MOTIVATION:

Why study differential equations?

Suppose we know how a certain quantity changes with time (for example, the temperature of coffee in a cup, the number of people infected with a virus). The rate of change of this quantity is the derivative, so we can work out how quickly the temperature changes, how quickly the number of infected people changes.

Suppose instead we know the value of the quantity now and we wish to predict its value in the future. To do this, we must know how quickly the quantity is changing. But the rate of change of a quantity will depend on the quantity itself: this gives rise to a differential equation.

EXAMPLE: Suppose that N(t) is the number of bacteria growing on a plate of nutrients. At the start of the experiment, suppose that there are 1000 bacteria, so N(0) = 1000. The rate of change of N will be proportional to N itself: if there are twice as many bacteria, then N will grow twice as rapidly. So we have:

$$\frac{dN}{dt} = cN$$

where c is a constant, and $\frac{dN}{dt}$ is the derivative (rate of change) of N with respect to time. We would have to do further experiments to find out the value of c. We can easily verify that: $N(t) = 1000e^{ct}$

is a solution of this differential equation with the given initial condition. To do this, first calculate N(0) and verify that it is the same as the number given:

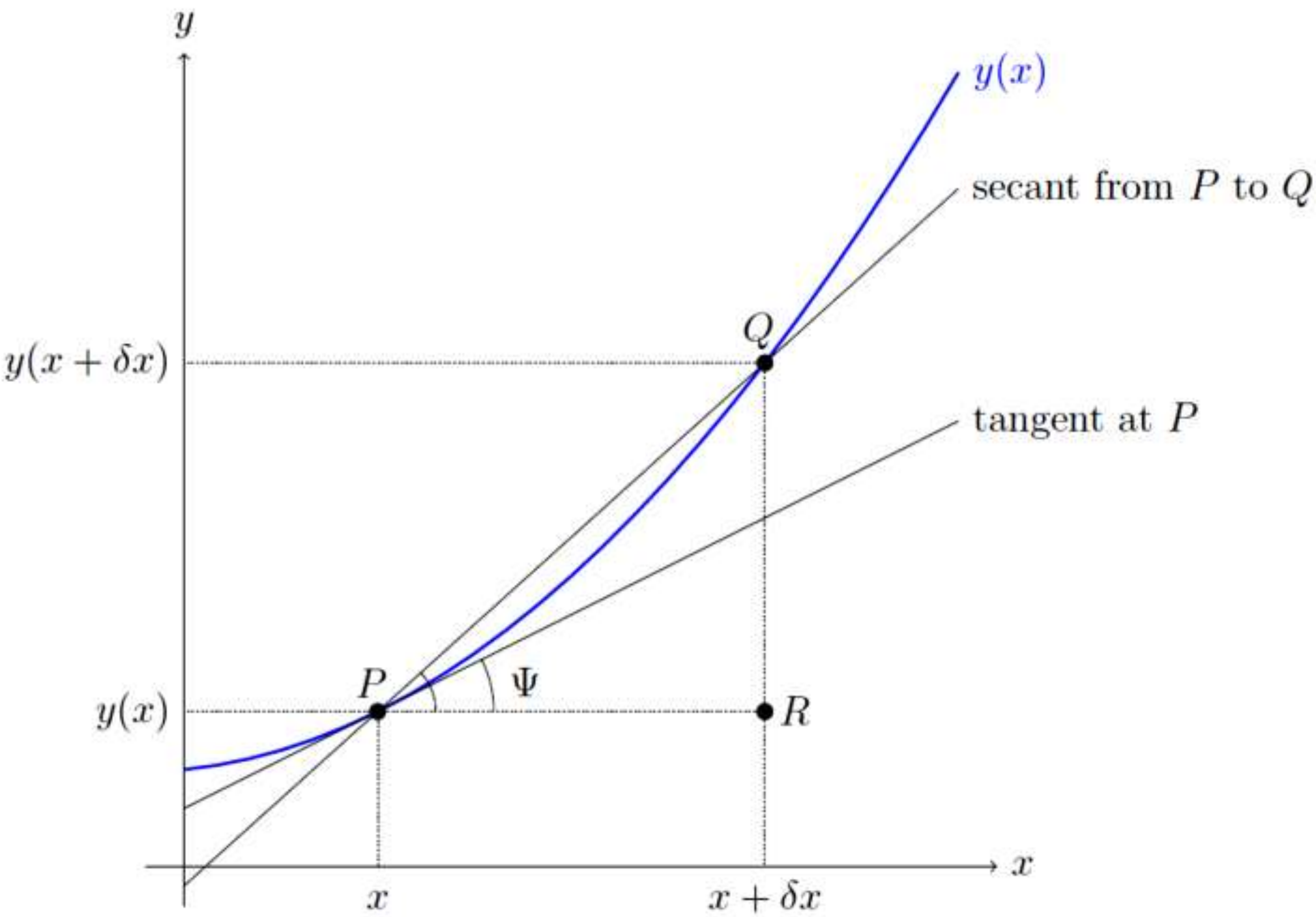
$$N(0) = 1000e^{c(0)} = 1000(1) = 1000$$

