Integration التكامل

Introduction

we shall discuss a problem of finding a function $y = F(x)$ a derivative is given by the equation:

$$\frac{dy}{dx} = f(x) \qquad \text{———} (1)$$

equation (1) usually called a differential equation.

we shall restrict our attention on differential equations that contain a single derivative.

A function $y = F(x)$ is called a solution of the differential equation (1) if $\frac{d}{dx} F(x) = f(x)$.

The function $F(x)$ is also called a solution of $f(x)$.

Solve equation means to find all the functions that are solutions $f(x)$.

Theorem (1) If $F(x)$ is ~~an~~ a solution of $f(x)$, then $F(x) + C$ is also a solution, where C is any constant.

Proof:

Since $F(x)$ is a solution of $f(x)$ then $\frac{d}{dx} F(x) = f(x)$

$$\frac{d}{dx}[F(x) + C] = \frac{d}{dx} F(x) + \frac{d}{dx} C = f(x) + 0 = f(x)$$

Hence $F(x) + C$ is a solution of $f(x)$.

From theorem (1), we may say that, if $F(x)$ is any solution of the equation $\frac{dy}{dx} = f(x)$ then all solutions are given by the ~~function~~ formula

$$y = F(x) + C \qquad (2)$$

Indefinite Integral:

The set of all solutions of $f(x)$ is called indefinite integral of f with respect to x and denoted by:

$$y = \int f(x)\,dx = F(x) + C \qquad (3)$$

where the symbol $\int$ is called an "integral sign", the function $f(x)$ is called the integrand of the integral, C is called the constant of integration, and dx tell us that the variable of integration is x.

Ex.(1) Solve the differential equation $\frac{dy}{dx} = 3x^2$

Since $\frac{d}{dx} x^3 = 3x^2$

Then $y = \int 3x^2\,dx = x^3 + C$.

EX.(2) Solve $\frac{dy}{dx} = \sin x \cos x$

Since $\frac{d}{dx}\left(\frac{1}{2}\sin^2 x\right) = \sin x \cos x$

Then $y = \int \sin x \cos x\,dx = \frac{1}{2}\sin^2 x + C$.

Some Integration Formulas

If $u = u(x)$

(1) $\int \frac{du}{dx}\, dx = u(x) + C$

(2) $\int a\, u(x)\, dx = a \int u(x)\, dx$, where a is constant.

(3) $\int \{u_1(x) + u_2(x) + \cdots + u_n(x)\}\, dx = \int u_1(x)\,dx + \int u_2(x)\,dx + \cdots + \int u_n(x)\,dx$

(4) $\int u^n \frac{du}{dx} \cdot dx = \frac{u^{n+1}}{n+1} + C$, $n \neq -1$

<u>EX.(3)</u> Evaluate $I = \int (3x^4 - 2x^3 + x^{\frac{1}{2}} + 2x^{-2} + 5x^{-\frac{1}{2}} - \sqrt{2})\, dx$

$$= 3\int x^4 dx - 2\int x^3 dx + \int x^{1/2} dx + 2\int x^{-2} dx + 5\int x^{-\frac{1}{2}} dx - \sqrt{2}\int dx.$$

$$= \frac{3}{5}x^5 - \frac{1}{2}x^4 + \frac{2}{3}x^{3/2} - 2x^{-1} + 10x^{+\frac{1}{2}} - \sqrt{2}x + C.$$

<u>EX.(4)</u> Evaluate $I = \int \sqrt{(3x-1)^3}$

$$I = \int (3x-1)^{3/2} dx = \frac{1}{3}\int (3x-1)^{3/2}\, 3\, dx = \frac{1}{3}\frac{2}{5}(3x-1)^{5/2} + C$$

$$= \frac{2}{15}(3x-1)^{5/2} + C.$$

H.W

<u>EX.(5)</u> $\int (x^2+1)^3\, 2x\, dx$.

