Next, we will study some important methods that can be used to find the numerical solutions of initial value problems

Euler Algorithm

The general idea of this method, is to divide the interval $[a, b]$, into n of subintervals, as follows:

$$x_0 = a\ , x_1 = x_0 + h \dots \dots \quad x_n = x_{n-1} + h = b$$

$$h\ = \frac{(b-a)}{n}$$

In order to find the approximate solution of the initial value problem at the point $x_{i+1}, i = 0,1,2, \dots n-1$, we consider the definition of $y'(x_i)$, as follows:

$$y'(x_i) = \lim_{x \to x_i} \left(\frac{y(x_{i+1}) - y(x_i)}{x_{i+1} - x_i} \right) \cong \frac{y(x_{i+1}) - y(x_i)}{x_{i+1} - x_i}$$

since, $y' = f(x, y)$, we obtain

$$\frac{y(x_{i+1}) - y(x_i)}{x_{i+1} - x_i} \cong f(x_i, y_i), \quad i = 0,1,2, \dots \dots n$$

$$\therefore \ y(x_{i+1}) \cong y(x_i) + hf\big(x_i, y(x_i)\big)$$

Assume that, y_i is the approximate value of $y(x_i)$, we get

$$y_{i+1} = \ y_i + hf(x_i, y_i) \qquad i = 0,1,2, \dots \dots \dots, n-1$$

The last equation is called **Euler formula.**

Remark:-

Euler formula, can only be used only for finding the numerical solutions at the points $x_i)_0^{\mathrm{n}}$, while if we would like to find the approximate value of $y(x^*)$, where $x^* \ \neq \ x_i)_0^{\mathrm{n}}$, $x^* \in [a, b]$, then in this case, we can use an interpolation method.

Steps of Euler algorithm

1-Input a, b

2-Input the points, $x_1^*, x_2^*, \dots.$

3-Define $f(x, y)$

4- Input, n, $h = \frac{(b-a)}{n}$

5- Set $x_{i+1} = x_i + h, \;\; i = 0,1,2, \dots . n - 1$

6-Compute the approximate values of $y(x_i), i = 1,2, \dots n,$ using Euler formula:

$$y_{i+1} = y_i + hf(x_i, y_i) \qquad i = 0,1,2, \dots \dots . . , n - 1$$

6- If $x_k^* \in [a, b], \quad x_k^* \neq x_i, \forall i,$ then use a Newton's finite difference formula, to find the approximate value of the initial value problem at this point.

Example: - Consider that, we have the following initial value problem

$$y' = -xy, \qquad y(0) = 0.5$$
$$x \in [0,0.2],$$

Use Euler method to find the approximate values of y(0.1),y(0.15), y(0.2) and compare these values with the exact solution (find the absolute errors)

Solution :-

$$h = \frac{b-a}{2} = \frac{0.2-0}{2} = 0.1, \quad x_0 = 0 \, , x_1 = 0.1 \, , x_2 = 0.2$$

by using Euler formula

$$y_1 = y_0 + h\, f(x_0, y_0)$$
$$= 0.5 + (0.1)(-(0)(0.5)) = 0.5$$
$$y_2 = y_1 + h\, f(x_1, y_1)$$
$$= 0.5 + (0.1)(-(0.1)(0.5)) = 0.495$$

Thus, we get the following database

x	0	0.1	0.2
y	0.5	0.5	0.495

We can show that, by using separation of variables, the exact solution of this problem takes the form: $y = \frac{1}{2} e^{\frac{-x^2}{2}}$

which leads to $y(0.1) = 0.4975, \;\; y(0.2) = 0.4901$

Thus, the absolute errors can be computed as follows:

$$E_1 = |y(0.1) - y_1| = |0.4975 - 0.5| = 0.0025$$
$$E_2 = |y(0.2) - y_1| = |0.4901 - 0.495| = 0.0049$$

From the above database, we can compute the approximate value of $y(0.15)$, by using Backward Newton's formula studied in Chapter 4. (**H.W**.)

Modified Euler method

In order to get more accurate results than Euler method, we define the modified Euler method, which has the following general formula

$y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^*)\right),$ **Modified Euler**

where

$y_{i+1}^* = y_i + h\ f(x_i, y_i)$ **Normal Euler**

$$i = 0,1,2, \dots \dots . n - 1$$

In fact, the first equation depends on the second equation, and this way is called (**Estimation–Correction**), because from the first equation, we get estimate value for $y(x^*)$, and then in the second equation, we get a correction for this value.

The steps of Modified Euler algorithm are the same as the steps of normal Euler algorithm and we only have to add one step more after step 5, which is:

6- Correct the estimated value y_{i+1}^*, that we get from using normal Euler formula, by using the modified Euler formula.

