كلية المنصور الجامعة

Civil Engineering

1st. Year

Mathematics I

مدرس المادة / د. عماد

ملزمة (١)

[Calculus And Analytic Geometry] آم الطهر

By Thomas

Inequalities الترايحات

If a and b are real no.'s, then one of the following is true: $a > b$ or $a = b$ or $a < b$

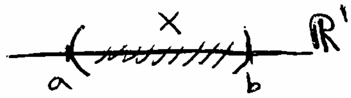
Notes: (1) If $a > b$ then $-a < -b$.

(2) If $a > b$ then $\frac{1}{a} < \frac{1}{b}$.

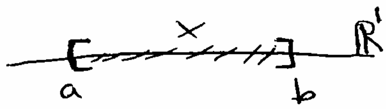
Intervals الفترات

Defn. An interval is a set of no.'s x having one of the following form:

(i) Open interval: $a < x < b \equiv (a, b)$



(ii) Closed interval: $a \leq x \leq b \equiv [a, b]$



(iii) Half open from the left or half close from the right:

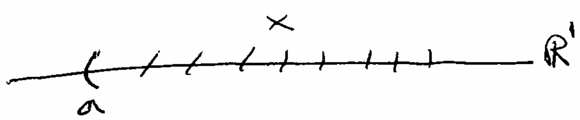
$a < x \leq b \equiv (a, b]$.

(iv) Half close from the left or half open from the right:

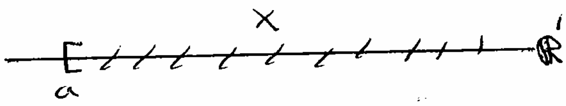
$a \leq x < b \equiv [a, b)$.

Notes:

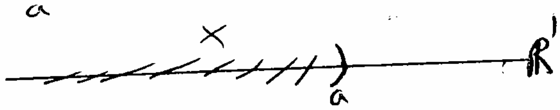
(1) $a < x < \infty \equiv a < x \equiv (a, \infty)$



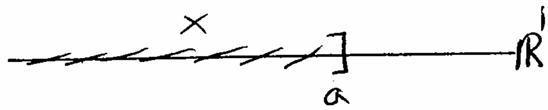
(2) $a \leq x < \infty \equiv a \leq x \equiv [a, \infty)$



(3) $\infty < x < a \equiv x < a \equiv (-\infty, a)$



(4) $\infty < x \leq a \equiv x \leq a \equiv (-\infty, a]$



Absolute Value القيمة المطلقة

Defn: The absolute value of a real no. x is define as

$$|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$

Properties of Absolute Values : خصائص القيم المطلقة

1. $|x \cdot y| = |x| \cdot |y|$ and $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.
2. $|-x| = |x|$.
3. $|x+y| \leq |x| + |y|$.
4. $|x| < a$ mean $-a < x < a$.
5. $|x| \leq a$ mean $-a \leq x \leq a$.
6. $|x| > a$ mean $x < -a$ or $x > a$.
7. $|x| \geq a$ mean $x \leq -a$ or $x \geq a$.

Example Find the solution set of the following ineq.s :

(1) $\left|\frac{3x+1}{2}\right| < 1$, (2) $|x-1| \geq 5$.

Solu.

(1) $\left|\frac{3x+1}{2}\right| < 1 \Rightarrow -1 < \frac{3x+1}{2} < 1 \Rightarrow -2 < 3x+1 < 2$

$\Rightarrow -3 < 3x < 1 \Rightarrow -1 < x < \frac{1}{3}$

(2) $|x+1| \geq 5 \Rightarrow x-1 \leq -5$ or $x-1 \geq 5 \Rightarrow x \leq -4$

or $x \geq 6$

Graphs And Functions :

Defn. : The solution set or Locus of an equation in two unknown consists of all points in the plane whose coordinates satisfy the eq.

A geometrical representation of the locus is called the graph of the equation.

EX. Sketch the graph of the following eq.'s :

(1) $2x+3y=6$.

(2) $y = \begin{cases} x & 0 \leq x \leq 1 \\ 2-x & 1 \leq x \leq 2 \end{cases}$

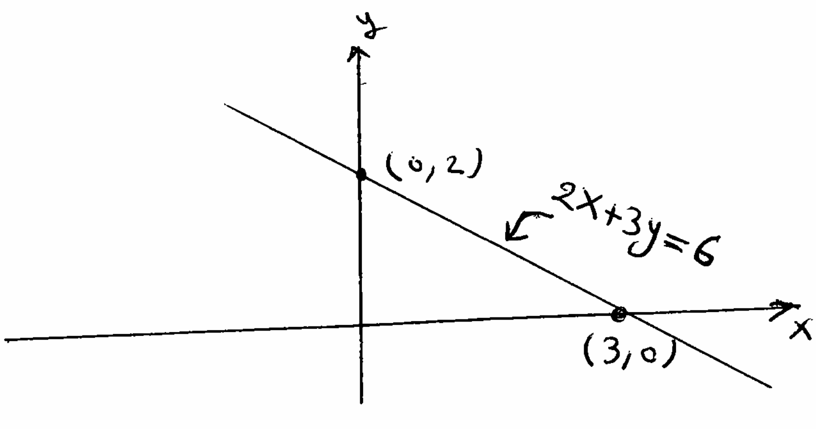
(3) $y = \begin{cases} -x, & x<0 \\ x^2, & 0<x\leq 1 \\ 1, & 1<x \end{cases}$

(4) $y = |x^2-1|$

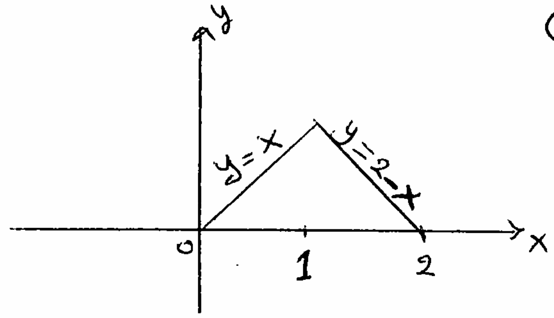
(5) $16x^2+25y^2=400$.