Definite Integrals

The integral $\int_a^b f(x)\,dx$ is called the definite integral of $f(x)$ over the interval $[a,b]$.

Later, we shall show that this integral is a number ~~defined~~ defined in a certain way as a limit of approximating sums over the interval from a to b on the x-axis.

Properties of definite Integrals

If $f(x)$ is a continuous function on $[a,b]$, then

① $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

② $\int_a^a f(x)\,dx = 0$

③ $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$, k is constant.

④ $\int_a^b \{f_1(x) + f_2(x) + \cdots + f_n(x)\}\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx + \cdots + \int_a^b f_n(x)\,dx$

⑤ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any $c \in [a,b]$.

"The fundamental Theorem of Integral Calculus"

If $f(x)$ is continuous function on $[a,b]$ and $F(x)$ is any solution of $f(x)$ over $[a,b]$, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

Ex.(6) Evaluate $I = \int_{-3}^{2} (6 - x - x^2)\,dx$

$$I = \left(6x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_{-3}^{2} = \left(12 - 2 - \tfrac{8}{3}\right) - \left(-18 - \tfrac{9}{2} + 9\right)$$
$$= \frac{125}{6}.$$

Ex.(7) Find $I = \int_{0}^{\pi} \sin x\,dx = -\cos x \Big|_{0}^{\pi} = -(\cos\pi - \cos 0) = -(-1-1) = 2.$

H.w

Ex.(8) Find $I = \int_{0}^{\frac{\pi}{6}} \frac{\sin 2x}{\cos^2 2x}\,dx.$

Ex.(9) If $f(x)$ is a continuous, show that
$$\int_0^1 f(x)\,dx = \int_0^1 f(1-t)\,dt.$$

Proof

Let $x = 1 - t \Rightarrow dx = -dt$

at $x = 0 \Rightarrow t = 1$
at $x = 1 \Rightarrow t = 0$

then $\int_0^1 f(x)\,dx = \int_1^0 f(1-t)\,\bar{d}t = -\int_1^0 f(1-t)\,dt = \int_0^1 f(1-t)\,dt.$

Methods of Integration

Integral formulae (Standard Forms)

① $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1.$

② $\int \dfrac{du}{u} = \ln u + C.$

③ $\int e^u\,du = e^u + C, \quad e = 2.7183.$

④ $\int a^u\,du = \dfrac{a^u}{\ln a} + C.$ $a > 0$

5) $\int \sin u\,du = -\cos u + C.$

6) $\int \cos u\,du = \sin u + C.$

7) $\int \sec^2 u\,du = \tan u + C.$

8) $\int \csc^2 u\,du = -\cot u + C.$

9) $\int \sec u \tan u\,du = \sec u + C$

10) $\int \csc u \cot u\,du = -\csc u + C$

⑪ $\int \sinh u\,du = \cosh u + C.$

⑫ $\int \cosh u\,du = \sinh u + C.$

⑬ $\int \operatorname{sech}^2 u\,du = \tanh u + C.$

⑭ $\int \operatorname{csch}^2 u\,du = -\coth u + C.$

⑮ $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C.$

⑯ $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C.$

17) $\int \dfrac{du}{\sqrt{a^2+u^2}} = \sin^{-1}\dfrac{u}{a} + C$

18) $\int \dfrac{du}{a^2+u^2} = \dfrac{1}{a}\tan^{-1}\dfrac{u}{a} + C$

19) $\int \dfrac{du}{u\sqrt{u^2+a^2}} = \dfrac{1}{a}\sec^{-1}\dfrac{u}{a} + C$

20) $\int \dfrac{du}{\sqrt{u^2+a^2}} = \sinh^{-1}\dfrac{u}{a} + C$

21) $\int \dfrac{du}{a^2\,[\text{illegible}]\,u^2} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\dfrac{u}{a} + C, & |u| < a \\ \dfrac{1}{a}\coth^{-1}\dfrac{u}{a} + C, & |u| > a \end{cases}$