Example:- Use Modified Euler method for the last example.
Solution

$h = \frac{b-a}{2} = \frac{0.2-0}{2} = 0.1,$ n=2

Estimation: $y_1^* = y_0 + h\ f(x_0, y_0)$

$$= 0.5 + (0.1)(0) = 0.5$$

Correction: $y_1 = y_0 + \frac{h}{2}(f(x_0, y_0) + f(x_1, y_1^*))$

$$= 0.5 + \frac{0.1}{2}\left(-(0.1)(0.5)\right) = 0.4975$$

Estimation: $y_2^* = y_1 + h\ f(x_1, y_1)$

$$= 0.4975 + (0.1)\left(-(0.1)(0.4975)\right) = 0.4925$$

Correction: $y_2 = y_1 + \frac{h}{2}(f(x_1, y_1) + f(x_2, y_2^*))$

$$= 0.4975 + \frac{(0.1)}{2}(-(0.1)(0.4975) - (0.2)(0.4925)) = 0.4901$$

Thus, we get the following data base

x	0	0.1	0.2
y	0.5	0.4975	0.4901

Next, we compute the absolute errors:

$$E_1 = |y(0.1) - y_1| = |0.4975 - 0.4975| = 0$$

$$E_2 = |y(0.2) - y_1| = |0.4901 - 0.4901| = 0$$

It is clear that, this database is much different from that we have got from using normal Euler method. Moreover, it is more accurate.

Iteration modified Euler method

In order to increase the speed of convergence with much more accurate results, we can use the Iteration modified Euler method**,** which takes the following form:

$$y_{i+1}^{0} = y_i + h\ f(x_i, y_i)$$

$$y_{i+1}^{(k+1)} = y_i + \frac{h}{2}(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(k)})\qquad k = 0,1,2, \ldots.$$

$$i = 0,1,2, \ldots\ldots, n-1$$

Remark:-

The steps of Iteration modified Euler method, are the same as those of modified Euler method, but the only different is after computing the estimated value y_i^0, we compute $y_i^{(1)}, y_i^{(2)}, \ldots.$

Until, we get the convergence condition is satisfied :

$$|y_i^{(k+1)} - y_i^{(k)}| <\in$$

And then, we continue to compute the other points $y_{\mathrm{i}+1}, y_{\mathrm{i}+2}, \ldots\ldots.,$ using the same way.

Example: - Use Iteration modified Euler methods for the last example

h$=\frac{b-a}{2} = \frac{0.2-0}{2} = 0.1, \quad x_0 = 0\,, x_1 = 0.1\,, x_2 = 0.2$

$$y_1^0 = y_0 + h\, f(x_0, y_0)$$

$$= 0.5 + (0.1)(0) = 0.5$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\,(\mathrm{f}(x_0, y_0) + \mathrm{f}(x_1, {y_1}^0))$$

$$= 0.5 + \left(\frac{0.1}{2}\right)\Big(0 + \big(-(0.1)(0.5)\big)\Big) = 0.4975$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\,(\mathrm{f}(x_0, y_0) + \mathrm{f}(x_1, {y_1}^{(1)}))$$

$$= 0.5 + \frac{(0.1)}{2}\big(-(0.1)(0.4975)\big) = 0.4975$$

Since the convergence condition is satisfied, which is

$|{y_1}^{(2)} - {y_1}^{(1)}| < \in$, we move on computing y_2

$y_2^0 = y_1 + h\, f(x_1, y_1) = 0.4975 + (0.1)\big(-(0.1)(0.4975)\big) = 0.4925$

$$y_2^{(1)} = y_1 + \frac{h}{2}\,(\mathrm{f}(x_1, y_1) + \mathrm{f}(x_2, {y_2}^0))$$

$$= 0.4975 + \frac{(0.1)}{2}\big(-(0.1)(0.4975) - (0.2)(0.4925)\big) = 0.4901$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\,(\mathrm{f}(x_1, y_1) + \mathrm{f}(x_2, {y_2}^{(1)}))$$

$$= 0.4975 + \frac{(0.1)}{2}\big(-(0.1)(0.4975) - (0.2)(0.4901)\big) = 0.4901$$

Since $|\, {y_2}^{(2)} - {y_2}^{(1)}| < \in$, we stop here.

Explicit & Implicit Methods

Normal Euler method, belongs to group of methods called **explicit methods**, and that because of , computing y_{i+1} depends only on x_i ,while modified Euler methods belongs to **implicit methods**, and that because of , computing y_{i+1} depends on both of x_i & x_{i+1}