* n *and* s*, and let* $H=\langle b\rangle$*. Then* $|H|=n/d$*. In particular,* b *generates all of G if and only if r is relatively prime with* n*.*

Example 7. The following subgroup diagram is obtained from Theorem 6 by direct calculations.



Generating sets

Let $(G,\cdot)$ be a group, and let S be a subset of G.

Theorem 7. *Let $\langle S\rangle$ be the set of elements of G consisting of all products $x_1\ldots x_n$ such that x_i or x_i^{-1} is an element of S for each i, and also containing the unit element. It is the smallest subgroup of G containing S.*

Definition 8. The elements of S are called the **generators** of $\langle S\rangle$. If $\langle S\rangle = G$, we say that S **generates** G. If there exists a finite set S that generates G, then G is **finitely generated**.

Example 8. $(\mathbb{Z},+) = \langle 1\rangle$ is a finitely generated group. Its subgroup $\langle r,s\rangle$ is also generated by one element d, which is the greatest common divisor of r and s.

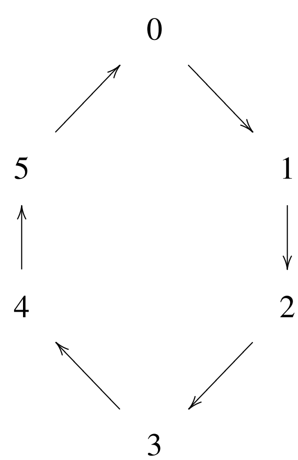
Directed graphs: definition

Definition 9. A **directed graph** (or just digraph) is a finite set of points called **vertices** and some **arcs** (with a direction denoted by an arrowhead or without a direction) joining vertices.

For each generating set S of a *finite* group G, we can construct the following **Cayley digraph** $\mathscr{D}$. The number of vertices in $\mathscr{D}$ is $|G|$. For any $a \in S$, there exist arcs of type a. An arc of type a points from $x \in G$ to $y \in G$ if and only if $y = xa$. If $a \in S$ and $a^2 = e$, it is customary to omit the arrowhead from the arc of type a.

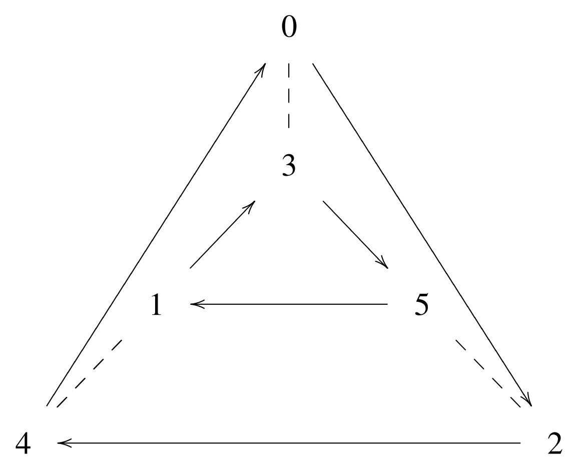
Example: Cayley digraph for $G = \mathbb{Z}_6$ and $S = \{1\}$

Example 9. Let $G = \mathbb{Z}_6$ and $S = \{1\}$. The Cayley digraph has the form



Example: Cayley digraph for $G = \mathbb{Z}_6$ and $S = \{2, 3\}$

Example 10. Let $G = \mathbb{Z}_6$ and $S = \{2, 3\}$. Let $\longrightarrow$ be an arrow of type 2. Because $3^2 = 0$ in $\mathbb{Z}_6$, the arrow of type 3 must be $- - -$. The Cayley digraph has the form



A characterisation of Cayley digraphs

Theorem 8. *A digraph $\mathscr{G}$ is a Cayley digraph of some generating set H of a finite group G if and only if the following four properties are satisfied.*

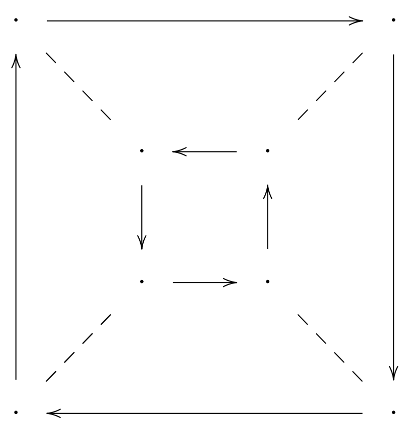
① *$\mathscr{G}$ is connected.*

② *At most one arc goes from vertex g to a vertex h.*

③ *Each vertex g has exactly one arc of each type starting at g, and one of each type ending at g.*

④ *If two different sequences of arc types starting from vertex g lead to the same vertex h, then those same sequences of arc types starting from any vertex u will lead to the same vertex v.*

Cayley used this theorem to construct new groups. For example, the following digraph satisfies all conditions of Theorem 8.



If we label $\longrightarrow$ by a and $---$ by b, we obtain a Cayley digraph of a new group of order 8: