Gauss – Sidel algorithm

The main difference between this method and Jacobi method, is for any iterative step ,k, the new approximate values of the x_j, $\mathrm{j}=1,\ldots \mathrm{i}-1$, are used directly, to compute the approximate values of x_i, while in Jacobi we don't use the new approximate values x_j until, we consider the next iterative step, $k+1$.

Remark: The steps of Gauss-Sidel algorithm are the same as the step of Jacobi method expect for in step (4), we use the Gauss-Sidel iterative system:

$$x_i^{k+1} = (\mathrm{bi} - \sum_{\substack{\mathrm{j}=1 \\ \mathrm{j}\neq\mathrm{i}}}^{i-1} \mathrm{a}_{ij}\ x_j^{k+1} - \sum_{\substack{\mathrm{j}=\mathrm{i} \\ \mathrm{j}\neq\mathrm{i}}}^{n} \mathrm{a}_{ij}\, x_j^{k})\ /a_{ii}$$

for $i = 1,2,3,\ldots n, k = 0,1,2,\ldots..$

Example:- Use Gauss-Sidel algorithm to find the approximate solution to the following system for two iterative steps, with x^0=(0.8, 1.8, 2.8), and find the iterative errors at each step.

$$4x_1 + 2x_2 + x_3 = 11$$
$$-x_1 + 2x_2 \qquad = 3$$
$$2x_1 + x_2 + 4x_3 = 16$$

It is clear that, $a_{ii\ \neq 0}$ and the last system satisfies the **diagonal control condition.**
We can rewrite the last system as follows:

$x_1 = \frac{11}{4} - \frac{1}{2}x_2 - \frac{1}{4}x_3$

x_2=$\frac{3}{2} + \frac{1}{2}x_1$

x_3=4 - $\frac{1}{2}x_1$ - $\frac{1}{2}x_2$

We write the system in the iterative form as follows

$x_1{}^{\mathrm{k}+1}$= $\frac{11}{4}$ - $\frac{1}{2}x_2^k$ - $\frac{1}{4}$ x_3^k

x_2^{k+1}= $\frac{3}{2}$ + $\frac{1}{2}$ x_1^{k+1} $\qquad\qquad k = 0,1,\ldots....$

x_3^{k+1}= 4- $\frac{1}{2}x_1^{k+1}$- $\frac{1}{4}x_2^{k+1}$

Set k=0, we get

$x_1^{(1)} = \frac{11}{4} - \frac{1}{2}(1.8) - \frac{1}{4}(2.8) = 1.15$

$x_2^{(1)} = \frac{3}{2} + \frac{1}{2}(1.15) = 2.075$

$x_3^{(1)} = 4 - \frac{1}{2}(1.15) - \frac{1}{4}(2.075) = 2.9063$

$x^{(1)} = (1.15, 2.075, 2.9063)$

$$E_1 = \max\{ |1.15 - 0.8|, |2.075 - 1.8|, |2.9063 - 2.8|\} = 0.35$$

Set k=1, we can, in the same way, compute

$x_1^{(2)} = \frac{11}{4} - \frac{1}{2}(2.075) - \frac{1}{4}(2.9063) = 0.9859$

$x_2^{(2)} = \frac{3}{2} + \frac{1}{2}(0.9859) = 1.993$

$= 4 - \frac{1}{2}(0.9859) - \frac{1}{4}(1.9930) = 3.0088$

$$x^{(2)} = (0.9859, 0.9859, \ \ 3.0088)$$

$$E_2 = \max\{ |0.9859 - 1.15|, |1.993 - 2.075|, |3.0088 - 2.9063|\} = 0.1641$$

We continue iteratively, until the convergent condition can be satisfied:

$\|x^{(k+1)} - x^k\| < \in$ or $\left\|\mathrm{b} - \mathrm{A} * x^{(\mathrm{k})}\right\| < \in$

Remark: From last example, we see that, the approximate results that we get by using Gauss-Sidel are more accurate and closer to the exact solution, compared with the results that we got by using Jacobi method. Moreover, for $k \geq 2$,the iterative errors those arise from using Gauss-Sidel method are less than the iterative errors those arise from using Gauss-Sidel. Which means Gauss-Sidel method is faster than Jacobi method in convergence.