Newton-Cotes formulas

Newton-Cotes rules are a group of formulas for numerical integration, also called quadrature. These methods were first suggested by **Isaac Newton & Roger Cotes.**

As we have mentioned before the general formula of integration methods takes the form:

$$\int_a^b f(x)dx \cong \int_a^b p_n(x)\,dx = \sum_{i=0}^{n} a_i\, f(xi)$$

Where p_n can be considered as Lagrange polynomial

For n=1, we have showed that, the general form above is the **Trapezoidal formula**

$$\int_a^b f(x)dx \cong \sum_{i=0}^{1} a_i\, f(x_i) = \frac{h}{2}f(x_0) + \frac{h}{2}f(x_1),$$

while for n=2, we get the **Simpson** formula:

$$\int_a^b f(x)dx \cong \sum_{i=0}^{2} a_i\, f(x_i) = \frac{h}{3}\mathrm{f}(\mathrm{x}_0) + \frac{4h}{3}\mathrm{f}(\mathrm{x}_1) + \frac{h}{3}\mathrm{f}(\mathrm{x}_2),$$

For n=3, we can show using a similar way that the integral formula takes the following form (which called **Simpson's 3/8** rule):

$$\int_a^b f(x)dx \cong \frac{3h}{8}\left(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right)$$

And so on.. We can find a Newton-cotes formula for any n.

All these methods above are called **closed Newton-Cotes formulas,** because it use the function value at all points. While , there are another type of Newton-Cotes methods, which does not use the function values at the endpoints (x_0 or x_n) , these methods are called **open Newton-Cotes formulas**

Next, we state some of these methods, that can be derived in a manner similar to the closed formulas **:**

$$\text{for } n = 2, \qquad \int_a^b f(x)dx \cong 2hf(x_1),$$

$$\text{for } n = 3, \quad \int_a^b f(x)dx \cong \frac{3h}{2}(f(x_1) + f(x_2))$$

$$\text{for } n = 4, \quad \int_a^b f(x)dx \cong \frac{4h}{3}(2f(x_1) - f(x_2) + 2f(x_3))$$

It is well known that, any Newton-Cotes method of order n, gives exact results where f is polynomial of order less than or equal n, and that because the truncation error formula equal zero, where

$$E_n = \int_{x_0}^{x_n} \frac{f^{(n+1)}(\delta(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i)dx,$$

This fact can be used to derive closed Newton-Cotes formulas, without using Lagrange polynomial as follows:

For $n = 1$,

$$\int_a^b f(x) = a_0 f(x_0) + a_1 f(x_1) + E$$

Where E is the truncation error, which equal zero for $f = \{1, x\}$, $0 \leq x \leq h$

$\int_0^h 1dx = h = a_0 + a_1$,……(1)

$\int_0^h xdx = \frac{h^2}{2} = a_0 x_0 + a_1 x_1 = a_0(0) + a_1(h)$……..(2)

From equation (2), we get $a_1 = \frac{h}{2}$,

If we substitute a_1 in equation (1) , we get $a_0 = \frac{h}{2}$

Thus for any function f,

$$\int_a^b f(x) \cong \frac{h}{2} f(x_0) + \frac{h}{2} f(x_1),$$

which is the Trapezoidal formula

For $n = 2$, since it is known that E=0, where , $f = \{1, x, x^2\}$,

$$0 \leq x \leq 2h$$

$$\int_a^b f(x) = a_0 f(x_0) + a_1 f(x_1) + a_3 f(x_2) + E$$

Let $x_0 = 0,\ x_1 = h,\ x_2 = 2h$

by using similar way to that above, we have

$$\int_0^{2h} 1dx = 2h = a_0 + a_1 + a_3,$$

$$\int_0^{2h} xdx = 2h^2 = a_0(0) + a_1(h) + a_2(2h) = a_1 h + 2a_2 h$$

$$\int_0^{2h} x^2 dx = \frac{8h^3}{3} = a_0(0) + a_1(h^2) + a_2(4h^2) = a_1 h^2 + 4a_2 h^2$$

From last three equations, we get

$$a_0 = \frac{h}{3}\ , \quad a_1 = \frac{4h}{3}\ , \quad a_2 = \frac{h}{3}$$

Thus for any function f, we have

$$\int_a^b f(x) \cong \frac{h}{3} f(x_0) + \frac{4h}{3} f(x_1) + \frac{h}{3} f(x_2),$$

which is the Simpson formula.

By using similar way we can get a closed Newton-Cotes formula for any n.

Remark:- For Newton-cotes methods, to be accurate, the step size h needs to be small, which means that the interval of integration must be small itself. For this reason, composite Newton-Cotes rules give more accurate results with small h, such as composite Trapezoidal or composite Simpson formula.