Solu.

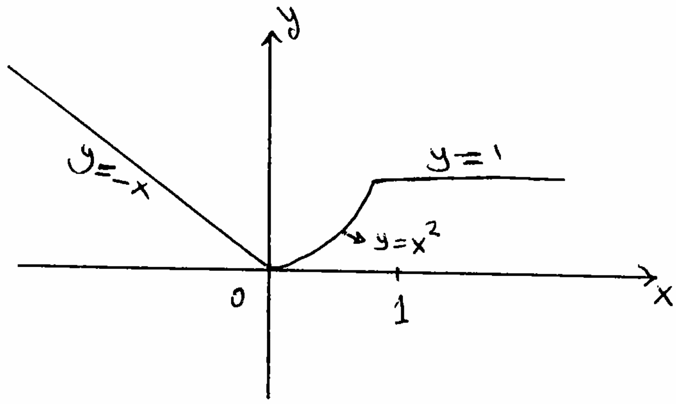
(1) $2x+3y=6$



(2) $y = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \end{cases}$



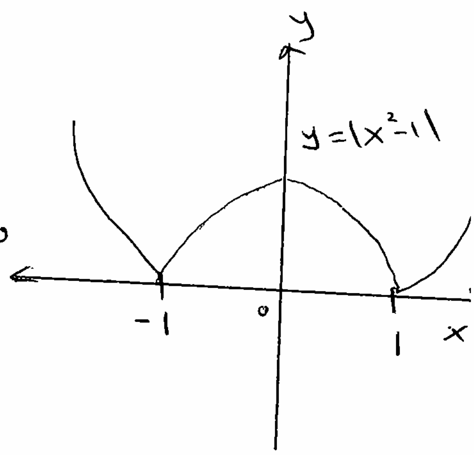
(3) $y = \begin{cases} -x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & 1 < x \end{cases}$



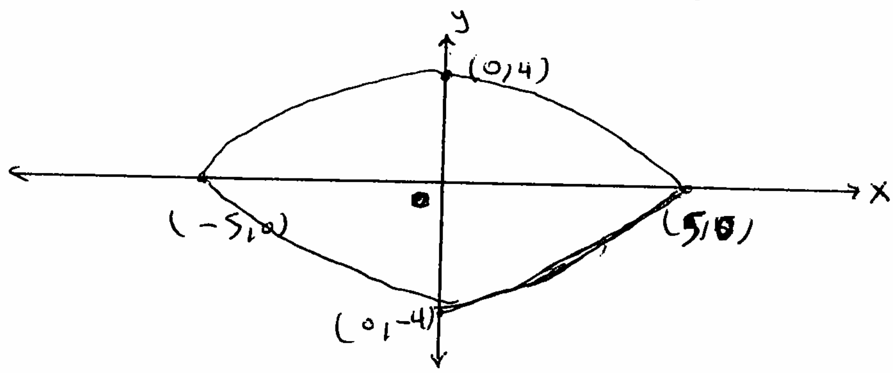
(4) $y = |x^2-1| = \begin{cases} x^2-1, & x^2-1 \ge 0 \\ -(x^2-1), & x^2-1 < 0 \end{cases}$

$= \begin{cases} x^2-1, & (x-1)(x+1) \ge 0 \\ 1-x^2, & (x-1)(x+1) < 0 \end{cases}$

$= \begin{cases} x^2-1, & x \le -1 \text{ or } x \ge 1 \\ 1-x^2, & -1 < x < 1 \end{cases}$



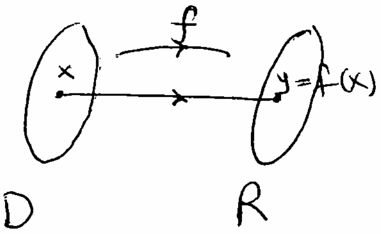
(5) $16x^2+25y^2=400 \Rightarrow \frac{x^2}{25}+\frac{y^2}{16}=1$, ellipse



الدالة

Defn. (Function): A function f from a set D to a set R is a rule that assigns a single element $y \in R$ to each element $x \in D$.

Note The element $y \in R$ denoted by $f(x)$, the set D is called the domain of f, and the set R is called the range of f.



Defn. $f(x)$ is an even function if $f(-x)=f(x)$.
$f(x)$ is an odd function if $f(-x)=-f(x)$.

EX.

(1) $f(x)=x^2\cos x \Rightarrow f(-x)=(-x)^2\cos(-x)$
$= x^2\cos x = f(x)$

$\therefore f(x)$ is even function.

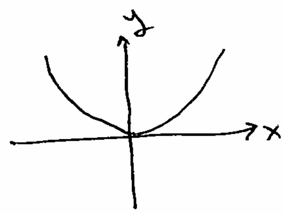
(2) $f(x)=\dfrac{x^2-1}{\sin x} \Rightarrow f(-x)=\dfrac{(-x)^2-1}{\sin(-x)} = \dfrac{x^2-1}{-\sin x} = -f(x)$

$\therefore f(x)$ is odd function.

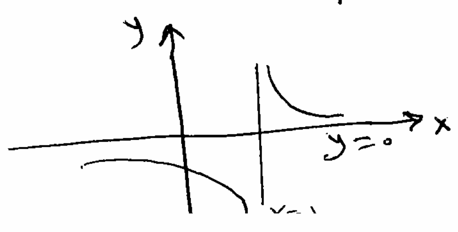
Note: We may define
The domain D is the set of all values of x for which y is define
The range R " " " " " " y " x " ".

