Error Analysis for Iterative Methods

Definition 1:- Suppose $\{x_n\}$ is a sequence that can be get from using any iterative method (such as Bisection, N.R., Fixed point) and assume that it converges to the exact root α, and let $e_n = |x_n - \alpha|$ is the absolute error at the iterative step n.

If there exists λ and k such that

$$\lim_{n\to\infty} \frac{e_{n+1}}{{e_n}^k} = \lambda$$

Then $\{\alpha_n\}$ is said to convergence to α of order k with constant error λ

In general, a sequence with a high order of convergence will converge more rapidly than a sequence with low order.

In this lectures notes, two cases of order will be given special attention

1-if $\mathrm{k} = 1, \textit{the method is called linear}$

2-if $\mathrm{k} = 2, \textit{the method is quadratic}$

Theorem 1:- Fixed point algorithm exhibits linear convergence.

In order to prove the last theorem, we need to recall mean value theorem:

Mean value theorem:

Let $h \in C[a,b]$, $(h \text{ and } h' \text{ are continous functions on } [a,b])$. Then exists $c \in (a,b)$ $\textit{such that}$ $h(b) - h(a) = h'(c)(b-a)$.

To prove **Theorem 1**, suppose we would like to find the approximate solution to $f(x) = 0$, on $[a,b]$ using fixed point algorithm with $x_0 \in [a,b]$,

$$x = g(x), \text{ such that, } 0 < |g'(x)| \le M < 1 \quad \forall\, x \in [a,b],$$

which means we have the iterative scheme $x_{n+1} = g(x_n)$, $\forall n \ge 0$

Thus, according to fixed point Theorem, g has a unique fixed point α (the exact root of $f(x) = 0$), where $\alpha \in [a,b]$, & $\{x_n\} \xrightarrow{n\to\infty} \alpha$.

By mean value theorem , we get

$$e_{n+1} = |x_{n+1} - \alpha| = |g(x_n) - g(\alpha)| = |g'(z_n)||x_n - \alpha| = \lambda_n e_n \ldots\ldots(1)$$

where $x_n \leq z_n \leq \alpha, \ \lambda_n = |g'(z_n)|, \quad \forall\, n$

since $\{x_n\} \xrightarrow{n\to\infty} \alpha$, we get $\{z_n\} \xrightarrow{n\to\infty} \alpha$

and since g' is continuos **,** by the definition of continuity, we get

$$\lambda_n = |g'(z_n)| \xrightarrow{n\to\infty} |g'(\alpha)| = \lambda < 1$$

Thus $$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = \lim_{n\to\infty} \lambda_n = \lambda \ldots\ldots\ldots\ldots (2)$$

Therefore, from equations (1) & (2) and by definition 1, we conclude that, fixed point iteration algorithm exhibits linear convergence.

Theorem 2: Newton Raphson Algorithm converges quadratically

In order to prove Theorem 2, we need to state first the following theorem without giving its proof.

Theorem:- Let α is a solution of $x = g(x)$ such that $g'(\alpha) = 0$ and g'' is continuous on an open interval contains α . Then there exists $\delta > 0$ such that for $x_0 \in [\alpha - \delta, \alpha + \delta]$, the sequence defined by

$x_{n+1} = g(x_n), \ \forall n \geq 0$ is quadratically convergent .

Proof of Th. 2: From using N.R. algorithm to find the approximate root of the equation $f(x) = 0$, where $f \in C^2[a,b]$ and $f'(x) \neq 0, \forall\, x \in [a,b]$, we get the iterative scheme:

$$x_{n+1} = g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \forall n \geq 0,$$

i.e. $$g(x) = x - \frac{f(x)}{f'(x)}$$

Thus $g'(x) = 1 - \frac{f'^2(x) - f(x)f''(x)}{f'^2(x)} = 1 - 1 + \frac{f(x)f''(x)}{f'^2(x)} = \frac{f(x)f''(x)}{f'^2(x)}$

which leads to $g'(\alpha) = \frac{f(0)f''(0)}{f'^2(0)} = 0$

Therefore, by last Theorem, if we choose $x_0 \in [\alpha - \delta, \alpha + \delta]$, then N.R. scheme is is quadratically convergent