Gaussian Quadrature Methods

Since Newton-Cotes formulas of order n give exact results, where f is of order less than or equal n, we need to study another type of methods such as Gauss Quadrature method that give the exact results where f is of degree less than or equal $2n-1$.

The quadrature form for a function f can be given as follows

$$Q(f)=\sum_{i=1}^{n} w_i f(x_i),$$

where $\{w_i\}_{i=1}^{n}$ are called the weights,

$\{x_i\}_{i=1}^{n}$ are the roots of a certain polynomial.

Gauss-Legendre Method

Suppose that, we need to find the integral

$\int_{-1}^{1} f(x)dx,$ and let $\{x_i\}_{i=1}^{n}$ are the roots of Gauss-Legendre polynomial, which takes the general form

$$P_n(x)=(\frac{1}{2^n n!})\frac{d^n}{dx^n}(x^2-1)^n, \quad -1\le x\le 1.$$

So, $P_1(x)=x,\;\; P_2(x)=\frac{1}{2}(3x^2-1),\;\; P_3(x)=\frac{1}{2}(5x^3-3x)$ and so on

The problem becomes

$$\int_{-1}^{1} f(x)dx \cong Q(f)=\sum_{i=1}^{n} w_i f(x_i),$$

with error of order $2n$th derivative of the function f

Therefore, our aim is to find the weights $\{w_i\}_{i=1}^{n}$

For n=2, the problem is

$$\int_{-1}^{1} f(x)dx \cong \sum_{i=1}^{2} w_i f(x_i)=w_1 f(x_1)+w_2 f(x_2),$$

where x_1, x_2 are the roots of $P_2(x)=\frac{1}{2}(3x^2-1)=0$.

Thus $x_1 = -\frac{1}{\sqrt{3}}$, $x_2 = \frac{1}{\sqrt{3}}$

So,

$$\int_{-1}^{1} f(x)dx \cong w_1 f\left(-\frac{1}{\sqrt{3}}\right) + w_2 f(\frac{1}{\sqrt{3}})$$

Since the truncation error equal zero where f is of order $2n-1=3$,

for $f = \{1, x, x^2, x^3\}$,

$$\int_{-1}^{1} 1dx = 2 = w_1 + w_2 \,,$$

$$\int_{-1}^{1} x\,dx = 0 = w_1x_1 + w_2x_2 = -\frac{w_1}{\sqrt{3}} + \frac{w_2}{\sqrt{3}} \,,$$

$$\int_{-1}^{1} x^2\,dx = \frac{2}{3} = w_1{x_1}^2 + w_2{x_2}^2 = \frac{w_1}{3} + \frac{w_2}{3} \,,$$

$$\int_{-1}^{1} x^3\,dx = 0 = w_1{x_1}^3 + w_2{x_2}^3 = -\frac{w_1}{(\sqrt{3})^3} + \frac{w_2}{(\sqrt{3})^3} \,,$$

From these equations, we get $w_1 = w_2 = 1$

Thus, the problem becomes

$$\int_{-1}^{1} f(x)dx \cong f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \ldots\ldots(1)$$

For n=3, by using similar way, we can show that

$$\int_{-1}^{1} f(x)dx \cong \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) \ldots\ldots(2)$$

where, $-\sqrt{\frac{3}{5}}$, 0 , $\sqrt{\frac{3}{5}}$ are the roots of Legendre polynomial of order 3.

Coordinate transformation from [a,b] to [-1,1]

Set $t = \frac{b-a}{2}x + \frac{b+a}{2}$, $dt = \frac{b-a}{2}\,dx$

$x = -1 \;\rightarrow\; t = a$,

$x = 1 \;\rightarrow\; t = b$

Thus

$$\int_a^b f(t)dt = \int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\left(\frac{b-a}{2}\right) dx$$

Example:-

Find the following integral, using the 2 points Gauss-Legendre formula

$$\mathrm{I} = \int_0^4 te^{2t}dt = 5216.9$$

$$t = \frac{b-a}{2}x + \frac{b+a}{2} = 2x + 2, \quad dt = 2dx$$

$$\mathrm{I} = \int_0^4 te^{2t}dt = \int_{-1}^{1} (4x+4)\, e^{4x+4}dx = \int_{-1}^{1} f(x)$$

Use equation (1), we get

$$\mathrm{I} = \int_{-1}^{1} f(x) \cong f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \left(4 - \frac{4}{\sqrt{3}}\right) e^{4-\frac{4}{\sqrt{3}}} + \left(4 + \frac{4}{\sqrt{3}}\right) e^{4+\frac{4}{\sqrt{3}}} = 3477.5$$

If we use 3 point Gauss- Legendre formula (2), we get much more accurate results

$$I = 5197.5$$