22) $\int \dfrac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\dfrac{u}{a} + C$

23) $\int \dfrac{du}{\;} = \dfrac{-1}{a}\operatorname{sech}^{-1}\dfrac{u}{a} + C$

24) $\int \dfrac{du}{u\sqrt{a^2+u^2}} = \dfrac{-1}{a}\operatorname{csch}^{-1}\dfrac{u}{a} + C.$

Method [1] Integration By Substitution

The goal of this method is to transform the integral into a standard ~~form~~ form. To evaluate the integral

$$I = \int f[g(x)]\, g'(x)\, dx$$

Carry out the following steps:

1. Substitute $u = g(x)$ then $du = g'(x)dx$ to obtain $I = \int f(u)du$
2. Evaluate $I = \int f(u)du$ by integrating with respect to u.
3. Replace u by $g(x)$ in the final result.

EX.(1) Evaluate $\displaystyle\int \frac{dx}{\sqrt[3]{1-2x}}$

Solu.

$I = \int (1-2x)^{-\frac{1}{3}} dx$. Let $u = 1-2x \Rightarrow du = -2\,dx$

$$I = -\frac{1}{2}\int (1-2x)^{-\frac{1}{3}} (-2)\,dx = -\frac{1}{2}\int u^{-\frac{1}{3}} du = \frac{-1}{2}\,\frac{u^{2/3}}{2/3} + C$$

$$= -\frac{3}{4}(1-2x)^{2/3} + C .$$

EX.(2) Find $I = \int \sin^2 5x \cos 5x\, dx$

Solu. Let $u = \sin 5x \Rightarrow du = 5\cos 5x\, dx$

$$I = \frac{1}{5}\int u^2 du = \frac{1}{5}\,\frac{u^3}{3} + C = \frac{1}{15}\sin^3 5x + C .$$

EX.(3) Find $I = \int x\, e^{x^2+1} dx = \frac{1}{2}\int e^{x^2+1}\, 2x\, dx$

$u = x^2+1 \Rightarrow du = 2x\,dx$

$$I = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2+1} + C .$$

EX.(4) Find $I=\frac{1}{3}\int \frac{3\cosh 3x}{4+\sinh 3x}\,dx$

solu.

$u=4+\sinh 3x \Rightarrow du = 3\cosh 3x\,dx$

$$I=\frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln u + C = \frac{1}{3}\ln(4+\sinh 3x)+C.$$

EX.(5) Evaluate $I=\frac{1}{3}\int \frac{3\cosh 3x}{4+\sinh^2 3x}\,dx$

solu.

$u=\sinh 3x \Rightarrow du = 3\cosh 3x\,dx$

$$I=\frac{1}{3}\int \frac{du}{4+u^2} = \frac{1}{3}\cdot\frac{1}{2}\tan^{-1}\frac{u}{2} + C = \frac{1}{6}\tan^{-1}\left[\frac{\sinh 3x}{2}\right]+C.$$

EX.(6) $I=\int \frac{dx}{1+e^x} = \int \frac{dx}{1+\frac{1}{e^{-x}}} = -\int \frac{-e^{-x}\,dx}{e^{-x}+1} = -\ln(e^{-x}+1)+C.$

EX.(7) Find $I=\int \sec x\,dx = \int \sec x \cdot \frac{(\sec x+\tan x)}{(\sec x+\tan x)}\,dx$

$u=\sec x+\tan x \Rightarrow du = (\sec x\tan x + \sec^2 x)\,dx$

$= \sec x\,(\tan x + \sec x)\,dx$

$$I=\int \frac{du}{u} = \ln u + C = \ln(\sec x+\tan x)+C.$$

EX.(8) Evaluate $I=\int_0^{1/\sqrt{2}} \frac{x\,dx}{\sqrt{1-x^4}} = \frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}} \frac{2x\,dx}{\sqrt{1-(x^2)^2}}$