EX. Find the domain and the range of the following functions:

(1) $y=f(x)=x^2$, D: all x , R: $y \geq 0$.

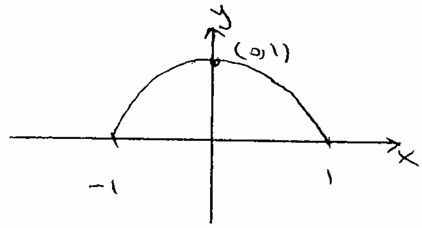


(2) $y=\dfrac{1}{x-1}$, D: $x \neq 1$



$X = \dfrac{y+1}{y}$, $R: y \neq 0$

③ $y = \sqrt{4-x^2}$, $D: -2 \leq X \leq 2$
$R: 0 \leq y \leq 2$



④ $y = f(x) = \sqrt{x^2-4x+3}$

$x^2-4X+3 \geq 0 \implies D: X \leq 1$ or $X \geq 3$

$y^2 = x^2-4X+3 \implies x^2-4x+3-y^2=0$

$$X = \frac{4 \pm \sqrt{16-4(3-y^2)}}{2} = \frac{4 \pm \sqrt{4+4y^2}}{2} = 2 \pm \sqrt{1+y^2}$$

$\therefore$ R: all y .

⑤ $y = \sqrt{2-\sqrt{x}}$

For $\sqrt{x}$ it must be $x \geq 0$

$2-\sqrt{x} \geq 0 \implies 2 \geq \sqrt{x} \implies 4 \geq x$

$\therefore$ $D: 0 \leq X \leq 4$

$X = (2-y^2)^2$, R: all y .

Intercepts , Symmetry , and Asymptotes

(1) To find x-intercepts , set $y=0$ and solve for y .
To find y-intercepts, set $x=0$ and solve for x .

(2) The locus is symmetric w.r.t the

(i) x-axis $(X,-y) \iff (x,y)$
(ii) y-axis $(-x,y) \iff (x,y)$
(iii) origin $(-x,-y) \iff (x,y)$

(3) (i) A line $x=a$ near which a locus goes of f to ∞ is called V. Asy.

(ii) A line $y=b$ near which a locus goes of f to ∞ is called H. Asy.

EX. Find the domain, the range, intercepts, symmetry, and asymptotes if they exist for the following functions. Sketch.

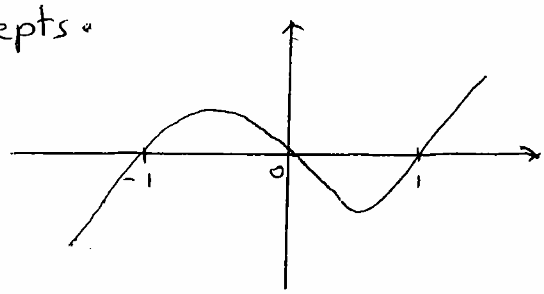
(1) $y=f(x)=x^3-x$, D: all x, R: all y

$(0,0)$, $(1,0)$, $(-1,0)$ are x-intercepts.

$(0,0)$ is y-intercept.

Symmetric w.r.t. origin only

No asymptotes.



(2) $y=f(x)=\dfrac{1}{x^2-1}$, D: $x\neq\pm 1$

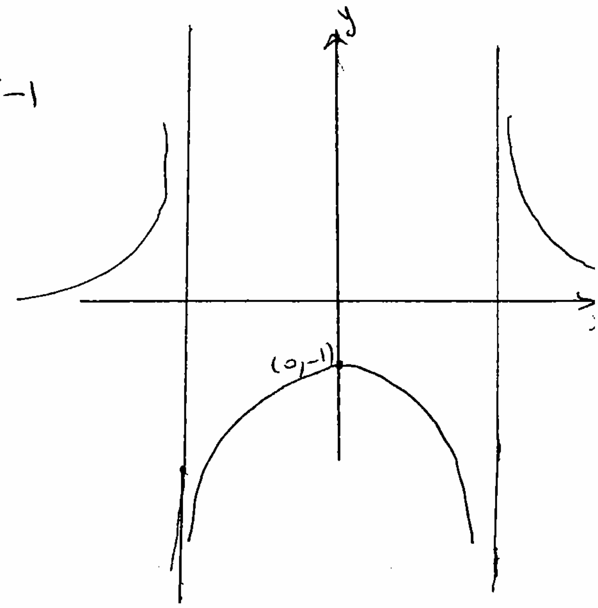
$x=\pm\sqrt{\dfrac{y+1}{y}}$, R: $y>0$ or $y\leq -1$

$(0,-1)$ is y-intercepts

Symm. w.r.t. y-axis only

$x=\pm 1$, V. Asy.

$y=0$, H. Asy.



Limit And Continuity الغاية والاستمرارية

Notation

When $f(x)$ tends to the number L as x tends to the number a

we write $f(x) \longrightarrow L$ as $x \longrightarrow a$

or $\lim\limits_{x \to a} f(x) = L$.

EX.(1) let $f(x) = 2x+5$

Evaluate $f(x)$ at $x = 1.1, 1.01, 1.001, 1.0001, \ldots$

$$f(1.1) = 2(1.1)+5 = 7.2$$
$$f(1.01) = 2(1.01)+5 = 7.02$$
$$f(1.001) = 2(1.001)+5 = 7.002$$
$$f(1.0001) = 2(1.0001)+5 = 7.0002$$
$$\vdots$$

We see that $f(x) \longrightarrow 7$ as $x \longrightarrow 1$

or $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} (2x+5) = 2(1)+5 = 7$

EX.(2) If $f(x) = \dfrac{x^2-3x+2}{x-2}$, $x \neq 2$. find $\lim\limits_{x \to 2} f(x)$.

