Derivatives of Hyperbolic Functions :

1. If $y=f(x)=\sinh x$ then $\frac{dy}{dx}=\cosh x$.
2. If $y=f(x)=\cosh x$ then $\frac{dy}{dx}=\sinh x$.
3. If $y=f(x)=\tanh x$ then $\frac{dy}{dx}=\operatorname{sech}^2 x$.
4. If $y=f(x)=\coth x$ then $\frac{dy}{dx}=-\operatorname{csch}^2 x$.
5. If $y=f(x)=\operatorname{sech} x$ then $\frac{dy}{dx}=-\operatorname{sech} x \tanh x$.
6. If $y=f(x)=\operatorname{csch} x$ then $\frac{dy}{dx}=-\operatorname{csch} x \coth x$.

proof(1) $y=\sinh x=\frac{1}{2}(e^{x}-e^{-x})$

$$\frac{dy}{dx}=\frac{1}{2}\left[e^{x}-e^{-x}(-1)\right]=\frac{1}{2}(e^{x}+e^{-x})=\cosh x.$$

proof(2) $y=\cosh x=\frac{1}{2}(e^{x}+e^{-x})$

$$\frac{dy}{dx}=\frac{1}{2}(e^{x}+e^{-x}(-1))=\frac{1}{2}(e^{x}-e^{-x})=\sinh x.$$

proof(3) $y=\tanh x=\frac{\sinh x}{\cosh x}$

$$\frac{dy}{dx}=\frac{\cosh x\cosh x-\sinh x\sinh x}{\cosh^2 x}=\frac{\cosh^2 x-\sinh^2 x}{\cosh^2 x}$$

$$=\frac{1}{\cosh^2 x}=\operatorname{sech}^2 x.$$

Now, if $u(x)=u$ is a differentiable function of x and

1. $y=\sinh u$ then $\frac{dy}{dx}=\cosh u\frac{du}{dx}$.
2. $y=\cosh u$ then $\frac{dy}{dx}=\sinh u\frac{du}{dx}$.
3. $y=\tanh u$ then $\frac{dy}{dx}=\operatorname{sech}^2 u\frac{du}{dx}$.
4. $y=\coth u$ then $\frac{dy}{dx}=-\operatorname{csch}^2 u\frac{du}{dx}$.
5. $y=\operatorname{sech} u$ then $\frac{dy}{dx}=-\operatorname{sech} u\tanh u\frac{du}{dx}$.
6. $y=\operatorname{csch} u$ then $\frac{dy}{dx}=-\operatorname{csch} u\coth u\frac{du}{dx}$.

EX.(5) Find $\frac{dy}{dx}$ of the following functions:

(1) $y = f(x) = \sinh(x^2 + 3\sin x + \ln x)$

$$y' = \frac{dy}{dx} = \cosh(x^2 + 3\sin x + \ln x) \cdot (2x + 3\cos x + \frac{1}{x}).$$

(2) $y = f(x) = \tanh^{-2}[e^{\tan^{-1}2x} + \operatorname{sech}^2 2x + \sin(e^{2x})]$

$$\frac{dy}{dx} = -2\tanh^{-3}[e^{\tan^{-1}2x} + \operatorname{sech}^2 2x + \sin(e^{2x})] \cdot \operatorname{sech}^2[e^{\tan^{-1}2x} + \operatorname{sech}^2 2x + \sin(e^{2x})] \cdot [e^{\tan^{-1}2x} \cdot \frac{2}{1+4x} - 4\operatorname{sech}^2 2x \tanh 2x + 2\cos(e^{2x}) \cdot e^{2x}].$$

H.W

(3) $y = f(x) = \operatorname{csch}^{-3}(\tan 2x + \tan^{-1}2x) \cdot \csc^{-3}(\tanh 2x)$.

EX.(6) If $y = x^{x\sinh x}$. find $\frac{dy^3}{dx^3}$

Solu.

