Trapezoidal method

From the general form of integration, with choosing Lagrange polynomial, and n=1, we get

$h = x_1 - x_0$,$b = x_1$, $a = x_0$

$$\int_a^b f(x)dx = \sum_{i=0}^{n}(\int_{x_0}^{x_n} L_i(x)dx) f(x_i)dx + \int_{x_0}^{x_1} \frac{f''(\delta(x))}{2}(x-x_0)(x-x_1)dx$$

$$\text{Set , } \mathrm{a}_0 = \int_{x_0}^{x_1} L_0(x)dx = \int_{x_0}^{x_1} \frac{(x-x_1)}{(x_0-x_1)}dx = \frac{(x-x_1)^2}{2(x_0-x_1)} = |_{x_0}^{x_1} = \frac{h}{2}$$

$$\mathrm{a}_1 = \int_{x_0}^{x_1} L_1(x)dx = \int_{x_0}^{x_1} \frac{(x-x_0)}{(x_1-x_0)}dx = \frac{(x-x_0)^2}{2(x_1-x_0)} = |_{x_0}^{x_1} = \frac{h}{2}$$

Substitute a_0, a_1 in equation (1), we get

$$\int_a^b f(x)dx \cong \sum_{i=0}^{1} a_i\, f(x_i) = \frac{h}{2} f(x_0) + \frac{h}{2} f(x_1),$$

which is the **Trapezoidal formula.**

where the truncation error formula takes the form

$$T(h) = \int_{x_0}^{x_1} \frac{f''(\delta(x))}{2}(x-x_0)(x-x_1)\mathrm{dx} = -\frac{h^3}{12} f''(\delta),$$

Remark:- We note that, if f is polynomial of order less than or equal one, then the truncation error equal zero, which means:

$$\int_a^b f(x)dx = \sum_{i=0}^{1} a_i\, f(x_i) = \frac{h}{2} f(x_0) + \frac{h}{2} f(x_1),$$

Example:- Use the Trapezoidal method to find the approximate value of the following integral

$$\int_a^b (x^3+1)\, dx$$

Solution

$$n = 1\ ,\ x_0 = 0\ ,\ x_1 = 1\ ,\ h = \frac{x_1 - x_0}{1} = 1$$

$\int_a^b f(x)dx \cong \frac{1}{2}\big(f(x_0) + f(x_1)\big)$

$$I_n = \int_a^b (x^3 + 1)\,dx \cong \frac{1}{2}\big((x^3+1)|_0 + (x^3+1)|_1\big) = \frac{1}{2}(1+2) = \frac{3}{2} = 1.5$$

While, we can find the exact value as follows:

$$I_e = \int_a^b (x^3 + 1)\,dx = \left(\frac{x^4}{4} + x\right)|_0^1 = \frac{1}{4} + 1 = 1.25$$

Thus, the absolute error is

$$E = |I_n - I_e| = |1.5 - 1.25| = 0.25$$

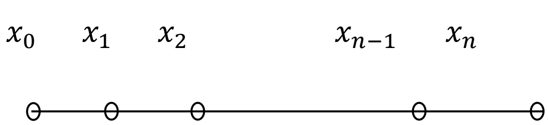
In order to get more accurate value to the integration, we use the composite Trapezoidal methods.

Composite Trapezoidal methods

The idea is, we divide the interval $[\mathrm{a}, \mathrm{b}]$ into *n* of subintervals

$$[a, b] = [x_0, x_1] \cup [x_1, x_2] \ldots\ldots \cup [x_{n-1}, x_n],$$

Since the summation of all the integrals on the subintervals is equal the integral on the whole interval $[a, b]$, we can apply the Trapezoidal formula, on each of the integrals as follows:



$I_N = \ I_1 + I_2 + \cdots I_n$

$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx$

$$= [\frac{h}{2}\big(f(x_0) + f(x_1)\big)] + \ [\frac{h}{2}\big(f(x_1) + f(x_2)\big)] + \cdots + \ [\frac{h}{2}\big(f(x_{n-1}) + f(x_n)\big) - \sum_{i=1}^{n} \frac{h^3}{12} f''(\delta_i),$$

where $\ x_{i-1} \le \delta_i \le x_i$

$$\therefore \int_a^b f(x)dx \cong \left[\frac{h}{2}\left(f(x_0)+f(x_n)\right)\right] + h\sum_{i=2}^{n} f(x_{i-1})$$

The last equation is called, the **Composite Trapezoidal Formula.**

Steps of composite Trapezoidal algorithm

1-Input a, b

2- Input the number of partitions n

3-Define the function $f(x)$

4-Find $h = \frac{b-a}{n}$

5- Use composite Trapezoidal formula

$$\int_a^b f(x)dx \cong \left[\frac{h}{2}\left(f(x_0)+f(x_n)\right)\right] + h\sum_{i=2}^{n} f(x_{i-1})$$

Example: For the last example, find the approximate value of the integral, using composite Trapezoidal methods with taking $n = 2$

$$\int_0^1 (x^3+1)\mathrm{dx}$$

Solution

$$h = \frac{b-a}{n} = \frac{1}{2}$$
$$[0,1] = [0,0.5] \cup [0.5,1]$$

$$\int_0^1 (x^3+1)\mathrm{dx} = \int_0^{1/2} (x^3+1)\mathrm{dx} + \int_{1/2}^1 (x^3+1)\mathrm{dx}$$

$$I_n \cong \frac{h}{2}[(x^3+1)dx|_0 + (x^3+1)dx|_{0.5}] + \frac{h}{2}[(x^3+1)dx|_{0.5} + [(x^3+1)dx|_1]$$

=1/4(1+1/8+1+1/8+1+2)= 1/4[21/4]=[21/16]=1.3125

E=|I_n-I_e|=|1.31-1.25|=0.06

Remark

In last example, it is clear that, from the absolute errors, the result of the composite Trapezoidal method is more accurate than the result that we get by using normal Trapezoidal method**.**