Numerical Differentiations

Let f, be a differentiable function on $[a,b]$, and let $x_i)_0^n \in [a,b]$, and $f(x_i)\,]_0^n$ are known . So, we have the following database

$$\begin{array}{c|c} x & x_0 \;\; x_1 \dots \dots .. x_n \\ \hline y & y_0 \;\; y_1 \dots \dots \dots \dots y_n \end{array}$$

Let $x^* \in [a,b]$

Our aim is to find the derivative of f at x^*, which is $f'(x^*)$

Here, we have to study two cases, first when the distances between the points in the data base are not equal. In this case we can use Lagrange formula to find $\mathrm{f}(x)$ as follows

$$\mathrm{f(x)} \cong P_n(x) = \sum_{k=0}^{n} f(x_k) L_k(x)$$

$$f'(\mathrm{x}) \cong p_n'(x) = \sum_{k=0}^{n} f(x_k) L'_k(x)$$

Example:- Find the approximate value of $f'(2.3)$, from the following data base.

X	1.1	1.7	3
Y	10.6	5.2	20.3

Solution:- Since the spaces between the points are not equal, we use Lagrange algorithm

$$f(x) \cong p_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Thus

$$f'(x) \cong p_2'(x) = \frac{(x-x_1)+(x-x_2)}{(x_0-x_1)(x_0-x2)} f(x_0) + \frac{(x-x_0)+(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)+(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Thus $f'(2.3) = (\quad)$

Remark:- as we have mentioned before, Lagrange method needs lots of calculations, moreover, in case of, we add more data to the data base, we need to start from the beginning, therefore, we have to think about another method to find, $f'(x^*)$ which is Newton's method with divided finite differences**,** as follows

$$f(x) = f(x_0) + (x - x_0)\mathrm{D}y_0 + (x - x_0)\,(x - x_1)\,\mathrm{D}^2 y_0 + (x - x_0)\,(x - x_1)\,(x - x_2)\,\mathrm{D}^3 y_0$$

$$f'(x) = \mathrm{D}y_0 + [(x - x_0) + (x - x_1)]\,\mathrm{D}^2 y_0 + [(x - x_0)(x - x_1) + (x - x_2)\big((x - x_0) + (x - x_1)\big)]\,\mathrm{D}^3 y_0$$

Example:- Find $f'(2)$, from the following data base

x	0	1	4	6
y	-10	20	14	30

Going back to previous example, we have

$$\mathrm{f(x)} \cong pn(x) = -10 + 30x - 8x(x - 1) + \frac{10}{6}x(x - 1)(x - 4)$$

$$\mathrm{f(x)} \cong pn(x) = -10 + 30x - 8x^2 + 8x + \frac{10}{6}(x^3 - 5x^2 + 4x)$$

$$f'(x) \cong p'n(x) = 30 - 16x + 8 + \frac{10}{6}(3x^2 - 10x + 4)$$

$$f'(2) = 30 - 32 + 8 + \frac{10}{6}(12 - 20 + 4) = -0.6667$$

Using Finite difference operators to find derivatives

In case of the distances between the points, $x_i)_0^n$, are equal , $x_{i+1} - x_i = h$, we can also use Forward, Backward, centre Newton's formulas, to find
$f'(\,x^*)$, we will study two cases :

Case1: Interpolation Problems

If $x^* \neq x_i)_{i=0}^n$, $x^* \in [x_0, x_n]$, we use Newton's finite difference formulas as follows:
Set $x^* = x_p = x_j + ph$, where

$$j = \left\{ \begin{array}{cc} 0 & in\ Forward\ \ formula, \\ n & in\ Backward\ \ formula \\ 1 \leq j \leq n-1 & in\ Center\ \ formula \end{array} \right\}$$

So,

$$\Longrightarrow \qquad p = \frac{x_p - x_j}{h} \qquad \qquad \frac{dp}{dx_p} = \frac{1}{h}$$

$$f'(x_p) = \frac{df}{dx_p} = \frac{df}{dp} . \frac{dp}{dx_p} = \frac{df}{dp}\left(\frac{1}{h}\right)$$

$$\therefore \quad f'\left(x_p\right) = \left(\frac{1}{h}\right)\frac{df(x_p)}{dp}$$

Therefore, Forward, Backward and Center Newton formulas for differentiation take the forms, respectively:

$$f'\left(x_p\right) = \left(\frac{1}{h}\right)\frac{df(x_p)}{dp} = \frac{1}{h}(\Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0),$$

$$f'\left(x_p\right) = \left(\frac{1}{h}\right)\frac{df(x_p)}{dp} = \frac{1}{h}(\nabla y_n + \frac{(2p+1)}{2}\nabla^2 y_n),$$

$$f'\left(x_p\right) = \left(\frac{1}{h}\right)\frac{df(x_p)}{dp} = y_2 + \frac{(3p^2-1)}{6}\delta^2 y_2 - y_1 + \frac{[3(1-p)^2+1]}{6}\delta^2 y_1$$

Example: Going back the example that we have considered before, find $f'(4.5), f'(9), f'(6.4)$ by using (forward, backward, center) Newton formulas.

Solution:-

Since 4.5 close to the beginning of the data base, we use forward formula to find $f'(4.5)$

$$p = \frac{x_p - x_0}{h} = \frac{1}{4}, \quad h = 2$$

$$f'\left(x_p\right) = \frac{1}{h}(\Delta y_0 + \frac{(2p-1)}{2!}\Delta^2 y_0)\ ,$$

where $\Delta y_0 = 2, \quad \Delta^2 y_0 = 3$

$$\therefore \quad f'(4.5) = \frac{1}{2}\left(2 + \left(\frac{-0.5}{2}\right)3\right) = 0.6250$$

H.W. in the same way we can find $f'(9)$ and $f'(6.4)$, by using the backward and center formulas, respectively.