3. Newton-Raphson Method

This algorithm can be used to find the approximate toots for the equation $f(x) = 0$, when it easy to find the derivative, f'.

Deriving the Newton-Raphson's formula:

Let $f \in C^2[a,b]$,

i.e. f and f' are continuos functions on $[\mathrm{a},\mathrm{b}]$,

let $\{x_n\}$, is a sequence of approximate roots for f , such that:

$$f'(x_n) \neq 0, \qquad \text{and} \quad |x_n - \alpha| < \epsilon, \qquad \forall\, n$$

where, α is the exact root for $f(x) = 0$.

Use Taylor expansion for f, around x_n , we get

$$f(x) = f(x_n) + (x - x_n)f'(x_n) + \frac{(x-x_n)^2}{2!}f''(x_n) \dots \dots.$$

substitute $x = \alpha$

$$0 = f(\alpha) = f(xn) + (\alpha - x_n)f'(x_n) + (\,small\ terms)$$

$$\alpha - x_n = -\frac{\mathrm{f}(x_n)}{f'(x_n)} \quad \Longrightarrow \alpha = x_n - \frac{\mathrm{f}(x_n)}{f'(x_n)}$$

Set $\alpha = x_{n+1}$

We get the **Newton-Raphson's equation**

$$x_{n+1} = x_n - \frac{f(xn)}{f'(x_n)}, \quad n = 0,1,2, \dots \dots.$$

Remark:- In order to guarantee that, the iterative process is convergent, the initial root , x_0 , should be chosen close to the exact root α, which means:

$$|x_0 - \alpha| < \epsilon$$

Steps of Newton-Raphson algorithm

1- Choose appropriate initial root $x_0 \in [a,b]$

2- Set $n = 0$

3- Calculate $, x_{n+1} = x_n - \frac{f(xn)}{f'(x_n)}$

4- Set $n = n + 1$ and continue in the iterative processes, until we get the stop condition is satisfied:

$$|x_{n+1} - x_n| < \epsilon$$

Example: Use Newton-Raphson algorithm to find the approximate root of the following equation

$$f(x) = \sin x - \frac{(x+1)}{(x-1)}, \quad \text{with } x_0 = -0.2$$

For two iterative steps (only find x_1, x_2). Also, find the iterative error at each step, where $E_n = |x_{n+1} - x_n|$

solution

$n = 0$

$$f'(\mathrm{x}) = \cos \mathrm{x} - \frac{(x-1)-(x+1)}{(x-1)^2} = \cos x + \frac{2}{(x-1)^{2}}$$

$$f(x_0) = \sin(-0.2) - \frac{(-0.2+1)}{(-0.2-1)} = (0.42) = 0.4680$$

$$f'(x_0) = \cos(-0.2) + \frac{2}{(-0.2-1)^2} = 2.3690$$

Thus

$$x_1 = x_0 - \frac{f(x0)}{f'(x_0)} = (-0.3976)$$

$n = 1$

$$f(x_1) = 0.0439$$

$$f'(x_1) = 1.9460$$

$$x_2 = x_1 - \frac{f(x1)}{f'(x_1)} = (-0.4201)$$

By the same way, we can find $x_3 = -0.4204$, $x_4 = -0.4203$

We continue, iteratively, until we get the stop condition is satisfied:

$$|x_{n+1} - x_n| < \epsilon$$

N	x_n	$f(xn)$	$f'(x_n)$	E_n
0	-0.2000	0.4680	2.3690	0.1976
1	-0.3976	0.0439	1.9460	0.0225
2	-0.4201			0.0003
3	-0.4204			0.0001
4	-0.4203			