Let $u = y^3$ and $v = x^3 \Rightarrow \frac{dy^3}{dx^3} = \frac{du}{dv} = \frac{du}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dv}$

Now, $u = y^3 \Rightarrow \frac{du}{dy} = 3y^2$

$v = x^3 \Rightarrow \frac{dv}{dx} = 3x^2$

$\ln y = x \sinh x \ln x$

$$\frac{1}{y}\frac{dy}{dx} = (x\sinh x) \cdot \frac{1}{x} + \ln x\,(x\cosh x + \sinh x)$$

$$\therefore \frac{dy}{dx} = y[\sinh x + x\ln x \cosh x + \ln x \sinh x]$$

so $\frac{dy^3}{dx^3} = \frac{du}{dv} = 3y^2 \cdot y[\sinh x + x\ln x\cosh x + \ln x \sinh x] \cdot \frac{1}{3x^2}$
where $y = x^{x\sinh x}$.

EX. (7) Evaluate the following Limits:

1. $$\lim_{x\to 0} \frac{\sinh x - x}{x^3} = \lim_{x\to 0} \frac{\cosh x - 1}{3x^2} = \lim_{x\to 0} \frac{\sinh x}{6x}$$

$$= \lim_{x\to 0} \frac{\cosh x}{6} = \frac{\cosh 0}{6} = \frac{1}{6}$$

H.W
2. $$\lim_{x\to 0} \frac{\tanh 2x - 2x}{3x - \sinh 3x}$$

H.W
3. $$\lim_{x\to 0} (\operatorname{csch} x - \coth x)$$

H.W
4. $$\lim_{x\to 0} \frac{x - \sinh x}{(1-\cosh x)^2}$$

The Inverse of The Hyperbolic Functions معكوس الدوال الزائدية

1. For $-\infty < x < \infty$, we define $y = \sinh^{-1} x$ if and only if $x = \sinh y$ for which $-\infty < y < \infty$

2. For $x \geq 1$, we define $y = \cosh^{-1} x$ iff $x = \cosh y$ for which $y \geq 0$.

3. For $|x| < 1$, we define $y = \tanh^{-1} x$ iff $x = \tanh y$ for which $-\infty < y < \infty$.

4. For $|x| > 1$, we define $y = \coth^{-1} x$ iff $x = \coth y$ for which $y \neq 0$

5. For $0 < x \leq 1$, we define $y = \operatorname{sech}^{-1} x$ iff $x = \operatorname{sech} y$ for which $0 \leq y < \infty$.

6. For $x \neq 0$, we define $y = \operatorname{csch}^{-1} x$ if and only if $x = \operatorname{csch} y$ for which $y \neq 0$

Relations

1. $\sinh^{-1} x = \operatorname{csch}^{-1} \frac{1}{x}$ 2. $\cosh^{-1} x = \operatorname{sech}^{-1} \frac{1}{x}$ 3. $\tanh^{-1} x = \coth^{-1} \frac{1}{x}$

proof

Let $y = \sinh^{-1} x \Rightarrow x = \sinh y = \frac{1}{\operatorname{csch} y} \Rightarrow \operatorname{csch} y = \frac{1}{x}$

$\Rightarrow y = \operatorname{csch}^{-1} \frac{1}{x} \Rightarrow \sinh^{-1} x = \operatorname{csch}^{-1}\left(\frac{1}{x}\right)$

Expressions for the inverse of the hyperbolic functions

(1.) $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$ (2.) $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$

(3.) $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$ (4.) $\coth^{-1} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right)$

(5.) $\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ (6.) $\operatorname{csch}^{-1} x = \frac{1}{2} \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)$.

proof(1) let $y = \sinh^{-1} x \Rightarrow x = \sinh y = \frac{e^{y} - e^{-y}}{2} \Rightarrow 2x = \frac{e^{y} - e^{-y}}{1}$

$\Rightarrow 2x = e^{y} - \frac{1}{e^{y}} \Rightarrow 2x e^{y} = e^{2y} - 1 \Rightarrow (e^{y})^2 - 2x e^{y} - 1 = 0$

Let $z = e^{y} \Rightarrow z^2 - 2x z - 1 = 0 \Rightarrow z = \frac{2x \pm \sqrt{4x^2 - 4(-1)(1)}}{2}$

$z = \frac{2x \pm 2\sqrt{x^2+1}}{2} = x \pm \sqrt{x^2+1} = e^{y}$

since $e^{y} > 0 \Rightarrow x - \sqrt{x^2+1} > 0 \Rightarrow x > \sqrt{x^2+1}$ impossible.

$\therefore e^{y} = x + \sqrt{x^2+1} \Rightarrow y = \ln(x + \sqrt{x^2+1})$.