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2-3x+2}{x-2} = \frac{4-6+2}{2-2} = \frac{0}{0} \text{ meaning less}$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2-3x+2}{x-2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x-1)}{\cancel{(x-2)}} = \lim_{x \to 2} (x-1) = 2-1 = 1.$$

EX.(3) Evaluate the following limits, if they exist.

1. $L = \lim\limits_{x \to -1} \dfrac{\sqrt{2+x} - 1}{x+1}$, $x \neq -1$, $x \geq -2$

$$L = \lim_{x \to -1} \frac{\sqrt{2+x} - 1}{x+1} \cdot \frac{\sqrt{2+x}+1}{\sqrt{2+x}+1} = \lim_{x \to -1} \frac{(2+x-1)}{(x+1)(\sqrt{2+x}+1)}$$

$$= \frac{1}{\sqrt{2-1}+1} = \frac{1}{1+1} = \frac{1}{2}.$$

2. $\lim\limits_{x \to 2} \dfrac{2-x}{2-\sqrt{2x}}$, $x \neq 2$, $x \geq 0$

$$L = \lim_{x \to 2} \frac{2-x}{2-\sqrt{2x}} \cdot \frac{2+\sqrt{2x}}{2+\sqrt{2x}} = \lim_{x \to 2} \frac{(2-x)(2+\sqrt{2x})}{4-2x}$$

$$= \lim_{x \to 2} \frac{(2-x)(2+\sqrt{2x})}{2(2-x)} = \frac{2+\sqrt{4}}{2} = \frac{2+2}{2} = 2.$$

H.W
3. $\lim\limits_{x \to 3} \dfrac{\sqrt{3x}-3}{x-3}$, $x \neq 3$

H.W
4. $\lim\limits_{x \to 2} \dfrac{x^4-2x^2-8}{x^2-4}$, $x \neq 2$

H.W
5. $\lim\limits_{x \to a} \dfrac{\sqrt{x^2+1} - \sqrt{a^2+1}}{x-a}$, $x \neq a$

H.W
6. $\lim\limits_{x \to 0} \dfrac{1}{x}\left(\dfrac{1}{x-2} - \dfrac{1}{2}\right)$, $x \neq 0, 2$.

H.W
7. $\lim\limits_{x \to 0} \dfrac{(1+x)^{3/2} - 1}{x}$, $x \neq 0$.

Theorems On Limits (Calculation Technique)

1. Uniqueness of limit

If $\lim\limits_{x \to a} f(x) = L_1$, and $\lim\limits_{x \to a} f(x) = L_2 \Rightarrow L_1 = L_2$.

2. Limit of Constant

If $f(x) = C$, C is constant then $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} C = C$.

3. Obvious limit

If $f(x)=x$ then $\lim_{x\to a} f(x) = \lim_{x\to a} x = a$.

4. limit of Sum

If $f(x)=f_1(x) \pm f_2(x) \pm f_3(x) \pm \cdots \pm f_n(x)$ and $\lim_{x\to a} f_i(x) = L_i$,

$i=1,2,\cdots,n$ then

$$\lim_{x\to a} f(x) = \lim_{x\to a} [f_1(x) \pm f_2(x) \pm \cdots \pm f_n(x)]$$

$$= \lim_{x\to a} f_1(x) \pm \lim_{x\to a} f_2(x) \pm \cdots \pm \lim_{x\to a} f_n(x)$$

$$= L_1 \pm L_2 \pm \cdots \pm L_n = \sum_{i=1}^{n} L_i$$

5. Limit of a product

If $f(x)=f_1(x) \cdot f_2(x) \cdot \cdots \cdot f_n(x)$ and $\lim_{x\to a} f_i(x) = L_i$

$i=1,2,\cdots,n$ then $\lim_{x\to a} f(x) = \lim_{x\to a} [f_1(x) \cdot f_2(x) \cdot \cdots \cdot f_n(x)]$

$$= \lim_{x\to a} f_1(x) \cdot \lim_{x\to a} f_2(x) \cdot \cdots \cdot \lim_{x\to a} f_n(x)$$

$$= L_1 \cdot L_2 \cdot \cdots \cdot L_n = \prod_{i=1}^{n} L_i .$$

6. Limit of a Quotient

If $f(x) = \dfrac{g(x)}{h(x)}$ and $\lim_{x\to a} g(x) = L_1$, and $\lim_{x\to a} h(x) = L_2 \neq 0$

then $\lim_{x\to a} f(x) = \lim_{x\to a} \dfrac{g(x)}{h(x)} = \dfrac{\lim_{x\to a} g(x)}{\lim_{x\to a} h(x)} = \dfrac{L_1}{L_2}$.

EX. (4) Evaluate the following limits:

(i) $\lim_{x\to 1} \dfrac{x^3-1}{x-1}$, $x \neq 1$

$$\lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} = \lim_{x\to 1} (x^2+x+1) = (1)^2+1+1 = 3$$

(ii) $\lim\limits_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right)$, $h \neq 0$

$$\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \lim_{h\to 0} \frac{1}{h}\,\frac{x-x-h}{(x+h)\,x} = \lim_{h\to 0} \frac{-h}{h\,(x+h)\,x}$$

$$= -\lim_{h\to 0} \frac{1}{x(x+h)} = -\frac{1}{x(x+0)} = -\frac{1}{x^2}.$$

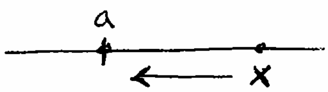
H.W

(iii) $\lim\limits_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$, $h \neq 0$.

