Runge-Kutta Methods

Since modified Euler method needs two steps to get the solutions, it is considered a two-steps method. Moreover, Euler methods need to approximate the derivatives by using special forms. Thus, we will use *Runge-Kutta* method , which is a one-step method and it can be used to avoid determining higher order derivatives.

Runge-Kutta Method of order 2:

Set $k_1 = hf(x_i, y_i)$

$k_2 = hf(x_i + h, y_i + k_1)$

Recall modified Euler formula

$$y_{i+1} = y_i + \frac{1}{2}\left(hf(x_i, y_i) + hf(x_{i+1}, y_{i+1}^*)\right),$$

where $y_{i+1}^* =$
$y_i + h\ f(x_i, y_i)$

This equation can rewrite as follows:

$$y_{i+1} = y_i + \frac{1}{2}(\,k_1 + k_2)\ \ldots..\ \textbf{Ruga}-\textbf{kutta formula of order2}$$

Runge-Kutta Method of order 4:

Set $k_1 = hf(x_i, y_i),$

$$k_2 = hf\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1\right),$$

$$k_3 = hf\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2\right),$$

$$k_4 = hf(x_i + h, y_i + k_3)$$

$$y_{i+1} = y_i + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

which is **Ruga-Kutta** formula of order4.

Example:- Consider that, we have the following intial value problem:

$$y' = x + y, \qquad y(0) = 1$$

Use Runga-Kutta method of order 2 and order 4 to compute the approximate values of $y(0.25), y(0.5)$

Solution

$$h = 0.25, \quad x_0 = 0, \quad x_1 = 0.25, \ \ x_2 = 0.5$$

$$i = 0, \quad k_1 = hf(x_0, y_0) = 0.25(0 + 1) = 0.25$$

$$k_2 = hf(x_0 + h, y_0 + k_1) = 0.25(0.25 + 1 + 0.25) = 0.3750$$

$$y_1 = y_0 + \frac{1}{2}(\,k_1 + k_2) = 1 + 0.5(0.25 + 0.3750) = 1.3125$$

$$i = 1, \quad k_1 = hf(x_1, y_1) = 0.25(0.25 + 1.3125) = 0.3906$$

$$k_2 = hf(x_1 + h, y_1 + k_1) = 0.25(0.25 + 0.25 + 1.3125 + 0.3906) = \ 0.5508$$

$$y_2 = y_1 + \frac{1}{2}(\,k_1 + k_2) = 1.3125 + 0.5(0.3906 + 0.5508) = \ 1.7832$$

Next, we compute these values by using Euler method of order4

$$i = 0, \quad k_1 = hf(x_0, y_0) = 0.25(0 + 1) = 0.25$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.25\left(\frac{1}{8} + 1 + \frac{1}{8}\right) = 0.3125$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.25\left(\frac{1}{8} + 1 + \frac{0.3125}{2}\right) = 0.3203$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.25(0.25 + 1 + 0.3203) = 0.3926$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + \frac{1}{6}(0.25 + 2(0.3125) + 2(0.3203) + 0.3926)$$
$$= 1.3180$$

$$i = 1, \quad k_1 = hf(x_1, y_1) = 0.25(0.25 + 1.3180) = 0.3920$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.25\left(\frac{1}{4} + \frac{1}{8} + 1.3180 + \frac{0.3920}{2}\right) = \ 0.4723$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.25\left(\frac{1}{4} + \frac{1}{8} + 1.3180 + \frac{0.4723}{2}\right) = 0.4823$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.25(0.25 + 0.25 + 1.3180 + 0.4823) = 0.5751$$

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.3180 + \frac{1}{6}(0.3920 + 2(0.4723) + 2(0.4823) + 0.5751) = 1.7974$$

Since $y' = x + y$ is linear equation, we can show that, by using integration factor, the exact solution of problem in the last example takes the form:

$$y = 2e^x - x - 1$$

Thus $y(0.25) = 1.3181,\ y(0.5) = 1.7974$

Thus, the absolute errors, obtained by Ruge-Kutta of order 2 are as follows:

$$E_1 = |y(0.25) - y_1| = |1.3181 - 1.3125| = 0.0056$$

$$E_2 = |y(0.5) - y_1| = |1.7974 - 1.7832| = 0.0142$$

while, the absolute errors, obtained by Ruge-Kutta of order 4 are as follows:

$$E_1 = |y(0.25) - y_1| = |1.3181 - 1.3180| = 0.0001$$

$$E_2 = |y(0.5) - y_1| = |1.7974 - 1.7974| = 0$$

It is clear that, the results of Ruga-Kutta method of order 4 are much accurate than the results of Ruga-Kutta method of order 2.