Next, we study some known numerical algorithms those can be used to find the approximate solutions (roots) for non-linear equations, which are Bisection algorithm, Newton–Raphson algorithm and fixed point algorithm.

1. Bisection Algorithm

Let $f \in C[a,b]$,

i.e. f is continuous function on the closed interval $[a,b]$.

Assume that the condition $f(a)f(b) < 0$ is satisfied, we follow the following steps:

1- We bisect the space as follows:

$$c = \frac{a+b}{2} \quad \text{or} \quad c_n = \frac{a_n+b_n}{2}$$

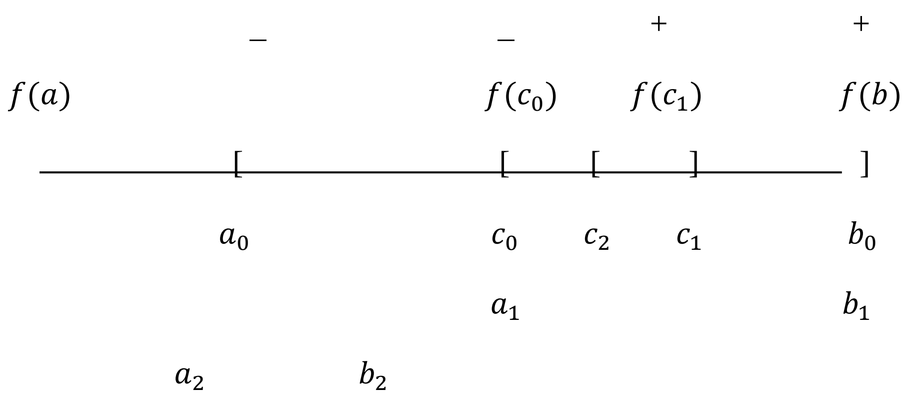
2- If $f(c) = 0$, it follows that c is the exact root, otherwise check the sign of $f(c)$, if $f(c) < 0, f(b) > 0$, then the exact root belong to $[c,b]$, and we set $a = c, b = b$, and then we repeat the same steps.

While, if $f(c) > 0, f(a) < 0$, then the exact root belong to $[a,c]$, and we set $a = a$, $b = c$, and then we repeat the same steps.

3- We continue iteratively, until we get the following condition is satisfied

$$|f(c_n)| < \in, \quad or \quad |c_{n+1} - c_n| < \in,$$

$$or \quad |b_n - a_n| < \in$$



Remarks:

1- From the iterative processes of Bisection algorithm, we get a sequence of closed intervals $I_i = [a_i, b_i]$, and for i<j, the length of $I_j = [a_j, b_j]$ is shorter than the length of $[a_i, b_i]$ and $I_i \supseteq I_j$. Therefore, according to known real analysis theorems, the intersection of all these intervals contains only one point, which is the exact root of $f(x) = 0$.

2- From the iterative processes of Bisection algorithm, we get a sequence of approximate roots, $\{c_n\}$ for the nonlinear equation $f(x) = 0$, which is convergent to the exact root α

i.e. $\{c_n\} \xrightarrow{n\to\infty} \alpha$

Example: consider that, we have the following equation

$f(x) = x^2 - 2 = 0, \qquad x \in [1,2],$

1- Compute the approximate roots of this equation, with using Bisection algorithm, for three iterative steps.

2- Find the iterative errors as each step.

3- Since the exact solution is known for this equation, find also the absolute errors at each steps.

Solution

Clearly, $\alpha = \sqrt{2} \approx 1.414$, is the exact root of f on the interval [1,2]

$f(2) = 2$, , $f(1) = -1$, thus there exists a root in this interval

$a_0 = 1,\ b_0 = 2,\ c_0 = \frac{a_0+b_0}{2} = 1.5,\ \ f(c_1) = 0.25 > 0$

So, $a_1 = 1,\ b_1 = 1.5,\ c_1 = \frac{a_1+b_1}{2} = 1.25$

The iterative errors can be found as follows:

$E_n = |c_{n+1} - c_n|,\quad E_1 = |1.25 - 1.5| = 0.25$

$$c_2 = \frac{a_2 + b_2}{2} = \frac{1.25 + 1.5}{2} = 1.375$$

Absolute error=$|c_n - \alpha|$

n	a_n	b_n	c_n	f(c_n)	E_n	Absolute Errors
0	1	2	1.5	0.25	0.25	0. 086

1	1	1.5	1.25	-0.4375	0.125	0.164
2	1.25	1.5	1.375	-0.1094		0.039

The iterative error, $E_2 = |c_3$-$c_2|$=1.25, clearly, it is, still too large, so we have to continue in the iterative processes until we get the convergent condition,

$$|c_{n+1} - c_n| < \in, \quad \text{is satisfied}.$$

or $|b_n - a_n| < \in$

H.w:- Find the approximate roots of the following equation:

$f(x) = x^3 - 8 = 0,$ on the closed interval [0,3].