$u=x^2 \Rightarrow du = 2x\,dx$. at $x=0 \Rightarrow u=0$

at $x=\frac{1}{\sqrt{2}} \Rightarrow u=\frac{1}{2}$

$$I=\frac{1}{2}\int_0^{1/2} \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u\Big|_0^{1/2} = \frac{1}{2}\left[\sin^{-1}\frac{1}{2} - \sin^{-1}0\right] = \frac{1}{2}\left(\frac{\pi}{6}-0\right) = \frac{\pi}{12}.$$

Exercises To Solve [No. 1]

1. $\int \left(x - \frac{1}{x}\right)^2 dx$

2. $\int \frac{1 + e^{2x}}{e^x} dx$

3. $\int \frac{\sec^2 x \, dx}{1 + \tan^2 x}$

4. $\int \frac{e^x}{1 + e^{2x}} dx$

5. $\int \frac{dx}{x[1 + (\ln x)^2]}$

6. $\int \tan^2 3x \, dx$

7. $\int \operatorname{sech} x \, dx$

8. $\int \frac{\sec^2(\ln x)}{x} dx$

9. $\int \tan x \ln(\cos x) \, dx$

10. $\int_0^{\frac{\pi}{2}} \sin^3 x \cos x \, dx$

11. $\int_0^1 \frac{\sqrt{1 + e^{-2x}}}{e^{-3x}} dx$

12. $\int_4^9 \frac{dx}{x - \sqrt{x}}$

13. $\int_0^1 \frac{(1 + e^{-2x})^{\frac{1}{2}}}{e^{-3x}} dx$

14. $\int_0^\infty e^{-x - e^{-x}} dx$

15. $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$

16. $\int \tan 4x \, dx$

17. $\int \frac{x - 2}{\sqrt{9 - x^2}} dx$

18. $\int \frac{x\sqrt{x}}{1 + x^5} dx$

19. $\int \operatorname{csch} x \, dx$

20. $\int \csc x \, dx$.

Method[2] Certain Powers of Trigonometric And Hyperbolic Integral

Consider the following integrals forms:

(A) $\int \sin^m u \cos^n u \, du$ or $\int \sinh^m u \cosh^n u \, du$

(B) $\int \tan^m u \sec^n u \, du$ or $\int \tanh^m u \operatorname{sech}^n u \, du$

(C) $\int \cot^m u \csc^n u \, du$ or $\int \coth^m u \operatorname{csch}^n u \, du$.

Under Type (A), there are three Cases:

Case I: If m is odd and +ive, we factor out $\sin u$ ($\sinh u$) and change the remaining even power of $\sin u$ ($\sinh u$) to $\cos u$ ($\cosh u$) using the identities:

$$\sin^2 u = 1 - \cos^2 u \quad , \quad \sinh^2 u = \cosh^2 u - 1 .$$

EX.(1) Find $I = \int \sin^5 2x \cos^{-\frac{3}{2}} 2x \, dx = \int \sin^4 2x \sin 2x \cos^{-3/2} 2x \, dx$

$= \int (1-\cos^2 2x)^2 \sin 2x \cos^{-\frac{3}{2}} 2x \, dx = \int (1-2\cos^2 2x + \cos^4 2x) \cdot \sin 2x \cdot \cos^{-3/2} 2x \, dx$.

$= \int \left(\cos^{-\frac{3}{2}} 2x - 2\cos^{\frac{1}{2}} 2x + \cos^{\frac{5}{2}} 2x\right) \sin 2x \, dx$

$= -\frac{1}{2}\left[\frac{\cos^{-1/2} 2x}{-\frac{1}{2}} - 2\,\frac{\cos^{3/2} 2x}{3/2} + \frac{\cos^{7/2} 2x}{7/2}\right] + C = \cos^{-\frac{1}{2}} 2x + \frac{2}{3}\cos^{3/2} 2x - \frac{2}{7}\cos^{7/2} 2x + C.$

