Chapter 3

The Numerical Solutions of Linear Systems

It is well known from the linear algebra that, that there are many methods used to find the exact solutions of linear systems, $Ax = b$, where $A \in R^{n\times n}$, $b \in R^{n\times 1}$, such as Gauss elimination or, Gauss-Gordan, or Kramer's method. But using these methods becomes so difficult when the dimension n, of the matrix A, is large. Therefore, we need to compute the solutions numerically by using computers.

In general, there are two types of numerical methods, which can be used to find the numerical solutions of linear systems: **direct methods** and **indirect methods**.

Before starting to study these methods, let us revise some equivalent algebraic facts of the linear system: $Ax = b$, $A \in R^{n\times n}$, $b \in R^{n\times 1}$

1-The homogenous system $Ax = 0$, has only zero solution, iff $|A| \neq 0$, which means A is nonsingular matrix.

2-The linear system $Ax = b$, has a unique solution , iff $|A| \neq 0$, which means A is nonsingular matrix.

From above, before to think about finding a numerical solution of a linear system, we need to make sure that the exact solution exists, which means we should check whether $|A| \neq 0$, or A is nonsingular.

Stability of Linear systems

In some linear systems, $Ax = b$, small change on the elements of A or b, leads to big changes on the solution , x, of this system. We call these types of systems ill condition systems, while if any small change on A or b does not make big change on the solution , x, in this case the system is called well condition.

Example

$$x_1 + 2x_2 = 3$$

$$1.0001X_1 + 2X_2 = 3.0001$$

It is easy to show that the exact solution of this system is

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

While, If we make small change on the matrix A as follows

$$x_1 + 2x_2 = 3$$

$$0.99999x_1 + 2x_2 = 3.0001$$

We can show that, the new exact solution becomes

$$\text{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Therefore, we conclude that this system is not stable (ill condition), because small changes on **A** lead to big changes on the solution **X**.

Since when we compute the approximate solutions by using computers, we get some changes on the matrix ***A*** and the vector ***b*,** throughout iterative steps. And that causes big errors on solutions for ill condition systems. Therefore, it is important to check whether the system is stable or not, before starting to compute the numerical solutions.

The condition number

Definition :- let A^TA is a square matrix and let $\lambda_1, \lambda_2, \dots \dots, \lambda_n$ are the eigenvalues of A , then the norm of A and A^{-1} can be defined as follows, $||A||_2 = \sqrt{|\lambda_L|}$,

$$||A^{-1}||_2 = \frac{1}{\sqrt{|\lambda_S|}}$$

Where λ_L and λ_S are the smallest and the biggest eigenvalues of A, respectively

And the condition number $of\ A\ is\ K(A) = \left||A^{-1}|\right|_2 \left||A|\right|_2 = \sqrt{\frac{|\lambda_L|}{|\lambda_S|}}$,

If K(A) is big then the system $Ax = b$ is stable, while, if $K(A)$ is small , the system is stable.

Example: - Study the stability of the following linear system**.**

$$2x_1 + 6x_2 = 2$$

$$5x_1 + 3x_2 = 3$$

Solution

A=$\begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$, $A^T = \begin{bmatrix} 2 & 5 \\ 6 & 3 \end{bmatrix}$

$$\text{A}A^T = \begin{bmatrix} 40 & 28 \\ 28 & 34 \end{bmatrix}$$

The eigenvalues of AA^T are 8.8397 , 65.1603

$$K(A) = \sqrt{\frac{|\lambda_L|}{|\lambda_S\,|}} = \sqrt{\frac{65.1603}{8.8397}} = 2.$$

Since $K(A)$ is large, the system above is unstable.