Indirect methods

In these methods, we don't need to do lots of matrix operations as in the direct methods, but it is known that indirect methods are slower than direct methods in convergence. Moreover, the main different between direct and indirect methods that indirect methods needs an initial solution, $x^0 = (x_1^0, x_2^0, \dots x_n^0)$, in order to start and depending on this initial condition, we can get $x^1, x^2, \dots \dots$

Therefore, indirect methods are also called the *iterative methods*.

We will study two algorithms of indirect methods: *Jacobi & Gauss-Sidel*

Jacobi iterative algorithm

Consider that, we have the following the linear system:

$$A_{n\times n}x = b_{n\times 1}$$

which can be written as follows:

$a_{11}x_1+a_{12}x_2+\dots\dots\dots+a_{1n}x_n=b_1$

$a_{21}x_1+a_{22}x_2+\dots\dots\dots\dots+a_{21}x_n=b_2$

$a_{n1}x_1+a_{n2}x_2+\dots\dots\dots\dots+a_{nn}x_{n=b_n}$

where $a_{\text{ii}} \neq 0, \quad i = 1,2, \dots . n$

Remarks:

1- In case of $a_{\text{ii}} = 0$, for some i, we replace the equation number i, with another equation in order to get $a_{\text{ii}} \neq 0, \quad i = 1,2, \dots . n$

2- On the other hand, in order to get faster convergence, we should make sure that, **The diagonal control condition** is satisfied

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$$

In case of this condition is not satisfied, we can also switch places between the equations, until we get this condition is satisfied.

Steps of Jacobi algorithm

1-Make sure that the system has a unique solution, $|A| \neq 0$.

2-Choose the initial solution $x^0 = (x_1^0, x_2^0, \ldots\ldots x_n^0) \in R^n$

3-Make sure that $a_{ii} \neq 0, and\ |a_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$

and in case of one of these condition is not satisfied, switch places of the equations, until we get the two conditions are satisfied.

4- Write the system in iterative form: $x_i^{k+1} = (b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}\ x_j^k)/a_{ij}$, for

$$i = 1,2, \ldots . n$$

5-Compute the solution iterative iteratively, (for $k = 0,1,2, \ldots$), until we get the following the stop condition is satisfied:

$\|x^{(k+1)} - x^{(k)}\| < \in$, where

$$E_k = \|x^{(k+1)} - x^{(k)}\| = \max_{1\leq i\leq n} |x_i^{k+1} - x_i^k|$$

Example:- Use Jacobi algorithm to solve the following linear system, for two iterative step and find the iterative error at each step.

Note: The exact solution for this system is x=(1; 2; 3)

$$\begin{aligned} 4x_1 + 2x_2 + x_3 &= 11 \\ 2x_1 + x_2 + 4x_3 &= 16 \\ -x_1 + 2x_2 \quad &= 3 \end{aligned}$$

Solution

Since $a_{33} = 0$, we need to switch the places of equation 2 and 3:

$$\begin{aligned} 4x_1 + 2x_2 + x_3 &= 11 \\ -x_1 + 2x_2 \quad &= 3 \\ 2x_1 + x_2 + 4x_3 &= 16 \end{aligned}$$

It is clear that, the last system satisfies the **diagonal control condition.**

We can rewrite the last system as follows:

$$x_1 = \frac{11}{4} - \frac{1}{2}x_2 - \frac{1}{4}x_3$$

$$x_2 = \frac{3}{2} + \frac{1}{2}x_1$$

x_3=4 - $\frac{1}{2}\,x_1$ - $\frac{1}{2}\,x_2$

We write the system in the iterative form as follows:

$x_1{}^{k+1} = \frac{11}{4} - \frac{1}{2}x_2{}^{k} - \frac{1}{4}\,x_3{}^{k}$

$x_2^{k+1} = \frac{3}{2} + \frac{1}{2}\,x_1^k$ k=0,1,…….

$x_3^{k+1} = 4 - \frac{1}{2}x_1^k - \frac{1}{4}x_2^k$

Let x^0=(x_1^0,x_2^0,x_3^0)=(0.8, 1.8, 2.8)

Set k=0

$x_1^{(1)} = \frac{11}{4} - \frac{1}{2}(1.8) - \frac{1}{4}(2.8) = 1.15$

$x_2^{(1)} = \frac{3}{2} + \frac{1}{2}(0.8) = 1.9$

$x_3^{(1)} = 4 - \frac{1}{2}(0.8) - \frac{1}{4}(1.8) = 3.15$

$x^{(1)} = (1.15, 1.9, 3.15)$

$$E_1 = \max\{\,|1.15 - 0.8|, |1.9 - 1.8|, |3.15 - 2.8|\} = 0.35$$

set $k = 1$, by using the same way obtain:

$x_1^{(2)} = \frac{11}{4} - \frac{1}{2}(1.9) - \frac{1}{4}\,(3.15) = 1.0125$,

$x_2^{(2)} = \frac{3}{2} + \frac{1}{2}\,(1.15) = 2.0750$,

$x_3^{(2)} = 4 - \frac{1}{2}(1.15) - \frac{1}{4}(1.9) = 2.95$

$x^{(2)} = (1.0125, 2.075, 2.95)$

$$E_2 = \max\{|1.0125 - 1.15|, |2.075 - 1.9|, |2.95 - 3.15|\} = 0.175$$

We continue iteratively, until the convergent condition can be satisfied:

$||x^{(k+1)} - x^k|| < \in$