LU Algorithm :

This method is called by LU, because the matrix A in the linear system $Ax = b$, decomposes into the multiplication of two matrices : lower L and upper U, and this decomposition works for any vector b, which means $A = LU$.

Thus, we get lower triangular and upper triangular systems.

In order to get the solution of the system $Ax = b$, we need to solve these two systems.

Steps of LU Method

1- Decompose A, in the form $A = LU$, where U is an upper matrix and L is a lower matrix.

2- Set Ux=y, which leads to Ly=b

3-Solve first the lower triangular system, Ly=b, using the *forward substitutions* to get y, and then solve the lower triangular system, Ux=y, using the *backward substitutions* to get x .

Remarks:-

1-This method can be considered better than Gauss and Gauss-Gordan methods and that because the decomposition of the matrix A works for any vector b, while in Gauss and Gauss - Gordan, the mathematical operations, which we have to do on [A:b], should be redone again when we choose another vector b.

2-In fact, not any matrix A can be decomposed to LU, unless the following condition (**The diagonal control condition),** is satisfied.

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|, \; i=1,2,..n$$

Example:- Can we decompose the matrix A to LU ?, where

$$A=\begin{pmatrix} 3 & 1 & 1 \\ 5 & 6 & 0 \\ 0 & 1 & 7 \end{pmatrix}$$

Solution

$$2 = |a_{12}| + |a_{13}| \leq |a_{11}| = 3$$
$$5 = |a_{21}| + |a_{23}| \leq |a_{22}| = 6$$
$$1 = |a_{31}| + |a_{32}| \leq |a_{33}| = 7$$

Since, A satisfies the diagonal control condition

Therefore, it follows that A can be decomposed to LU.

The following example shows how to use LU algorithm to solve a linear system.

Example:- Use LU Algorithm to find the solution of the following system.

$$x_1 + 2x_2 = 3$$
$$3x_1 + x_2 = 5$$

Solution

Set

$$A=\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}}_{U}$$

$u_{11} = 1, u_{12} = 2,\ \ 3u_{12} + u_{22} = 1 \qquad u_{22} = 1 - (3)(2) = -5$

It follows that

$L=\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \quad u = \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$

Set $Ux = y\ , Ly = b$

We need to solve first the system $[Ly = b]$ by using Forward substitutions

$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, thus $\quad y_1 = 3$

$y_2 = 5 - (3)(3) = -4 \qquad y=\begin{bmatrix} 3 \\ -4 \end{bmatrix}$

Secondly, we solve the system $[Ux = y]$, by using Backward substitutions

$\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \qquad x_2 = -4/(-5) = 4/5$

$x_1 = 3 - 2(4/5) = 3 - 8/5 = 7/5$

Thus $\qquad x = \begin{bmatrix} 7/5 \\ 4/5 \end{bmatrix}$

H.W. Consider that, we have the following linear system

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 3x_1 \qquad + 3x_3 &= 0 \\ 2x_1 + 5x_2 + x_3 &= 1 \end{aligned}$$

Solve the system by using,

1- Gauss elimination (with Backward or Forward substitutions).
2- LU algorithm.