Case II

If n is odd and +ive, we factor out $\cos u$ ($\cosh u$) and change the remaining even power of $\cos u$ ($\cosh u$) to $\sin u$ ($\sinh u$) using the identities:-

$$\cos^2 u = 1 - \sin^2 u \quad , \quad \cosh^2 u = 1 + \sinh^2 u .$$

EX.(2) Find $I = \int \sinh^4 3x \cosh^3 3x \, dx = \int \cosh^2 3x \sinh^4 3x \cosh 3x \, dx$

$$I = \int (1 + \sinh^2 3x) \sinh^4 3x \cosh 3x \, dx = \int (\sinh^6 3x + \sinh^4 3x) \cosh 3x \, dx$$

$$= \frac{1}{3}\left[\frac{\sinh^7 3x}{7} + \frac{\sinh^5 3x}{5}\right] + C = \frac{1}{21}\sinh^7 3x + \frac{1}{15}\sinh^5 3x + C.$$

Case III

If both m and n are even and +ive (or one of them Zero) we reduce the degree of the expression by using the identities:-

$$\sin^2 u = \frac{1 - \cos 2u}{2} \quad , \quad \sinh^2 u = \frac{\cosh 2u - 1}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2} \quad , \quad \cosh^2 u = \frac{\cosh 2u + 1}{2}$$

EX.(3) $I = \int \sin^2 2x \cos^2 2x \, dx = \frac{1}{4}\int (1 - \cos 4x)(1 + \cos 4x) \, dx$

$$= \frac{1}{4}\int (1 - \cos^2 4x) \, dx = \frac{1}{4}\int \left(1 - \frac{1 + \cos 8x}{2}\right) dx = \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos 8x\right) dx$$

$$= \frac{1}{4}\left[\frac{x}{2} - \frac{1}{16}\sin 8x\right] + C.$$

Under Type (B), there are <u>two cases</u>

Case I: If n is even and +ive, we factor out $\sec^2 u$ ($\operatorname{sech}^2 u$) and change the remaining even power of $\sec u$ ($\operatorname{sech} u$) to $\tan u$ ($\tanh u$) using the identities: $\sec^2 u = 1 + \tan^2 u$, $\operatorname{sech}^2 u = 1 - \tanh^2 u$.

EX.(4) Find $I=\int \operatorname{sech}^4\frac{x}{2}\ \tanh^{-\frac{1}{3}}\frac{x}{2}\,dx$

$$=\int \operatorname{sech}^2\frac{x}{2}\ \tanh^{-\frac{1}{3}}\frac{x}{2}\ \operatorname{sech}^2\frac{x}{2}\,dx$$

$$=\int\left(1-\tanh^2\frac{x}{2}\right)\tanh^{-\frac{1}{3}}\frac{x}{2}\ \operatorname{sech}^2\frac{x}{2}\,dx=\int\left(\tanh^{-\frac{1}{3}}\frac{x}{2}-\tanh^{\frac{5}{3}}\frac{x}{2}\right)\operatorname{sech}^2\frac{x}{2}\,dx$$

$$=2\left[\frac{\tanh^{2/3}2x}{2/3}-\frac{\tanh^{8/3}\frac{x}{2}}{8/3}\right]+C=\frac{1}{3}\tanh^{2/3}\frac{x}{2}-\frac{3}{4}\tanh^{8/3}2x+C$$

Case II If m is odd and +ive, we factor out $\sec u\tan u$ ($\operatorname{sech}u\tanh u$) and change remaining even power of $\tan u$ ($\tanh u$) to $\sec u$ ($\operatorname{sech}u$) using the identities:-

$$\tan^2u=\sec^2u-1\quad,\quad \tanh^2u=1-\operatorname{sech}^2u.$$

EX.(5) $I=\int \tan^3 2x\ \sec^{-\frac{1}{4}}2x\,dx=\int\left(\tan^2 2x\ \sec^{-5/4}2x\right)\sec 2x\tan 2x\,dx$

$$=\int\left(\sec^2 2x-1\right)\sec^{-\frac{5}{4}}2x\ \sec 2x\tan 2x\,dx=\int\left(\sec^{3/4}2x-\sec^{-\frac{5}{4}}2x\right)\sec 2x\tan 2x\,dx$$

$$=\frac{1}{2}\left[\frac{\sec^{7/4}2x}{7/4}-\frac{\sec^{-\frac{1}{4}}2x}{-1/4}\right]+C=\frac{2}{7}\sec^{7/4}2x+2\sec^{-\frac{1}{4}}2x+C.$$