One Sided and Two Sided limits (Right limits and Left limits)

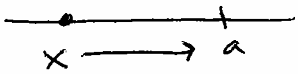
Some times the Values of a function $f(x)$ tend to different limits as x tends a from different sides. When this happens, we the limit of $f(x)$ as x approaches a from the right by the <u>Right-hand limit</u> and denoted by

$$\lim_{x\to a^+} f(x) = L$$



and the limit of $f(x)$ as x approaches a from the left by the <u>left-hand limit</u> and denoted by

$$\lim_{x\to a^-} f(x) = L$$

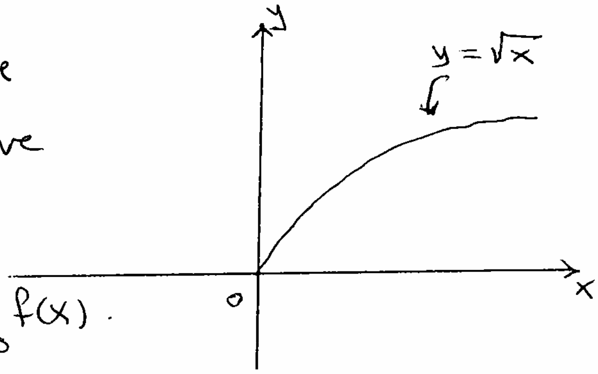


<u>Note</u> From uniqueness theorem of the limit, we know that if Limit exist then it is unique, so that

$\lim\limits_{x\to a} f(x) = L$ if and only if $\lim\limits_{x\to a^+} f(x) = L$ and $\lim\limits_{x\to a^-} f(x) = L$.

EX.(5) $f(x)=\sqrt{x}$, $D: x \geq 0$. Find $\lim\limits_{x\to 0} f(x) = ?$

since $\sqrt{x}$ is not define for -ive value of x, so we restrict to +ive value of x.

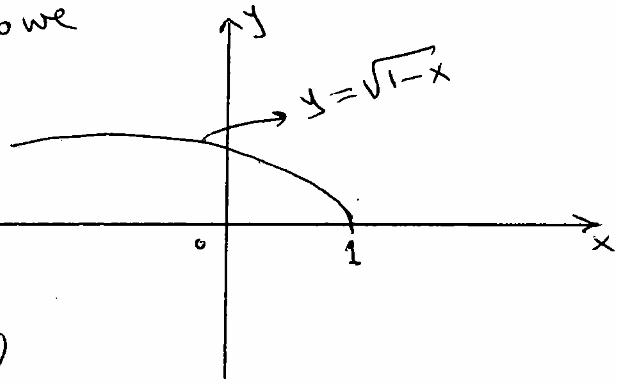


$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{x} = \sqrt{0} = 0 = \lim_{x\to 0} f(x).$$

(This example of one-sided limit).

EX.(6) $f(x)=\sqrt{1-x}$, $D: x \leq -1$. Find $\lim\limits_{x\to 1} f(x) = ?$

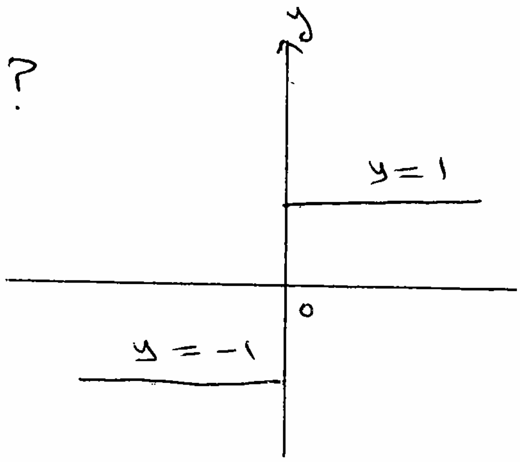
since $\sqrt{1-x}$ is not define for $x>1$, so we restrict to values of $x \leq 1$



$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{1-x} = \sqrt{1-1} = \sqrt{0} = 0 = \lim_{x\to 1} f(x)$$

(This example of one-sided limit)

EX.(7) $f(x) = \dfrac{x}{|x|}$. Find $\lim\limits_{x\to 0} f(x) = ?$



since $|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$

$\therefore \quad f(x) = \begin{cases} 1 & , x \geq 0 \\ -1 & , x < 0 \end{cases}$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} 1 = 1$ and $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (-1) = -1$

Since $\lim\limits_{x\to 0^+} f(x) \neq \lim\limits_{x\to 0^-} f(x)$, the $\lim\limits_{x\to 0} f(x)$ does not exist.

(This example of two-sided limit)

H.W

EX. (8) $f(x) = \dfrac{x\sqrt{x^2+1}}{|x|}$, $x \neq 0$. Find $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 0^-} f(x)$, and $\lim\limits_{x\to 0} f(x)$.

EX. (9) $f(x) = \dfrac{\sqrt{4-x^2}}{\sqrt{6-5x+x^2}}$. What is the domain. $\lim\limits_{x\to 2^-} f(x)$, $\lim\limits_{x\to 2^+} f(x)$, $\lim\limits_{x\to 2} f(x)$.

D: $-2 \le x < 2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{\sqrt{4-x^2}}{\sqrt{6-5x+x^2}} = \lim_{x\to 2^-} \frac{\cancel{\sqrt{2-x}}\,\sqrt{2+x}}{\cancel{\sqrt{2-x}}\,\sqrt{3-x}}$$

$$= \lim_{x\to 2^-} \frac{\sqrt{2+x}}{\sqrt{3-x}} = \frac{\sqrt{2+2}}{\sqrt{3-2}} = \sqrt{4} = 2.$$

$\lim\limits_{x\to 2^+} f(x)$ is not define, so $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2^-} f(x) = 2$.