<u>proof(3)</u> Let $y = \tanh^{-1} x \Rightarrow x = \tanh y = \dfrac{e^{y} - e^{-y}}{e^{y} + e^{-y}} = \dfrac{e^{y} - \frac{1}{e^{y}}}{e^{y} + \frac{1}{e^{y}}}$

$$x = \frac{e^{2y} - 1}{e^{2y} + 1} \Rightarrow x e^{2y} + x = e^{2y} - 1$$

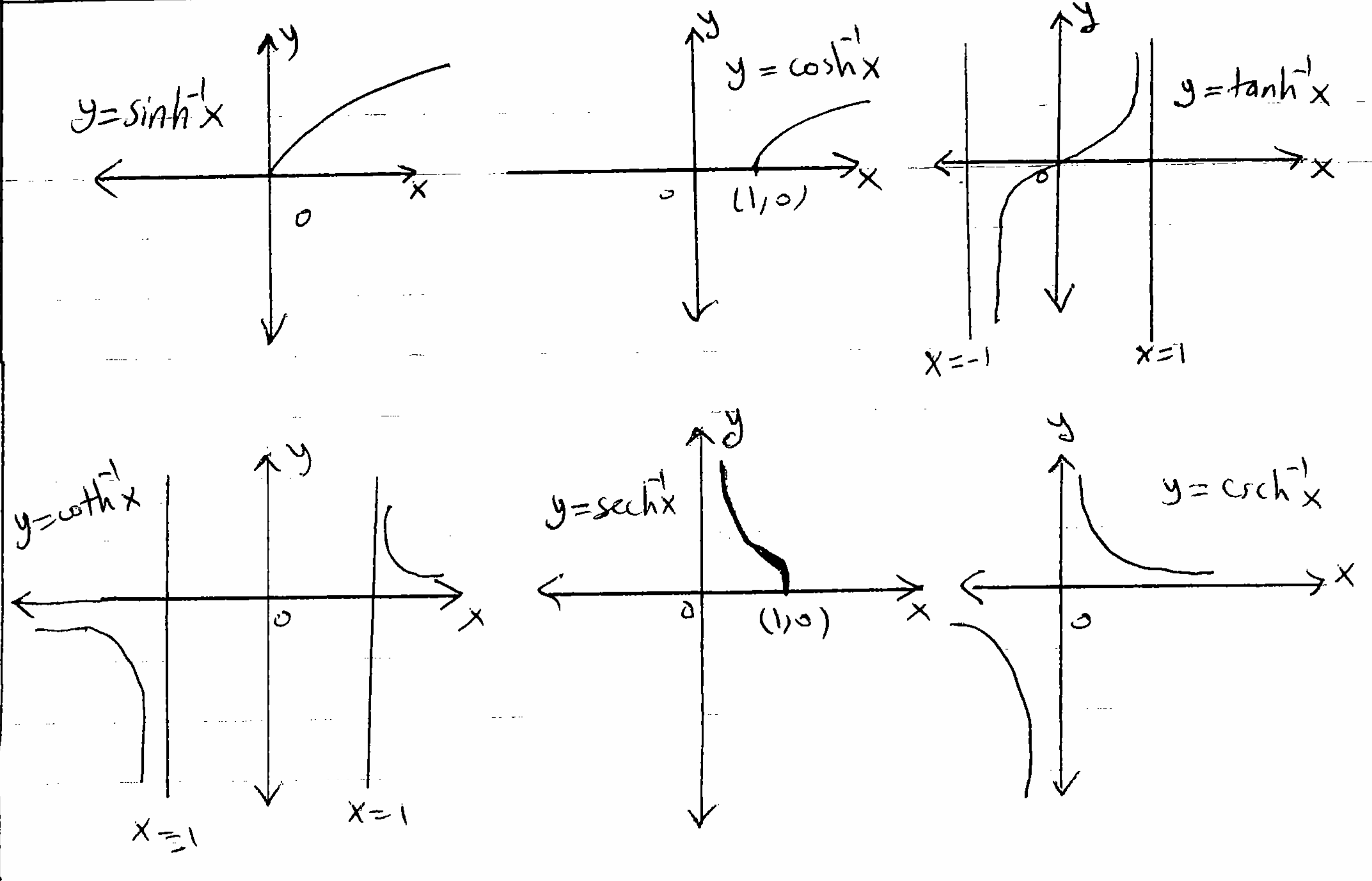
$$\Rightarrow 1 + x = e^{2x}(1 - x) \Rightarrow e^{2y} = \frac{1+x}{1-x} \Rightarrow$$

$$2y = \ln\left(\frac{1+x}{1-x}\right) \Rightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