Under Type (C), there are two cases similar to these of type (B) where the identities :-

$$\csc^2u=\cot^2u+1\quad,\quad \operatorname{csch}^2u=\coth^2u-1.$$

Exercises To Solve [NO. 2]

(1) $\int \sin^5 2x \, dx$

(2) $\int \cot^4 3x \, dx$

(3) $\int \cot^3 2x \, \csc^4 2x \, dx$

(4) $\int_0^{\pi} \cos^2 x \, dx$

(5) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^3 2x \, dx$

(6) $\int_0^1 \sinh^4 x \, dx$

(7) $\int_0^{\frac{\pi}{2}} \sin^2 x \, \cos^2 x \, dx$

(8) $\int \sin^4 x \, \cos^{-2} x \, dx$

(9) $\int_0^{\pi/2} \cos^4 x \, dx$

(10) $\int x \, \sin^3 x^2 \, dx$.

(11) $\int \cos^3 x \, \sin^{-\frac{1}{2}} x \, dx$

(12) $\int \sin^6 x \, dx$

(13) $\int \csc^6 x \, dx$

(14) $\int_0^{\frac{\pi}{3}} \tan^3 x \, \sec x \, dx$

(15) $\int \tan \frac{x}{3} \, \sec^3 \frac{x}{3} \, dx$.

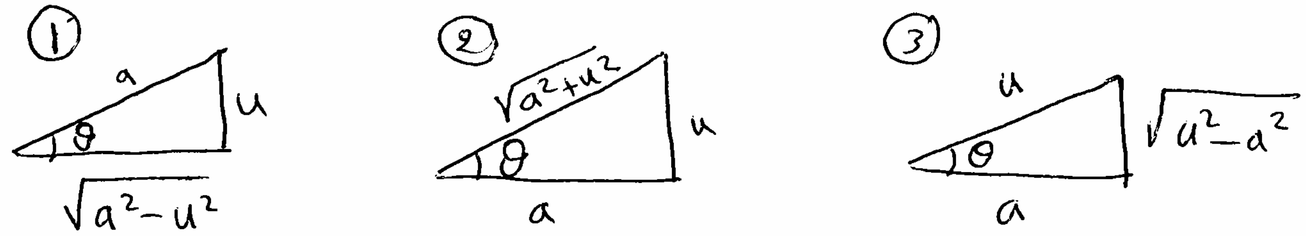
Method[3] Trigonometric Substitutions

If the integral involve one of the forms a^2+u^2, $\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$, or $\sqrt{u^2-a^2}$. Then the substitutions as follows:

① If $\sqrt{a^2-u^2}$, let $u=a\sin\theta \Rightarrow a^2-u^2=a^2\cos^2\theta$.

② If $\sqrt{a^2+u^2}$, let $u=a\tan\theta \Rightarrow a^2+u^2=a^2\sec^2\theta$.

③ If $\sqrt{u^2-a^2}$, let $u=a\sec\theta \Rightarrow u^2-a^2=a^2\tan^2\theta$.