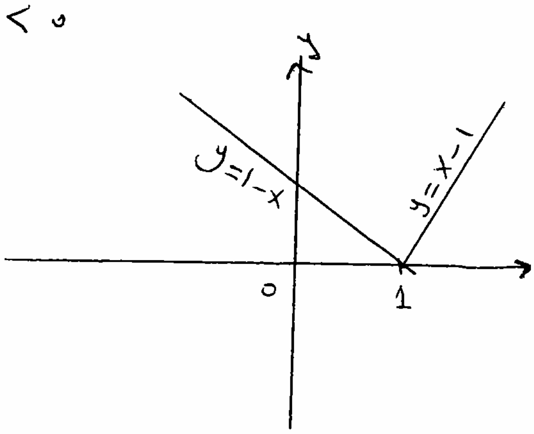
H.W

EX. (10) $f(x) = |x-1|$. Find $\lim\limits_{x\to 1^+} f(x)$, $\lim\limits_{x\to 1^-} f(x)$ and $\lim\limits_{x\to 1} f(x)$.

solu.

$$f(x) = |x-1| = \begin{cases} (x-1), & x-1 \ge 0 \\ -(x-1), & x-1 < 0 \end{cases}$$

$$= \begin{cases} x-1, & x-1 \ge 0 \\ & x \ge 1 \\ 1-x, & x-1 < 0 \\ & x < 1 \end{cases}$$



$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1) = 1-1 = 0$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1-x) = 1-1 = 0$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} |x-1| = 0$$

Limits At Infinity

We note that when the limit of a function $f(x)$ exist as x approachs infinity, we write $\lim_{x\to\infty} f(x) = L$.

So, we write

$\lim_{x\to\infty} f(x) = L$ for +ive values of x and $\lim_{x\to-\infty} f(x) = L$ for −ive values of x.

For one-sided and Two-sided limits, we have

$\lim_{x\to\infty} f(x) = L$ if and only if $\lim_{x\to+\infty} = L$ and $\lim_{x\to-\infty} f(x) = L$

Some Obvious limits

(1) If K is constant, then $\lim_{x\to+\infty} K = K$ and $\lim_{x\to-\infty} K = K$.

(2) $\lim_{x\to\infty} \frac{1}{x} = 0$, $\lim_{x\to+\infty} \frac{1}{x} = 0$, and $\lim_{x\to-\infty} \frac{1}{x} = 0$.

(3) $\lim_{x\to0} \frac{1}{x} = \infty$, $\lim_{x\to0^+} \frac{1}{x} = +\infty$, and $\lim_{x\to0^-} \frac{1}{x} = -\infty$.

EX.(11) Find the following limits:

(1) $$\lim_{x\to\infty} \frac{x}{2x+3} = \lim_{x\to\infty} \frac{1}{2+\frac{3}{x}} = \frac{1}{2+0} = \frac{1}{2}.$$

(2) $$\lim_{x\to\infty} \frac{2x^2+3x+5}{5x^2-4x+1} = \lim_{x\to\infty} \frac{2+\frac{3}{x}+\frac{5}{x^2}}{5-\frac{4}{x}+\frac{1}{x^2}} = \frac{2+0+0}{5-0+0} = \frac{2}{5}.$$

(3) $$\lim_{x\to\infty} \frac{2x^2+1}{3x^3-2x^2+5x-2} = \lim_{x\to\infty} \frac{\frac{2}{x}+\frac{1}{x^3}}{3-\frac{2}{x}+\frac{5}{x^2}-\frac{2}{x^3}} = \frac{0+0}{3-0+0-0} =$$

④ $$\lim_{x\to\infty} \frac{2x^3+2x-1}{x^2-5x+2} = \lim_{x\to\infty} \frac{2+\frac{2}{x^2}-\frac{1}{x^3}}{\frac{1}{x}-\frac{5}{x^2}+\frac{2}{x^3}} = \frac{2+0-0}{0-0+0} = \frac{2}{0} = \infty$$

That is the limit does not exist.

⑤ $$\lim_{x\to\infty} \sqrt{x} = \lim_{x\to+\infty} \sqrt{x} = +\infty \text{ or } \infty.$$

⑥ $$\lim_{x\to\infty} \left(2+\frac{\sin x}{x}\right) = \lim_{x\to\infty} 2 + \lim_{x\to\infty} \frac{\sin x}{x}$$, but $\lim_{x\to\infty} 2 = 2$

and $\lim_{x\to\infty} \frac{\sin x}{x} = 0$

because $-1 \leq \sin x \leq 1$ $\therefore \lim_{x\to\infty} \left(2+\frac{\sin x}{x}\right) = 2+0 = 2.$

⑦ $$\lim_{x\to-\infty} \left(2x+\frac{3}{x}\right) = -\infty + 0 = -\infty.$$

⑧ $$\lim_{x\to 2^-} \frac{1}{x^2-4} = \frac{1}{0} = -\infty$$ and $$\lim_{x\to 2^+} \frac{1}{x^2-4} = \frac{1}{0} = +\infty.$$

⑨ $\lim_{x\to\infty} (\sqrt{x^2+1} - x = \infty - \infty$ (meaning less).

$$\lim_{x\to\infty} (\sqrt{x^2+1}-x) \cdot \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} = \lim_{x\to\infty} \frac{x^2+1-x^2}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}+1} = \frac{0}{\sqrt{1+0}+1} = \frac{0}{2} = 0.$$

(10) $\lim_{x\to\infty}(\sqrt{x^2+2x}-x)=\lim_{x\to\infty}(\sqrt{x^2+2x}-x)\cdot\frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x}$

$=\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+2x}+x}=\lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{2}{x}}+1}=\lim_{x\to\infty}\frac{2}{\sqrt{1+0}+1}=\frac{2}{2}=1$

More About An Asymptotes

Given $y=f(x)$. A line $y=mx+b$ is an asymptote for $f(x)$

(1) $m=\lim_{x\to\infty}\frac{f(x)}{x}$ (2) $b=\lim_{x\to\infty}(f(x)-mx)$

EX.(12) Find the asymptotes of the following functions:

(1) $y=f(x)=x+\frac{1}{x}=\frac{x^2+1}{x}$

$x=0$ is V. Asy.