<u>Note</u> $\sinh^{-1}(\sinh x) = x$, $\sinh(\sinh^{-1} x) = x$

Similarly for the other hyperbolics

Graph of The Inverse of Hyperbolic Functions



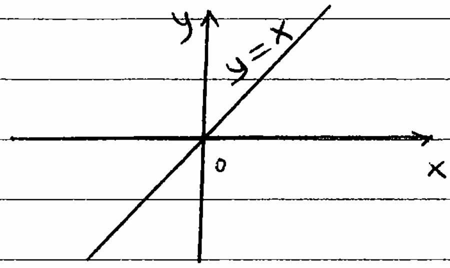
EX.(8) Evaluate $\tanh^{-1}(-\frac{1}{2}) = \frac{1}{2}\ln\left(\frac{1+(-\frac{1}{2})}{1-(-\frac{1}{2})}\right) = \frac{1}{2}\ln\left(\frac{1/2}{3/2}\right)$

$= \frac{1}{2}(\ln(1) - \ln(3)) = -\frac{1}{2}\ln 3.$

EX.(9) Sketch the graph of $y = \frac{1}{2}\ln\left(\frac{1+\tanh x}{1-\tanh x}\right)$

$$y = \frac{1}{2}\ln\left(\frac{1+\tanh x}{1-\tanh x}\right) = \tanh^{-1}(\tanh x) = x$$

$\therefore\ y = x$



EX.(10) Find the asymptotes of the function $y = \tanh(\ln\sqrt{x})$

Solu. $y = \dfrac{e^{2\ln\sqrt{x}} - 1}{e^{2\ln\sqrt{x}} + 1} = \dfrac{e^{\ln x} - 1}{e^{\ln x} + 1} = \dfrac{x-1}{x+1}$

$x = -1$ is V. Asymp.
$y = 1$ is H. Asymp.

Derivatives:

1. If $y = \sinh^{-1}x$ then $y' = \dfrac{1}{\sqrt{x^2+1}}$.

2. If $y = \cosh^{-1}x$ then $y' = \dfrac{1}{\sqrt{x^2-1}}$.

3. If $y = \tanh^{-1}x$ / $\coth^{-1}x$ then $y' = \dfrac{1}{1-x^2}$.

4. If $y = \operatorname{sech}^{-1}x$ then $y' = -\dfrac{1}{x\sqrt{1-x^2}}$.

5. If $y = \operatorname{csch}^{-1}x$ then $y' = -\dfrac{1}{x\sqrt{1+x^2}}$.

proof(1) $y = \sinh^{-1} x \Rightarrow x = \sinh y \Rightarrow 1 = \cosh y \frac{dy}{dx}$

$$\therefore \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{\sinh^2 y + 1}} = \frac{1}{\sqrt{x^2+1}}.$$

Now, if $u = u(x)$ is a differentiable function of x and

1. $y = \sinh^{-1} u$ then $\frac{dy}{dx} = \frac{1}{\sqrt{u^2+1}} \frac{du}{dx}$.

2. $y = \cosh^{-1} u$ then $\frac{dy}{dx} = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx}$

3. $y = \begin{cases} \tanh^{-1} u \\ \coth^{-1} u \end{cases}$ then $\frac{dy}{dx} = \frac{1}{1-u^2} \frac{du}{dx}$

4. $y = \operatorname{sech}^{-1} u$ then $\frac{dy}{dx} = \frac{-1}{u\sqrt{1-u^2}} \cdot \frac{du}{dx}$

5. $y = \operatorname{csch}^{-1} u$ then $\frac{dy}{dx} = -\frac{1}{u\sqrt{1+u^2}} \frac{du}{dx}$

EX. (11) Find $\frac{dy}{dx}$ of the following functions

(1) $y = \sinh^{-1}(x^2 + \sin^2 x) \Rightarrow y' = \frac{2x + 2\sin x \cos x}{\sqrt{(x^2+\sin^2 x)^2 + 1}}$.

H.W
(2) $y = \operatorname{sech}^{-3}\left[\operatorname{sech}^{-2}(\operatorname{sech}^{-1} 2x)\right]$.