Next, calculate $\frac{dN}{dt}$ and verify that it satisfies the differential equation:

$$\frac{dN}{dt} = 1000ce^{ct} = c(1000e^{ct}) = cN \qquad \text{as required.}$$

The derivative:

The derivative of a function y(x) at a particular value of x is the slope of the tangent



to the curve at the point P, or (x; y(x)).

Suppose y(x) is a function; then the derivative $\frac{dy}{dx}$ at a particular value of x is the

slope of the curve at the point P (or, the slope of the line that is tangent to the curve

at the point P):

$$\frac{dy}{dx} = \tan \Psi$$

If Q is a neighboring point on the curve, then we can take the limit as Q tends to P:

$$\frac{dy}{dx} = \lim_{Q \to P} \frac{QR}{PR} = \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{\delta x}$$

Review of Differentiation Formulas:

Powers of x rule:

If $f(x) = x^n$, then $f'(x) = n(x)^{n-1}$

Constant rule:

If $f(x) = C$, then $f'(x) = 0$

Coefficient rules:

If $f(x) = c \cdot u(x)$, then $f'(x) = c \cdot u'(x)$

If $f(x) = k \cdot x^n$, then $f'(x) = kn(x)^{n-1}$

If $f(x) = kx$, then $f'(x) = k$

Sum rule:

If $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$

Difference rule:

If $f(x) = u(x) - v(x)$, then $f'(x) = u'(x) - v'(x)$

Product rule:

If $f(x) = u(x) \cdot v(x)$, then $f'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$

Quotient rule:

If $f(x) = \frac{u(x)}{v(x)}$, then $f'(x) = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{\left(v(x)\right)^2}$

Power (Chain) rule:

If $f(x) = \left(u(x)\right)^n$, then $f'(x) = n\left(u(x)\right)^{n-1} \cdot u'(x)$

Derivative of Natural Log:

If $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$

If $f(x) = \ln u(x)$, then $f'(x) = \frac{1}{u(x)} \cdot u'(x)$

Derivative of Exp. Function:

If $f(x) = e^x$, then $f'(x) = e^x$

If $f(x) = e^{u(x)}$, then $f'(x) = e^{u(x)} \cdot u'(x)$

Properties of Natural Log:

$$\ln(a.b) = \ln(a) + \ln(b)$$

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$$

$$\ln(a^n) = n\ln(a)$$

Properties of Exp. Function:

$$e^a . e^b = e^{a+b}$$

$$\frac{e^a}{e^b} = e^{a-b}$$

$$(e^a)^b = e^{a.b}$$

$$\ln(e^x) = \mathrm{x}$$

$$e^{ln(x)} = \mathrm{x}$$

Trigonometric Functions:

