Fixed Point Algorithm

This method depends on the concept of fixed points for one variable functions

Definition :- The point x_0 which belongs to the domain of the function g is called a **fixed point** for g , iff $g(x_0) = x_0$

We can give the idea of this algorithm as follows:

We write the function f, where $f(x) = 0$ as follows:

$$f(x) = x - g(x)$$

such that , α is a root for f ,thus $f(\alpha) = 0$,

which means that, α is a fixed point for g, that is

$$\alpha - g(\alpha) = 0 \rightarrow g(\alpha) = \alpha$$

Therefore, the problem becomes, we have to look for the fixed point of g rather than, looking for the root of f.

Fixed point Theorem:-

Let $g \in c[a,b]$, $g(x) \in [a,b]$, $\forall x \in [a,b]$

Then g has a fixed point on [a,b], moreover, if g' exists on (a,b), such that

$$|g'(x)| \leq k \leq 1$$

Then g has a unique fixed point , $p \in [a,b]$.

Remark :- when we choose a certain form for g , we have to make sure that

$$|g'(x)| \leq 1, \forall\, x \in (a,b)$$

This condition can guarantee the convergence for the algorithm.

Fixed point algorithm steps

1-Choose the initial root x_0

2-Choose a form for the function g, *such that*

$$|g'(x)| \leq 1, \forall\, x \in (a,b),$$

3- Set $x_1 = g(x_0)$, $x_{n+1} = g(x_n)$, $n = 1,2,3 \ldots ..$

4- We continue in the iterative process until we get:

$$|x_{n+1} - x_n| < \in$$

Example: - Find the approximate root of the following equation $f(x) = x^2 + 4x - 10 = 0$, on the interval [1, 3.5], with considering that $x_0 = 3/2$, for two iterative steps (find only x_1, x_2).

Solution

Let us choose two forms for g as follows:

$$g_1(x) = x = \frac{10}{x+4} \ldots \ldots (1)$$

$$g_2(x) = x = \frac{10-x^2}{4} \ldots\ldots..(2)$$

$$g'_1(x) = \frac{-10}{(x+4)^2}$$

$$g'_2(x) = -\frac{x}{2}$$

It is clear that, $|g'_1(x)| \le 1$, $\forall x \in (1,3.5)$,

while $|g'_2(3)| = 3/2 > 1$

Therefore, we ignore g_2 and choose the convergent form $g = g_1 = \frac{10}{x+4}$

Next, we find $x_1, x_1, \ldots\ldots$

$$x_1 = g(x_0) = \frac{10}{\left(\frac{3}{2}\right) + 4} = \frac{10}{\frac{3+8}{2}} = \frac{20}{11} = 1.8182$$

$$x_2 = g(x_1) = \frac{10}{\left(\frac{20}{11}\right) + 4} = \frac{10}{\frac{64}{11}} = \frac{110}{64} = \frac{55}{32} = 1.7188$$

We continue, with n=3,4,…..until 29.

Where $x_{29} = 1.7416$ and $|x_{29} - x_{28}| < \in$

H.W. For last example find the iterative errors for three steps.

H.W. :- Find the approximate root of the following equation

$f(x) = cos\, x - xe^x = 0$, on the interval [0.25, 0.75], consider $x_0 = 0.5$