Direct methods

These methods can be convergent very fast, but when the dimension of A is large, it is not recommended to use these methods because, we need to compute lots of mathematical operations, which means, the errors become bigger.

Next, we will study some of direct methods.

Gauss Elimination Algorithm

The idea of this method, is to convert A in the linear system $Ax = b$, to upper U or lower matrix L. Thus, we get a lower or upper triangular system:

$$Ux = b_1 \text{ or } Lx = b_2$$

It is clear that, for solving lower triangular system we use **Forward substitutions** and for solving upper triangular system we use **Backward substitutions**.

In fact, solving lower (upper) triangular system is easier than solving the original system.

Steps of Gauss elimination algorithm:

1- Write the system $Ax = b$, in the matrix form [A: b].

2- Convert [A: b] to appear triangular form $[L: b_1]$ or lower triangular from $[U: b_2]$

3-Solve the lower (upper) triangular system, $Lx = b_1$ $(Ux = b_2)$, by using forward (backward) substitutions.

Example:- Solve the following linear system by using Gauss algorithm

$$5x_1 + 2x_2 = 3$$
$$4x_1 + 3x_2 = 1$$

Solution

Firstly, we write the system in matrix form $[A: b]$, as follows

$$\begin{bmatrix} 5 & 2 & : & 3 \\ 4 & 3 & : & 1 \end{bmatrix}$$

$(L_1)/5$, $-4(L_1/5)+L_2$

$$\begin{bmatrix} 1 & 2/5 & : & 3/5 \\ 0 & 7/5 & : & -7/5 \end{bmatrix}$$

Thus, we get the following **upper triangular system**

$$x_1 + (2/5)x_2 = 3/5$$
$$(7/5)x_2 = -7/5$$

Finally, we solve the last system by using the backward substitutions, to get

$$x_2 = (-7/5)/(7/5) = -1$$
$$x_1 = 3/5 + 2/5 = 5/5 = 1$$

Thus, the solution is

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

H.W. In the last example, try to change the system $Ax = b$, to a lower triangular system and find its solution by using the forward substitutions.

Gauss- Gordan algorithm

This algorithm is considered, as a modified to the Gauss elimination algorithm. We convert the linear system, $Ax = b$, to $I_n x = b_1$, where I_n is the identity matrix of order n. Thus to solve the last system we use the direct substitutions.

$$Ax=b \longrightarrow I_n x = b_1$$

Steps of Gauss- Gordan algorithm

1- Write the system $Ax = b$, in the matrix form [A: b].

2- Do some mathematical operations to convert [A: b] to the diagonal form $[I_n : b_1]$.

3- Find x by solving the system $I_n x = b_1$ using direct substitutions.

Example :- Find the solution of the following system, by using Gauss-Gordan Method

$$x_1 + 2x_2 = 1$$
$$2x_1 + x_2 = 3$$

Solution

Firstly, we write the system in matrix form $[A: b]$, as follows:

$$\begin{bmatrix} 1 & 2 & : & 1 \\ 2 & 1 & : & 3 \end{bmatrix} \ldots\ldots(1) \qquad -2(L_1)+L_2 \begin{bmatrix} 1 & 2 & : & 1 \\ 0 & -3 & : & 1 \end{bmatrix} \ldots\ldots..(2)$$

$$-(L_2/3) \begin{bmatrix} 1 & 2 & : & 1 \\ 0 & 1 & : & -1/3 \end{bmatrix} \ldots\ldots(3)$$

$$-2(L_2)+L_1\begin{bmatrix}1 & 0 & : & 5/3\\ 0 & 1 & : & -1/3\end{bmatrix}\ldots\ldots(4)$$

Thus, we get

$$I_n x = b_1$$
$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}5/3\\ -1/3\end{bmatrix}$$