If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$

If $f(x) = \cos(x)$, then $f'(x) = -sin(x)$

If $f(x) = \tan(x)$, then $f'(x) = \sec^2(x)$

If $f(x) = \cot(x)$, then $f'(x) = -\csc^2(x)$

If $f(x) = \sec(x)$, then $f'(x) = \sec(x)\tan(x)$

If $f(x) = \csc(x)$, then $f'(x) = -\csc(x)\cot(x)$

Some Rules of Integrals:

$\int kdx = kx + c$

$\int x^n dx = \dfrac{x^{n+1}}{n+1} + c$

$\int kx^n dx = \frac{k}{n+1} x^{n+1} + c$

$\int u(x)^n \cdot u'(x)dx = \int u^n du = \frac{u(x)^{n+1}}{n+1} + c,\ \ if\ n \neq -1.\,(du = u'dx)$

$\int e^x dx = e^x + c$

$\int e^{u(x)} u'(x)dx = \int e^u du = e^{u(x)} + c$

$\int \frac{1}{x} dx = \ln|x| + c$

$\int \frac{1}{u(x)} \cdot u'(x)dx = \int \frac{1}{u} du = \ln|u(x)| + c$

$\int \sin(u)\, du = -\cos(u) + c$

$\int \cos(u)\, du = \sin(u) + c$

$\int \sec^2(u)\, du = \tan(u) + c$

$\int \csc^2(u)\, du =$ -$\cot(u) + c$

$\int \sec(u) \tan(u)\, du = \sec(u) + c$

$\int \csc(u) \cot(u) du = -\csc(u) + c$

Integration by parts : $\int udv = uv - \int vdu$

Note: $\frac{rx+sy}{xy}$ can be transferred into $\frac{A}{x} + \frac{B}{y}$ for some values of A and B

Example: : $\frac{2x+3y}{xy} = \frac{A}{x} + \frac{B}{y} =: \frac{Ay+Bx}{xy} \rightarrow Ay + Bx = 2x + 3y \rightarrow A = 3, B = 2$

Hence, $\frac{2x+3y}{xy} = \frac{3}{x} + \frac{2}{y}$

<u>DEFINITION:</u> (**Differential Equation (DE)**)

An equation containing some derivatives of an unknown function (or dependent variable), with respect to one or more independent variables, is said to be a **differential equation (DE).**

To talk about them, we shall classify differential equations according to **type, order, degree, and linearity.**

<u>NOTE</u>: (Dependent and independent variables)

In general, for any given equation, there are two types of variables:

Independent variables - The values that can be changed or controlled in a given equation. They provide the "input" which is modified by the equation to change the "output."

Dependent variables - The values that result from the independent variables. For example,

unknown function
or dependent variable

$$\frac{d^2x}{dt^2} + 16x = 0$$

independent variable

<u>EXAMPLES:</u>

1- $y = f(x) = x^3 + 2x^2 - 5$

x: is the independent variable y: is the dependent variable

2- $N(t) = 1000e^{ct}$

t: is the independent variable N: is the dependent variable

3- $x = f(y, t) = x^3 + 2x^2 - 5$

y and t: are the independent variables x: is the dependent variable

<u>NOTE</u>: (Ordinary derivatives)

If the function $y = y(x)$ has a single (only one) independent variable, all derivatives of y are called ordinary derivatives and are defined as follows:

$\frac{dy}{dx} = y'$ is called the first derivative of y with respect of x.

$\frac{d^2y}{dx^2} = y''$ is called the derivative of y' with respect of x.

OR $\frac{d^2y}{dx^2} = y''$ is called the second derivative of y with respect of x.

$\frac{d^3y}{dx^3} = y'''$ is called the third derivative of y with respect of x.

$\frac{d^4y}{dx^4} = y^{(4)}$ is called the fourth derivative of y with respect of x.

