A. Relations

1. A **relation** is a set of ordered pairs. For example,

   $A = \{(-1, 3),\ (2, 0),\ (2, 5),\ (-3, 2)\}$

2. **Domain** is the set of all first coordinates: $\{-1, 2, 2, -3\}$

   so $dom(A) = \{-1, 2, -3\}$

3. **Range** is the set of all second coordinates: $\{3, 0, 5, 2\}$

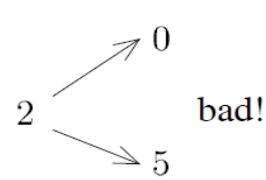
   so $rng(A) = \{3, 0, 5, 2\}$

B. Functions

A **function** is a relation that satisfies the following:

each x-value is allowed only **one** y-value

Note: A (below) is **not** a function, because 2 has y-values 0 and 5 **(violates our condition!)**



C. Testing Relations To See If They Are Functions

We make a "mapping table". We do this as follows:

1. List all the x-values on the left.
2. At each x-value, draw an arrow–one arrow pointing to each y-value it has.
3. If you see a situation where an x-value has two or more arrows branching to y-values, then it is **not** a function.

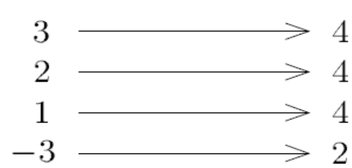
Examples:

Check to see if the following relations are functions:

$$B = \{(3,4),\ (2,4),\ (1,4),\ (-3,2)\}$$
$$C = \{(1,2),\ (-2,3),\ (5,1),\ (1,4)\}$$

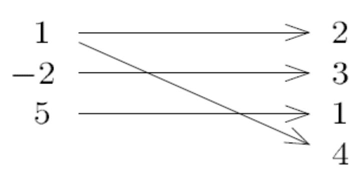
Solution

Make a mapping table for B:



Thus we see that B is a function.

Make a mapping table for C:



Thus we see that C is **not** a function!

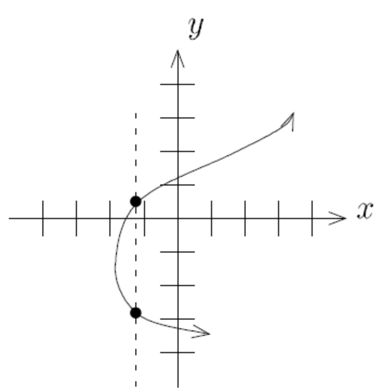
C. Graphs and Functions

To check to see if a **graph** determines a function, we apply the **Vertical Line Test**.

Vertical Line Test:

If a vertical line moved over allowed x-values intersects the graph exactly once (each time), the graph is a function; otherwise; it is not.

Example:



not a function!

Functions and its Algebra:

A function from a set A to set B is a rule that assigns each element $x \in A$ to only one element in B satisfy

$$f: A \to B \quad iff \ \forall x \in A \quad \exists! \quad y \in B \ , f(x) = y$$

The domain:

The set of all x ∈ first component occurring in the ordered pairs of f is called the domain of f , and it is denoted by dom (f)

Dom (f)=$\{x \in A\, ; (x, y) \in f \ , from\ some\ y \in B\}$

The codomain:

The codomain is the set of the second component occurring in the ordered pairs of f (all element in set B),and it is denoted by cod_f

$$cod_f = \{y: y \in B\}$$

The Range:

The Range of f is a subset of the codomain of f whose elements are assigned by elements of the domain of f, and it is denoted by R_f

$$R_f = \{f(x); x \in D_f\}$$

EX: find the domain of the functions:

1- $y = x + 2 \quad \Rightarrow D_f = R$

2- $y = \frac{x}{x^2-1} \Rightarrow x^2 - 1 = 0 \quad \Rightarrow x^2 = 1 \Rightarrow x = \mp 1 \Rightarrow D_f = R/\{1, -1\}$

3- $y = \sqrt{2x-1} \Rightarrow 2x - 1 \geq 0 \Rightarrow 2x \geq 1 \Rightarrow x \geq \frac{1}{2} \Rightarrow D_f = \left\{x: x \geq \frac{1}{2}\right\}$

4- $y = \frac{1}{\sqrt{1-x^2}} \Rightarrow 1 - x^2 > 0 \Rightarrow 1 > x^2 \Rightarrow x < \mp 1 \Rightarrow D_f = (-1,1)$

Some operations on functions:

1-$(f+g)(x)=f(x)+g(x)$; $x\in D_f\cap D_g \quad \Rightarrow d(f+g)=D_f\cap D_g$

2- $(f-g)(x)=f(x)-g(x)$; $x\in D_f\cap D_g \quad \Rightarrow d(f-g)=D_f\cap D_g$

3-$(f.g)(x)=f(x).g(x)$; $x\in D_f\cap D_g \quad \Rightarrow d(f.g)=D_f\cap D_g$

4-$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$; $x\in D_f\cap D_g$ $,\{x:g(x)\neq 0\}$

$$\Rightarrow d\left(\frac{f}{g}\right)=D_f\cap D_g \quad -\{x:g(x)=0\}$$

Ex: $let\ f(x)=\sqrt{x}$, $g(x)=\sqrt{1-x}$, find $f+g$, $f-g$, $f.g$, $\frac{f}{g}$, $\frac{g}{f}$,and their domains.

