Case2: Computing the Derivatives at the Points of Database

Consider that, we have the following data base:

x	$x_0 \;\; x_1 \ldots \ldots . . x_n$
y	$y_0 \;\; y_1 \ldots \ldots \ldots \ldots y_n$

where $x_{i+1} - x_i = h, \quad \forall\, i = 0,1, \ldots n-1$

The problem is: how to find the approximate values for $f'(x_i), f''(x_i)$, for $i = 0,1, \ldots . n$

It is well known that,

$$f'(x_i) = \lim_{h \to 0} \frac{f(x_i + h) - f(x_i)}{h} = \frac{\Delta f(x_i)}{h}$$

Thus, for small values of h

$$f'(x_i) \cong \frac{\Delta f(x_i)}{h} \;\;,\;\; i = 0,1, \ldots n-1 \;\;\ldots \ldots . . (1)$$

Or $\quad f'(x_i) = \lim_{-h \to 0} \frac{f(x_i - h) - f(x_i)}{-h} = \frac{f(x_i) - f(x_i - h)}{h} = \frac{\nabla f(x_i)}{h}$

For small values of h, we have

$$f'(x_i) \cong \frac{\nabla f(x_i)}{h}, \;\; i = 1,2, \ldots n \;\;\ldots \ldots . . (2)$$

From (1) & (2), for small values of h, we get

$$f'(x_i) \cong \frac{f(x_{i+1}) - f(x_{i-1})}{2h}, \qquad i = 1,2, \ldots n-1 \;\;\ldots \ldots (3)$$

Remarks:

1- equation (3) is more accurate than (1) & (2), therefore, we will use it to find $f'(x_i)$, for $1 \le i \le n-1$, while, we can only use equation (1) & (2) to find $f'(x_0)$ & $f'(x_n)$, respectively.

2- The formula (3) can be used to find $f'(x_i)$ even when $f(x_i)$ is unknown, therefore, it can be considered an interpolation formula.

Next, our aim is to derive a formula, which can be used to find the second derivatives, $f''(x_i)$, $1 \le i \le n-1$

From Taylor series, we have

$$f(x_i+h)=f(x_i)+hf'(x_i)+\frac{h^2}{2}f''(x_i)+\cdots\ldots\ldots\ldots(1)$$

and

$$f(x_i-h)=f(x_i)-hf'(x_i)+\frac{h^2}{2}f''(x_i)+\cdots\ldots\ldots\ldots(1)$$

From (1) & (2) , we get

$f''(x_i)=\frac{f(x_{i+1})-2f(x_i)+f(x_{i-1})}{h^2}$, for $1\le i\le n-1\ldots\ldots\ldots\ldots(4)$

Example:- Consider that, we have the following data base

X	0	0.2	0.4	0.6
$f(x)$	1	1.04	1.16	1.36

Find the approximate values for

1- $f'(x_i),\ 0\le i\le n$, & $f''(x_i),\ 1\le i\le n-1$

2- $f'(0.1),\ f'(0.3)$

solution

$$f'(x_0)\cong\frac{\Delta f(x_0)}{h}=\frac{f(x_1)-f(x_0)}{h}=\frac{1.04-1}{0.2}=0.2$$

$$f'(x_1)\cong\frac{f(x_2)-f(x_0)}{2h}=\frac{1.16-1}{0.4}=0.4$$

$$f'(x_2)\cong\frac{f(x_3)-f(x_1)}{2h}=\frac{1.36-1.04}{0.4}=0.8$$

$$f'(x_3)\cong\frac{\nabla f(x_3)}{h}=\frac{f(x_3)-f(x_2)}{h}=\frac{1.36-1.16}{0.2}=1$$

$$f''(x_1)=\frac{f(x_2)-2f(x_1)+f(x_0)}{h^2}=\frac{1.16-2(1.04)+1}{0.04}=2$$

$$f''(x_2)=\frac{f(x_3)-2f(x_2)+f(x_1)}{h^2}=\frac{1.36-2(1.16)+1.04}{0.04}=2$$

To compute, $f'(0.1), f'(0.3)$ assume now, $h = 0.1$

$$f'(0.1) \cong \frac{f(0.2) - f(0)}{2(0.1)} = \frac{1.04 - 1}{0.2} = 0.2$$

$$f'(0.3) \cong \frac{f(0.4) - f(0.2)}{2(0.1)} = \frac{1.16 - 1.04}{0.2} = 0.6$$