In general, $\frac{d^ny}{dx^n} = y^{(n)}$ is called the n-th derivative of y with respect of x.

NOTE: The ordinary derivative $\frac{dy}{dx}$ of a function $y = y(x)$ is itself another function $y' = y'(x)$ found by an appropriate rule.

EXAMPLE: Let $y = f(x) = x^3 + 2x^2 - 5$ be a function, so y' is another function such that: $\frac{dy}{dx} = y' = y'(x) = 3x^2 + 4x$

NOTE: (Partial derivatives)

If the function y has two or more independent variables, the derivatives of y are called partial derivatives. For example, if $y = y(x, t)$, the partial derivatives of y are defined as follows:

$\frac{\partial y}{\partial x} = y_x$ is called the partial derivative of y with respect of x.

$\frac{\partial y}{\partial t} = y_t$ is called the partial derivative of y with respect of t.

$\frac{\partial^2 y}{\partial x^2} = y_{xx}$ is called the partial derivative of y_x with respect of x.

$\frac{\partial^2 y}{\partial x \partial t} = y_{tx}$ is called the partial derivative of y_x with respect of t.

$\frac{\partial^2 y}{\partial t^2} = y_{tt}$ is called the partial derivative of y_t with respect of t.

EXAMPLE: Let $= f(x,t) = \ tx^3 + 2t^2 - 5$, we can find the partial derivatives as follows:

$$\frac{\partial y}{\partial x} = y_x = 3tx^2$$

$$\frac{\partial y}{\partial t} = y_t = x^3 + 4t$$

$$\frac{\partial^2 y}{\partial x^2} = y_{xx} = 6tx$$

$$\frac{\partial^2 y}{\partial x \partial t} = y_{tx} = 3x^2$$

$$\frac{\partial^2 y}{\partial t \partial x} = y_{xt} = 3x^2$$

$$\frac{\partial^2 y}{\partial t^2} = y_{tt} = 4$$

Classification by Type:

Type 1: Ordinary Differential Equation (ODE) : a differential equation contains some ordinary derivatives of an unknown function with respect to a single independent variable is said to be an ordinary differential equation (ODE).

EXAMPLE: The following Equations are examples of ordinary differential equations (ODEs) : $\frac{dy}{dx} + 5y = e^x$, $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$, $u'' + \sin(u) = t^2$

Type 2: Partial Differential Equation (PDE) : An equation involving some partial derivatives of an unknown function of two or more independent variables is called a partial differential equation (PDE).

EXAMPLE: The following equations are partial differential equations (PDEs)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Classification by Order: The order of a differential equation (either ODE
or PDE) is the order of the highest derivative in the equation. For example,

$$\overset{\text{second order}}{\frac{d^2y}{dx^2}} + 5\Bigg(\overset{\text{first order}}{\frac{dy}{dx}}\Bigg)^3 - 4y = e^x$$

is a second-order ordinary differential equation.

an ODE can contain more than one unknown function

$$\frac{dy}{dx} + 5y = e^x, \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \quad \text{and} \quad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

Classification by Degree: The degree of a differential equation is the power of the highest order derivative in the equation.

EXAMPLES:

$$\frac{d^3y}{dx^3} + 4x\left(\frac{dy}{dx}\right)^2 = y\frac{d^2y}{dx^2} + e^y$$ is an ODE of order 3 and degree 1

$$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} = \sin x$$ is an ODE of order 2 and degree 3

Remark: Not every differential equation has a degree. If the derivatives or the unknown function occur within radicals , fractions, radicals, trigonometric functions or logarithms, then the equation may not have a degree.

Note: In **Mathematics** the concept of **(**linear equation), with respect to a variable, just means that the variable in an equation appears only with a power of one and is not multiplied by a non-constant function of the same variable.

For examples,

$y^2 + x = 0$, is linear function with respect to x, whereas it is nonlinear with respect to y.

