Lagrange interpolation polynomial

Let, f(x) is a real function defined & continuous on I=[a,b], and assume that we have the following data base

$$y_i = f(x_i)\ ,\ i=1,2,\ldots\ldots n$$

X	x_0	x_1	x_2	....	x_n
Y	y_0	y_1	y_2	....	y_n

where $\{Xi\}_{i=0}^{n} \in$ I.

Our aim is to find the polynomial Pn(x), which satisfies the **interpolation condition**

$$f(x_i) = P_m(x_i)_{i=0.}^{n}$$

It has been proved that, there exists only one polynomial of order less than or equal n, such that the **interpolation condition** is satisfied. **Lagrange** showed that this polynomial can be written as follows:

$$P_n(x) = \sum_{k=0}^{n} f(x_k) L_k(x)\,,$$

$$L_k(x) = \prod_{\substack{i=0\\ i\neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)} \qquad k = 0,1,2\ldots\ldots n$$

Which is called the **Lagrange interpolation formula**

Theorem:- $f(x) = P_n(x) + T(x)$,

where $T(x)$ is the truncation error formula, which takes the form

$$T(x) = \frac{f^{(n+1)}(\delta(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i), \quad \delta \text{ is a number depends on x}$$

Remark:- from last theorem, if f is a polynomial of degree less than or equal n, then $T(x) = 0$, which means $f(x) = P_n(x), \forall x \in [a,b]$.

Steps of Lagrange algorithm

1-input the vectors $X = Xi)_{i=0}^{n}$, $Y = Yi)_{i=0}^{n}$

2-input x^*

3-find the approximate value of $f(x^*)$, using **Lagrange formula**

$$f(x^*) \cong P_n(x^*) = \sum_{k=0}^{n} f(x_k)L_k(x^*)$$

Example:- Let f is a continuous function, find f(4.5), from the following data base, by using Lagrange algorithm

x	2	4	5
y	3	15	24

Solution

Firstly, we write the general form of Lagrange formula

$$p_2(\mathrm{x}) = \sum_{k=0}^{2} f_{(xk)}\, L_k(\mathrm{x}) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x)$$

$$= \frac{(x-x1)(x-x2)}{(x0-x1)(x0-x2)}\mathrm{f}(x_0) + \frac{(x-x0)(x-x2)}{(x1-x0)(x1-x2)}\mathrm{f}(x_1) + \frac{(x-x0)(x-x1)}{(x2-x0)(x2-x1)}\mathrm{f}(x_2)$$

$$p_2(2) = \frac{(4.5-4)(4.5-5)}{(2-4)(2-5)}3 + \frac{(4.5-2)(4.5-5)}{(4-2)(4-5)}15 + \frac{(4.5-2)(4.5-4)}{(5-2)(5-4)}24 = \frac{77}{4}$$
$$= 19.25$$

Note that, we can get more accurate results if we increase the dimension of the data base.

Inverse Interpolation

Assume that $f(x^*)$ is known, where f is a continuous $x^* \neq x_i)_{i=0}^{n}$, $x^* \in [x_0, x_n]$ function. If we would like to find the value of x^*, this operation is called, the inverse interpolation.

In order to find a solution for this problem, we can use the Lagrange algorithm, with switch places of ***x*** and ***y.***

i.e., we consider **y** is the independent variable and **x** is the dependent variable. Therefore, the inverse Lagrange formula takes the form

$$x^* = p_n(y) = \sum_{k=0}^{n} x_k L_k\,(\mathrm{y})$$

$$L_{k\,(y)} = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(y - yi)}{(y_k - yi)}$$

Example:- Find the value of x^*such that $f(x) = 0$ (the root of f), from the following data base, by using Inverse Lagrange formula.

x	0	2	5
y	-1	3	24

$$p_2(\mathrm{x}) = \sum_{k=0}^{2} f_{(xk)}\, L_k(\mathrm{x}) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x)$$

$$x \;= \frac{(y-y1)(y-y2)}{(y0-y1)(y0-y2)}x_0 + \frac{(y-y0)(y-y2)}{(y1-y0)(y1-y2)}x_1 + \frac{(y-y0)(y-y1)}{(y2-y0)(y2-y1)}x_2$$

$$x^* = \frac{(0-3)(0-24)}{(-1-3)(-1-24)}(0) + \frac{\big(0-(-1)\big)(0-24)}{\big(3-(-1)\big)(3-24)}(2) + \frac{\big(0-(-1)\big)(0-3)}{\big(24-(-1)\big)(24-3)}(5)$$

$x^* = 1$

Remarks

1-in the last example, it is clear from the data base that the root of f belongs to (0,2), and that because the sign of $f(0)$ is negative and the sign of $f(2)$ is positive .

2-from the last example we conclude that, we can find the approximate root of $f(x) = 0$, when the form of the function f is unknown, and we only have the values of f at some points.