Chapter 4

Interpolation,Extrapolation & Numerical Differentiations

Sometimes, we need to estimate an unknown value depending on known values (Data base), for instance, consider that, the numbers of people who have lived in Iraq is known, for the years 57, 67, 77,87, 97,2007, 2017, if we would like to estimate the number of Iraq's people in the year 75, this operation is called the interpolation, because the number 75 belongs to the interval [37, 87], while , if we would like to estimate the number of Iraqi people in the year 2018, this operation is called the extrapolation, because the number 2018 does not belong to the range of the data base.

Mathematical problems of interpolation and extrapolation

Assume that, we have the following database, where f is a continuous function on $[x_0, x_1]$

$$y_i = f(x_i)\ ,\ i = 1,2, \dots \dots . n$$

x	x_0	x_1	x_2	....	x_n
y	y_0	y_1	y_2	....	y_n

$x_i = x_{i-1} + h,$ or $x_i = x_0 + ih, i = 0,1,2, \dots . . n$

If the aim is to find $f(x^*)$, where $x^* \neq x_i)_{i=0}^{n}{}_{,}$ $x^* \in [x_0, x_n]$, then this operation is called the **interpolation ,**

while, if the aim is to find $f(x^*)$, where $x^* \neq x_i)_{i=0}^{n}{}_{,}$ $x^* \notin [x_0, x_n]$,

then, the operation is called **Extrapolation**

Next, we will study some numerical techniques that can be used to find the solutions for interpolation problems, when the distances between the points in the database are equal and non-equal.

Interpolation by Polynomials forms

This method depends on writing the unknown function f as a polynomial of order m.

$$\text{f(x)} \cong P_m(x)\ ,\forall\ x\ \in [x_0, x_n], \quad \text{where}$$

$$P_m(x) = a_0 + a_1 x + a_2 x^2 \dots \dots + a_m x^{\text{m}}$$

such that the following condition is satisfied *(***interpolation condition***)*

$$f(x_i) = P_m(x_i)_{\text{i=0}}^{\text{n}}$$

which means, $f(x^*) = P_m(x^*)$

$$|f(x) - P_m(x)| < \in, \qquad \forall x \in [x_0, x_n]$$

Steps of the algorithm

1- Input the values of $x = x_i)_{i=0}^{n}$, $y = y_i)_{i=0}^{n}$, and x^*

2-find the coefficients, a_m,………,a_1, a_0 using the relation

$$f(x_i) = P_m(x_i)_{i=0.}^{n}$$

3-find the value of $f(x^*) = P_m(x^*)$

Example:- suppose that, we have the following data base, which has different distant space

X	1	2	4	5
Y	0	3	15	24

Find the approximate value for $f(3)$, by using linear interpolation (m=1). Secondly by using quadratic interpolation (m=2).

Solution

The linear form takes the following form

$$P_1(X) = a_0 + a_1 x$$

Since the last polynomial has two unknown constant coefficients a_0, a_1, we need two value of x_i such that 3 located between them (3 ∈ [2,4]).

$$P_1(2) = f(2) = a_0 + 2a_1 = 3 \ldots\ldots\ldots (1)$$

$$P_1(4) = f(4) = a_0 + 4a_1 = 15 \ldots\ldots\ldots (2)$$

From equations (1) &(2), we get

$$a_0 = -9 , \qquad a_1 = 6$$

$$P_1(X) = -9 + 6x$$

$$f(3) = P_1(3) = -9 + 6(3) = -8 + 18 = 10$$

Secondly, the quadratic polynomial takes the form

$$P_2(x) = a_0 + a_1x + a_2x^2$$

Since the last polynomial has three unknown constant coefficients a_0, a_1, a_2 we need three value of x_i such that 3 located between them. For instance we can choose 1,2,4 .

It follows that

$$P_2(1) = f(1) = a_0 + a_1 + a_2 = 0$$

$$P_2(2) = f(2) = a_0 + 2a_1 + 4a_2 = 3$$

$$P_2(4) = f(4) = a_0 + 4a_1 + 16a_2 = 15$$

Thus, we get a linear system can be solved by using Gauss algorithm or Gauss-Jordan algorithm.

We get

$$a_0 = -1\,, a_1 = 0\,, a_2 = 1$$

Therefore

$P_2(x) = -1 + x^2$

$$f(3) \cong P_2(3) = -1 + (9) = 8$$

Remarks:-

1-if we resolved the last example with taking the points 2,4,5, we would get the same result $f(3) = 8$

2-if we consider m=3,

$$P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

This means, in order to find the approximate value of $f(3)$, we need to find the coefficients, a_0, a_1, a_2, a_3 , therefore, we have to use all the points 1,2,4,5.

Moreover, it is clear that, we cannot consider $P(x)$ is a polynomial of order m where $n < m$, because the numbers of coefficient will be larger than the numbers of the points in the data base. In fact, the best approximation, when $m = n$ (the number of the points in the data base equal the order of the polynomial $P(x)$).

3-when the number of points in the data base, n, is large ,it is difficult to use the method of polynomial, and that because we need to deal with large order matrices, therefore, in this case we have to think about another method which does not depend on matrices .
4-polynomail method can be considered as an extrapolation method, for instance in the last example we can find the approximate value of f(6), by using the relation

$$f(6) \cong P_2(6) = -1 + 36 = 35$$