$ysin(y) + {}^{1}/_{x}$ is nonlinear function with respect to both of y and x.

Classification by Linearity: We say that the DE is **linear,** if it is linear with respect to the unknown function and its derivatives (each one of them appears with power of one and they are not multiplied by each other). Otherwise, it is called **Nonlinear**.

Here are some examples:

$y'' + y = e^x$ is linear $y'' + \log(x)y' + x\,y = 0$ is linear

$y' + 1/y = 0$ is non-linear because $1/y$ is not of power one

$y' + y^2 = 0$ is non-linear because y^2 is not of power one

$yy' + \sin(x) = 0$ is non-linear because y is multiplied by its first derivative.

DEFINITION: $y = y(t)$ is called a solution of the ODE on the interval $I \subseteq \mathbb{R}$ if it satisfies the equation, and it is defined on I.

EXAMPLE: $y = ce^{-x} + 2$ is the solution of the ODE $y' + y = 2$ on $\mathbb{R}$ because it is defined on $\mathbb{R}$ and it satisfies the ODE as follows:

$$y' + y = (ce^{-x} + 2)' + (ce^{-x} + 2) = -ce^{-x} + ce^{-x} + 2 = 2$$

NOTE: The general form of ordinary differential equations of order **n** takes the form: $y^{(n)} = f(x, y', y'', \dots\dots\dots..\, y^{(n-1)})$

DEFINITION: Consider that, we have an ODE of order **n**, if we know the value of y and some of its derivatives at a particular point x_0, such as:

$$y(x_0) = y_0\,,\quad y'(x_0) = y_1, \dots\dots\dots., y^{(n-1)}(x_0) = y_{n-1}$$

Then this problem is called initial value problem (IVP), while these values are called initial conditions.

NOTE: For ODEs of order one, the initial value problem takes the form:

$$y' = f(x, y), \qquad y(x_0) = y_0$$

SETRATIGY: The initial value problem **(IVP)** of order one can be solved in two steps:

1- Find the general solution of the ODE, $y = y(x, c)$

2- Using the initial condition, plug it into the general solution and solve for c.

EXAMPLE: Solve the initial value problem **(IVP)**:

$$y' = y^2$$

$$y(0) = -1$$

Solution: step 1: $y' = y^2 \quad \rightarrow \quad \frac{dy}{dx} = y^2 \quad \rightarrow \quad \frac{dy}{y^2} = dx$

$$\rightarrow \quad \int \frac{dy}{y^2} = \int dx \quad \rightarrow \quad \frac{-1}{y} = \mathrm{x} + \mathrm{c}$$

The general solution is $y = \frac{-1}{x+c}$ which is defined on $\mathbb{R}/\{-c\}$

Step 2: $y = \frac{-1}{x+c}$ and $y(0) = -1$

So $y(0) = \frac{-1}{0+c} = \frac{-1}{c} = -1 \quad \rightarrow \quad c = 1$

Finally, $y = \frac{-1}{x+c} = \frac{-1}{x+1}$ is the solution of the above IVP.

Exercises for Chapter Zero:

I. For each of the following DE, state the Type (ODE or PDE), the Order, the Degree, and Linearity (linear or nonlinear):

1. $(1 - x)y'' - 4xy' + 5y = \cos x$

2. $x\frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^4 + y = 0$

3. $t^5y^{(4)} - t^3y'' + 6y = 0$

4. $\frac{d^2u}{dr^2} + \frac{du}{dr} + u = \cos(r + u)$

5. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

6. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$

II. Verify whether or not that the indicated function is a solution of the given differential equation on the interval $(-\infty, \infty)$:

(a) $dy/dx = xy^{1/2}; \quad y = \frac{1}{16}x^4$ **(b)** $y'' - 2y' + y = 0; \quad y = xe^x$

III. Show that $y = 5e^x$, is the solution of the following IVP:

$$y' = y\ ,\ \ y(0) = 5$$