$x^2+1=yx \Rightarrow x^2-yx+1=0 \Rightarrow x=\frac{y\pm\sqrt{y^2-4}}{2}$

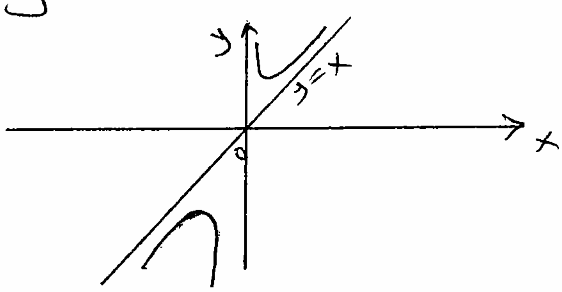
No H. Asy.

Let $y=mx+b$ be an asy.

$m=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{\frac{x^2+1}{x}}{x}=\lim_{x\to\infty}\frac{x^2+1}{x^2}=\lim_{x\to\infty}1+\frac{1}{x^2}=1+0=1$

$b=\lim_{x\to\infty}(f(x)-mx)=\lim_{x\to\infty}(\frac{x^2+1}{x}-x)=\lim_{x\to\infty}\frac{x^2+1-x^2}{x}$

$=\lim_{x\to\infty}\frac{1}{x}=0$ $\therefore$ $y=x$ is an asy.



(2) $y = f(x) = \dfrac{x^2 - 3}{2x-4}$, $x = 2$ is V. Asy.

$x^2 - 3 = 2yx - 4y \Rightarrow x^2 - 2yx + 4y - 3 = 0$

$$x = \frac{4y \pm \sqrt{4y^2 - 4(4y-3)}}{2} = \frac{4y \pm \sqrt{4y^2 - 16y + 12}}{2} = \frac{4y \pm 2\sqrt{y^2 - 4y + 3}}{2}$$

$x = 2y \pm \sqrt{y^2 - 4y + 3}$. No H. Asy.

Let $y = mx + b$ be an asymptote

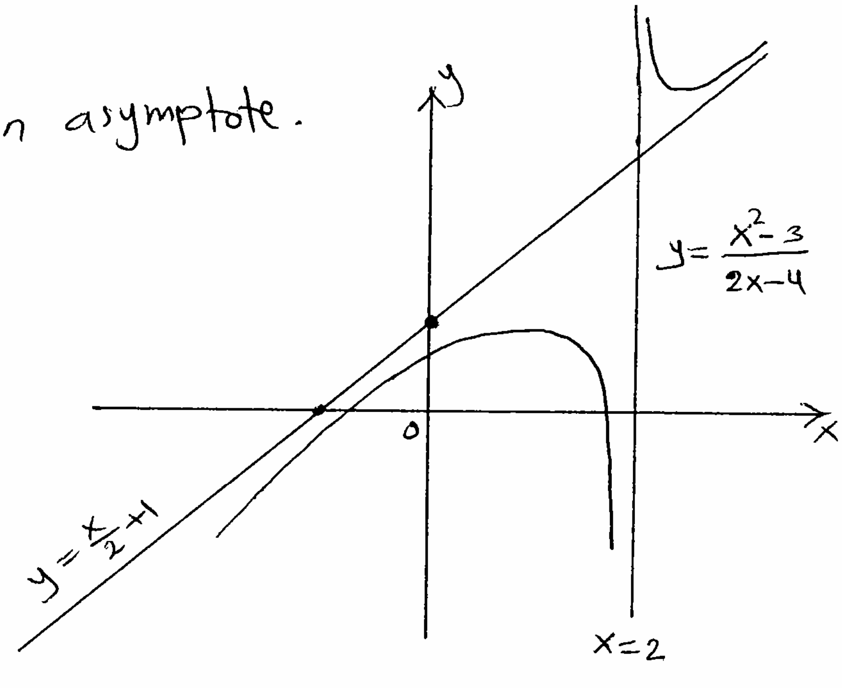
$$m = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{\frac{x^2-3}{2x-4}}{x} = \lim_{x \to \infty} \frac{x^2 - 3}{2x^2 - 3x}$$

$$= \lim_{x \to \infty} \frac{1 - \frac{3}{x^2}}{2 - \frac{3}{x}} = \frac{1-0}{2-0} = \frac{1}{2}$$

$$b = \lim_{x \to \infty} (f(x) - mx) = \lim_{x \to \infty} \left(\frac{x^2-3}{2x-4} - \frac{1}{2}x \right) = \lim_{x \to \infty} \frac{x^2 - 3 - x^2 + 2x}{2(x-2)}$$

$$= \lim_{x \to \infty} \frac{2x - 3}{2x - 4} = \lim_{x \to \infty} \frac{2 - \frac{3}{x}}{2 - \frac{4}{x}} = \frac{2-0}{2-0} = \frac{2}{2} = 1.$$

$\therefore$ $y = \frac{x}{2} + 1$ is an asymptote.



<u>Sandwich Theorem</u> If $g(x) \le f(x) \le h(x)$ and if $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$ then $\lim_{x \to a} f(x) = L$.