EX.(1) Find $I=\int \frac{dx}{4+x^2} = \frac{1}{2}\tan^{-1}\frac{x}{2}+C$.

or

$$x=2\tan\theta \Rightarrow \tan\theta=\frac{x}{2} \Rightarrow \theta=\tan^{-1}\frac{x}{2}$$
$$dx=2\sec^2\theta\, d\theta$$

$$I=\int\frac{2\sec^2\theta\, d\theta}{4+4\tan^2\theta}=\int\frac{2\sec^2\theta\, d\theta}{4\sec^2\theta}=\frac{1}{2}\int d\theta=\frac{1}{2}\theta+C$$

$$=\frac{1}{2}\tan^{-1}\frac{x}{2}+C.$$

EX.(2) Find $I=\int_{-\frac{1}{2}}^{\sqrt{3}/2}\sqrt{1-x^2}\,dx$

$x=\sin\theta$ at $x=-\frac{1}{2} \Rightarrow -\frac{1}{2}=\sin\theta \Rightarrow \theta=-\frac{\pi}{6}$

$dx=\cos\theta\, d\theta$ at $x=\frac{\sqrt{3}}{2} \Rightarrow \frac{\sqrt{3}}{2}=\sin\theta \Rightarrow \theta=\frac{\pi}{3}$

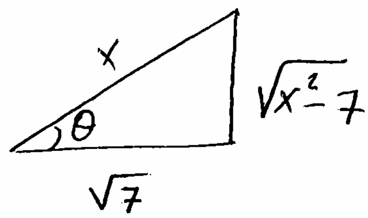
$$I=\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta=\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}}\cos^2\theta\,d\theta=\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1+\cos 2\theta}{2}\,d\theta$$

$$=\frac{1}{2}\left[\left(\theta+\frac{1}{2}\sin 2\theta\right)\right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}}=\frac{1}{2}\left[\left(\frac{\pi}{3}+\frac{1}{2}\sin\frac{2\pi}{3}\right)-\left(-\frac{\pi}{6}+\frac{1}{2}\sin\left(\frac{-\pi}{3}\right)\right)\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{3}+\frac{1}{2}\frac{\sqrt{3}}{2}+\frac{\pi}{6}+\frac{1}{2}\frac{\sqrt{3}}{2}\right]=\frac{1}{2}\left[\frac{\pi}{2}+\frac{\sqrt{3}}{2}\right]=\frac{\pi+\sqrt{3}}{4}.$$

EX.(3) Find

$$I=\int\frac{\sqrt{x^2-7}}{x}\,dx$$



$$x=\sqrt{7}\sec\theta \Rightarrow \sec\theta=\frac{x}{\sqrt{7}}$$

$$\Rightarrow \theta=\sec^{-1}\frac{x}{\sqrt{7}}$$

$$dx=\sqrt{7}\sec\theta\tan\theta\,d\theta.$$

$$I=\int\frac{\sqrt{7\sec^2\theta-7}}{\sqrt{7}\,\sec\theta}\cdot\sqrt{7}\,\sec\theta\tan\theta\,d\theta=\int\sqrt{7}\tan^2\theta\,d\theta$$

$$=\sqrt{7}\int(\sec^2\theta-1)\,d\theta=\sqrt{7}\,[\tan\theta-\theta]+C=\sqrt{7}\left[\tan\left[\sec^{-1}\frac{x}{\sqrt{7}}\right]-\sec^{-1}\frac{x}{\sqrt{7}}\right]+C.$$

$$=\sqrt{7}\left[\frac{1}{\sqrt{7}}\sqrt{x^2-7}-\sec^{-1}\frac{x}{\sqrt{7}}\right]+C.$$

H.W
EX.(4) Evaluate $I=\int x^3\sqrt{9+x^2}\,dx.$

Exercises To Solve [No. 3]

① $\displaystyle\int_{-6}^{-2\sqrt{3}} \frac{dx}{x\sqrt{x^2-9}}$ ② $\displaystyle\int_0^2 \frac{x^2\,dx}{x^2+4}$

③ $\displaystyle\int_0^{2\sqrt{3}} \frac{x^3\,dx}{\sqrt{x^2+4}}$ ④ $\displaystyle\int_0^1 \frac{x^2\,dx}{(4-x^2)^{3/2}}$

