Simpson method

Here, we set $n = 2$**,** which means

$$[a, b] = [x_0, x_1] \cup [x_1, x_2]\,, \qquad h = \frac{x_2 - x_0}{2}$$

We use Lagrange polynomial, of order 2, to find an approximate form for the function f .

$$\mathrm{f(x)} \cong P_2(x), \qquad \forall x \in [a, b]$$

$$\int_a^b f(x)dx = \sum_{i=0}^{n} (\int_{x_0}^{x_n} L_i(x)dx) f(x_i) + \int_{x_0}^{x_1} \frac{f'''(\delta(x))}{6}(x - x_0)(x - x_1)(x - x_2)dx$$

we can show that:

$a_0 = \int_{x_0}^{x_2} \mathrm{L_0(x)dx} = \frac{\mathrm{h}}{3}$
$a_1 = \int_{x_0}^{x_2} \mathrm{L_1(x)dx} = \frac{4\mathrm{h}}{3}$
$a_2 = \int_{x_0}^{x_2} \mathrm{L_2(x)dx} = \frac{\mathrm{h}}{3}$

$$T(h) = \int_{x_0}^{x_1} \frac{f'''\big(\delta(x)\big)}{6}(x - x_0)(x - x_1)(x - x_2)dx = -\frac{h^5}{90} f^{(4)}(\delta) \ldots \ldots .. (2)$$

Thus, we get

$$\therefore \int_a^b f(x)dx \cong \frac{h}{3}\mathrm{f(x_0)} + \frac{4h}{3}\mathrm{f(x_1)} \; + \frac{h}{3}\mathrm{f(x_2)}$$

which is the **Simpson integral formula**

Remark:- We note that, if f is polynomial of order less than or equal 3, then the truncation error equal zero, which means:

$$\int_a^b f(x)dx = \sum_{i=0}^{2} a_i\, f(x_i) = \frac{h}{3} f(x_0) + \frac{4h}{3} f(x_1) \; + \frac{h}{3} f(x_2),$$

Example: find the approximate value of the following integral , using Simpson method

$$\int_1^2 e^{x^2}\, dx$$

Solution:

$$h = \frac{b-a}{n} = \frac{2-1}{2} = \frac{1}{2}$$

$$[1,2] = [1,1.5]U[1.5,2]$$

$$I_n = \int_1^2 e^{x^2}\,dx = \frac{1}{6}[e^{x^2}\Big|_1 + 4e^{x^2}\Big|_{1.5} + e^{x^2}|_2]$$

$$=\frac{1}{6}[2.71 + 4(9.48) + 54.5] = 15.855$$

H.W. Find the following integral, by using Simpson method.

$$\int_0^1 (x^3 + 1)\mathrm{dx}$$

(answer 1.25, $I_n = I_e$)

Composite Simpson method

We divide, the interval $[a, b]$, into k, pairs of subintervals, where $n = 2k$ (n should be odd), as follows:

$[a, b] = ([x_0, x_1] \cup [x_1, x_2]) \cup ([x_2, x_3] \cup [x_3, x_4]) \dots \dots \cup ([x_{n-2}, x_{n-1}] \cup [\mathrm{x}_{n-1}, x_n])$

Apply Simpson formula for each of the pairs of subintervals

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \dots \dots . \int_{x_{n-2}}^{x_n} f(x)dx$$

$$= \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] + \frac{h}{3}[f(x_2) + 4f(x_3) + f(x_4)] \dots \dots . . + \frac{h}{3}[f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] - \sum_{i=1}^{k} \frac{h^5}{90} f^{(4)}(\delta_i)$$

where $x_{i-1} \le \delta_i \le x_i$

$$\int_a^b f(x)dx \cong \frac{h}{3}[f(x_0)+f(x_n)] + \frac{4h}{3}\sum_{i=1}^{k} f(x_{2i-1}) + \frac{2h}{3}\sum_{i=1}^{k-1} f(x_{2i})$$

which is the **'Composite Simpson Integral formula '**

Steps of composite Simpson algorithm

1-Input a,b

2-Input the numbers of the pairs of subintervals, k, and set $n = 2k$

3-Define the function *f*

4- $h = \frac{b-a}{n}$

5-Use the composite Simpson integral formula,

$$\int_a^b f(x)dx \cong \frac{h}{3}[f(x_0)+f(x_n)] + \frac{4h}{3}\sum_{i=1}^{k} f(x_{2i-1}) + \frac{2h}{3}\sum_{i=1}^{k-1} f(x_{2i})$$

Example:- Use Composite Simpson integral formula to find the value of the following integral, consider n=4

$$\int_1^2 e^{x^2} dx$$

solution

$$h = \frac{2-1}{4} = \frac{1}{4}$$

$[1,2] = ([1,1.25] \cup [1.25,1.5]) \cup ([1.5,1.75] \cup [1.75,2])$

$$\int_1^2 e^{x^2}\, dx = \int_1^{1.5} e^{x^2} dx + \int_{1.5}^2 e^{x^2}\, dx$$

Apply Simpson formula for each integral, we get that

$$\int_1^2 e^{x^2} dx \cong \frac{h}{3}(e^{x^2}\,|_1 + 4e^{x^2}|_{1.25} + e^{x^2}|_{1.5}) + \frac{h}{3}(e^{x^2}|_{1.5} + 4e^{x^2}|_{1.75} + e^{x^2}|_2)$$

$= \frac{1}{12}(2.71 + 4(4.77) + 9.48) + \frac{1}{12}(9.48 + 4(21.3) + 54.5) = 15.0375$

Remark: If we increase n, ($n = 6, or\ n = 8$), we can get more accurate results to the integration.

H.W. Compare between the two absolute errors, those can arise from using *Simpson* method with n=2 and 4, respectively, to find the approximate value for the following integral:

$$\int_0^1 (x^4 + 1)dx$$

Note that
$I_N = I_e = 1.25$, and that because f is a polynomial of order three.