Sol: $D_f=[0,\infty)$, $D_g=(-\infty,1]\rightarrow \quad d(f+g)=D_f\cap D_g \quad =[0,1]$

$(f+g)(x)=\sqrt{x}+\sqrt{1-x}$

$(f-g)(x)=\sqrt{x}-\sqrt{1-x}$

$(f.g)(x)=\sqrt{x}\,.\sqrt{1-x}=\sqrt{x(1-x)}=\sqrt{x-x^2}$

$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{\sqrt{x}}{\sqrt{1-x}}=\sqrt{\frac{x}{1-x}}$

$\left(\frac{g}{f}\right)(x)=\frac{g(x)}{f(x)}=\frac{\sqrt{1-x}}{\sqrt{x}}=\sqrt{\frac{1-x}{x}}$

composition of function

Let f and g be two functions, if the range of g is a subset of the domain of f, then there is a function fog defined as follow:

$$fog(x)=f(g(x))$$

$$d(fog)=\{x:g(x)\in D_f \ ,x\in D_g\}$$

$$d(gof)=\{x:f(x)\in D_g \ ,x\in D_f\}$$

Note: In general

$$fog\neq gof$$

Ex: Let $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$, find fog , gof , d_{fog} , d_{gof} .

Sol: $fog(x) = f\big(g(x)\big) = f(x^2 - 1) = \sqrt{x^2 - 1}$

$gof(x) = g\big(f(x)\big) = g\big(\sqrt{x}\big) = (\sqrt{x})^2 - 1 = x - 1$

$d_f = x \geq 0$, $d_g = \mathbb{R}$

$d(fog) = \big\{x : g(x) \in D_f , x \in D_g\big\} \Rightarrow x \in D_g = \mathbb{R}$

$g(x) \in D_f \Rightarrow x^2 - 1 \geq 0 \Rightarrow (x-1)(x+1) \geq 0$

1- X-1$\geq 0 \wedge x + 1 \geq 0$

X$\geq 1 \quad \wedge \quad x \geq -1 \quad \rightarrow \{x : x \geq 1\}$

2- X-1$\leq 0 \quad \wedge x + 1 \leq 0$

X$\leq 1 \quad \wedge \quad x \leq -1 \quad \rightarrow \{x : x \leq -1\}$

$d(fog) = \{x : x \leq -1, x \geq 1\} = \mathbb{R} \backslash (-1,1)$

$d(gof) = \big\{x : f(x) \in D_g , x \in D_f\big\}$

$= \{x : x \geq 0 , x \in \mathbb{R}\} = [0, \infty)$

Homework Assignments:

1- find the domain, codomain, and the Range of the following functions:

A- $y = \sqrt{x}$ B- $y = \frac{1}{\sqrt{X}}$ C- $y = \sqrt{x^2 - 9}$ D- $y = \sqrt{1 - x^2}$

2. Which of the following tables represent functions?

I.

Input	Output
0	1
0	2
2	3
3	4

III.

Input	Output
−3	2
−2	−2
2	−3
3	3

II.

Input	Output
0	0
1	0
2	1
3	1

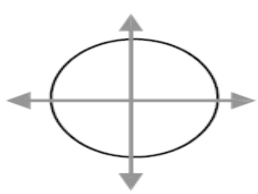
IV.

Input	Output
2	−11
0	8
2	4
4	7

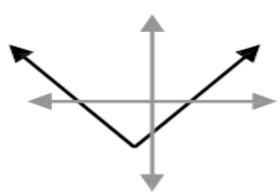
A. I and IV only
B. II and III only
C. I, II and III only
D. II, III and IV only

3. Determine which of the following are functions:

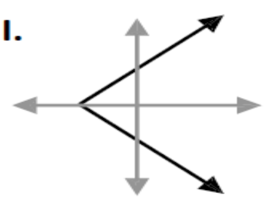
I.



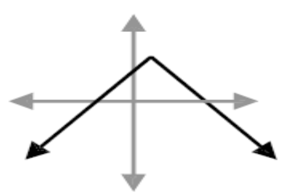
II.



III.



IV.



A. I and III only
B. II and IV only
C. II, III, and IV only
D. I, II, III and IV

4- $Find\ f+g\ ,\ f-g\ ,f.g,\ \frac{f}{g}\ ,\frac{g}{f}$,and their domains of the following functions:

A- $f(x)=\sqrt{x+1}\ ,\quad g(x)=\sqrt{4-x^2}$

B- $f(x)=\sqrt{x^2-1}\ ,\quad g(x)=x^2$

C- $f(x)=x^2+1\ ,\quad g(x)=\sqrt{2x-3}$

5- find $fog\ ,\ gof\ ,d_{fog}\ ,d_{gof}$ of the following functions:

A- $f(x)=\sqrt{2-x},\quad g(x)=\sqrt{x-2}$

B- $f(x)=\sqrt{x^2-1}\ ,\quad g(x)=x^2$

C- $f(x)=x^2+1\ ,\quad g(x)=\sqrt{2x-3}$

D- $f(x)=|x|\ ,\quad g(x)=-x$

E- $f(x)=\frac{x}{x+2}\ ,\quad g(x)=\frac{x-1}{x}$