Ex.(13) Find $\lim\limits_{x\to\infty} f(x)$ if $\dfrac{2x+3}{x} \leqslant f(x) \leqslant \dfrac{2x^2+5x}{x^2}$

$$\lim_{x\to\infty} \frac{2x+3}{x} = \lim_{x\to\infty} \left(2+\frac{3}{x}\right) = 2+0 = 2.$$

$$\lim_{x\to\infty} \frac{2x^2+5x}{x^2} = \lim_{x\to\infty} \left(2+\frac{5}{x}\right) = 2+0 = 2.$$

$\therefore$ By Sandwich Theorem $\lim\limits_{x\to\infty} f(x) = 2.$

Theorem(1) If θ is measured in radian, then

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Theorem(2) $\lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{\theta} = 0.$

Proof: $\lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{\theta} \cdot \dfrac{1+\cos\theta}{1+\cos\theta} = \lim\limits_{\theta\to 0} \dfrac{1-\cos^2\theta}{\theta(1+\cos\theta)}$

$$= \lim_{\theta\to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta\to 0} \frac{\sin\theta}{1+\cos\theta}$$

$$= (1) \cdot \frac{\sin(0)}{1+\cos(0)} = (1) \cdot \frac{0}{1+1} = \frac{0}{2} = 0.$$

Ex.(14) Find the following Limits :

(a) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x} = 3 \lim\limits_{3x\to 0} \dfrac{\sin 3x}{3x} = 3\cdot(1) = 3$

as $x\to 0 \Rightarrow 3x \to 0.$

(b) $\lim\limits_{x\to 0} \dfrac{\sin 5x}{\sin 3x} = \lim\limits_{x\to 0} \dfrac{\frac{\sin 5x}{x}}{\frac{\sin 3x}{x}} = \dfrac{\lim\limits_{x\to 0} \frac{\sin 5x}{x}}{\lim\limits_{x\to 0} \frac{\sin 3x}{x}}$

as $x\to 0 \Rightarrow$ and $3x\to 0$, $5x\to 0$

$$\lim_{x\to 0}\frac{\sin 5x}{\sin 3x}=\frac{5\lim\limits_{5x\to 0}\frac{\sin 5x}{5x}}{3\lim\limits_{3x\to 0}\frac{\sin 3x}{3x}}=\frac{5(1)}{3(1)}=\frac{5}{3}.$$

(c) $$\lim_{x\to \pi/2}\frac{\cos x}{x-\frac{\pi}{2}}=\lim_{x\to\frac{\pi}{2}}\frac{\sin(\frac{\pi}{2}-x)}{x-\frac{\pi}{2}}=\lim_{x\to\frac{\pi}{2}}\frac{-\sin(x-\frac{\pi}{2})}{x-\frac{\pi}{2}}$$

as $x\to\frac{\pi}{2} \Rightarrow x-\frac{\pi}{2}\to 0$

$$\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{x-\frac{\pi}{2}}=-\lim_{x-\frac{\pi}{2}\to 0}\frac{\sin(x-\frac{\pi}{2})}{x-\frac{\pi}{2}}=-1.$$

(d) $$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}$$

$$=(1)\cdot\frac{1}{\cos(0)}=(1)\cdot\frac{1}{1}=1.$$

(e) $$\lim_{x\to 0}\frac{\sin 2x}{2x^2+x}=\lim_{x\to 0}\frac{\sin 2x}{2x(x+\frac{1}{2})}=\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\lim_{x\to 0}\frac{1}{x+\frac{1}{2}}$$

$$=\lim_{2x\to 0}\frac{\sin 2x}{2x}\cdot\lim_{x\to 0}\frac{1}{x+\frac{1}{2}}=(1)\cdot\frac{1}{0+\frac{1}{2}}=2.$$

(f) $$\lim_{x\to\infty}x\sin\frac{1}{x}$$

Let $y=\frac{1}{x}$

as $x\to\infty \Rightarrow y=\frac{1}{x}\to 0$

$$\lim_{x\to\infty}x\sin\frac{1}{x}=\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=\lim_{y\to 0}\frac{\sin y}{y}=1.$$

(g) $$\lim_{x\to 0}\frac{\sin x}{|x|}$$

$$\lim_{x\to 0^+}\frac{\sin x}{|x|}=\lim_{x\to 0^+}\frac{\sin x}{x}=1 \quad\text{and}\quad \lim_{x\to 0^-}\frac{\sin x}{|x|}=\lim_{x\to 0^-}\frac{\sin x}{-x}=-1$$

Since $\lim\limits_{x\to 0^+}\frac{\sin x}{|x|}\neq\lim\limits_{x\to 0^-}\frac{\sin x}{|x|}$, $\therefore \lim\limits_{x\to 0}\frac{\sin x}{|x|}$ does not exist

Defn. (Continuous function) الدالة المستمرة

A function $y=f(x)$ is said to be cont. at $x=a$ if

(1) $f(a)$ is define.

(2) $\lim\limits_{x \to a} f(x) = f(a)$.

Ex.(15)

(a) Every polynomial of the form

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is cont. ~~at x=a~~ for all x.

(b) $f(x) = \dfrac{1}{x}$

$f(x)$ is cont. for all x except at $x=0$, because $f(0)$ is not define.

(c) $f(x) = \dfrac{x+3}{(x-5)(x+2)}$, $f(x)$ is discont. at $x=5,-2$.

(d) $f(x) = \dfrac{\sin x}{x}$, $f(x)$ is discont. at $x=0$

(e) $$f(x) = \begin{cases} \dfrac{\sin x}{x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$$

$f(x)$ is cont. at $x=0$

(f) $$f(x) = \begin{cases} \dfrac{x^2+x-6}{x^2-4} & , x \neq 2 \\ \dfrac{5}{4} & , x = 2 \end{cases}$$

$f(x)$ is cont. at $x=2$.