⑤ $\displaystyle\int \frac{\sin\theta}{\sqrt{2-\cos^2\theta}}\,d\theta$ ⑥ $\displaystyle\int \frac{dx}{(9-x^2)^{3/2}}$

⑦ $\displaystyle\int_0^{\sqrt{5}} x^2\sqrt{5-x^2}\,dx$ ⑧ $\displaystyle\int_1^3 \frac{dx}{x^4\sqrt{x^2+3}}$

⑨ $\displaystyle\int \frac{dx}{x^4\sqrt{4-x^2}}$ ⑩ $\displaystyle\int \frac{\sqrt{4x^2-9}}{x}\,dx$.

Method [4] Hyperbolic Substitutions

As in method [3], the hyperbolic substitutions can be used to the forms a^2+u^2, $\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$, or $\sqrt{u^2-a^2}$.

(1) If $\sqrt{a^2-u^2}$, let $u=a\tanh v$ or $u=a\,\text{sech}\,v$.
(2) If $\sqrt{a^2+u^2}$, let $u=a\sinh v$ or $u=a\,\text{csch}\,v$.
(3) If $\sqrt{u^2-a^2}$, let $u=a\cosh v$ or $u=a\coth v$.

EX.(1) Find $I=\int \sec x\,dx$ [See method [1], Ex.(7)]

$u=\sec x \Rightarrow du=\sec x\tan x\,dx \Rightarrow du=\sec x\sqrt{\sec^2 x-1}\,dx$

$$I=\int u\cdot\frac{du}{u\cdot\sqrt{u^2-1}}=\int\frac{du}{\sqrt{u^2-1}}=\cosh^{-1}u+C=\ln\left(u+\sqrt{u^2-1}\right)+C$$
$$=\ln\left(\sec x+\sqrt{\sec^2 x-1}\right)+C$$
$$=\ln(\sec x+\tan x)+C.$$

EX.(2) Evaluate $I = \int \frac{dx}{x^3 \sqrt{x^2+4}}$

Solu.

$$x = 2\sinh v \Rightarrow dx = 2\cosh v \, dv$$

$$I = \int \frac{2\cosh v \, dv}{8\sinh^3 v \cdot 2\cosh v} = \int \frac{dv}{8\sinh^3 v} = \frac{1}{8}\int \operatorname{csch}^3 v \, dv$$ is not easy to find.

we try $x = 2\operatorname{csch} v \Rightarrow v = \operatorname{csch}^{-1}\frac{x}{2}$

$$dx = -2 \operatorname{csch} v \coth v \, dv$$

$$I = \int \frac{-2\operatorname{csch} v \coth v \, dv}{8\operatorname{csch}^3 v \cdot 2\coth v} = -\frac{1}{8}\int \sinh^2 v \, dv = -\frac{1}{8}\int \left(\frac{\cosh 2v - 1}{2}\right) dv$$

$$= -\frac{1}{16}\left[\frac{1}{2}\sinh 2v - v\right] + C = -\frac{1}{16}\left[\sinh v \cosh v - v\right] + C$$

$$= -\frac{1}{16}\left[\frac{2}{x} \cdot \frac{\sqrt{x^2+4}}{x} - \operatorname{csch}^{-1}\frac{x}{2}\right] + C.$$

EX.(3) Find $I = \int \frac{x^3 \, dx}{(4-x^2)^{3/2}}$

Solu.

$$x = 2\tanh v \Rightarrow dx = 2\operatorname{sech}^2 v \, dv$$

$$I = \int \frac{8\tanh^3 v \cdot 2\operatorname{sech}^2 v \, dv}{(4 - 4\tanh^2 v)^{3/2}} = \int \frac{16\tanh^3 v \operatorname{sech}^2 v \, dv}{(4\operatorname{sech}^2 v)^{3/2}}$$

$$= \int \frac{16\tanh^3 v \operatorname{sech}^2 v \, dv}{8\operatorname{sech}^3 v} = 2\int \tanh^3 v (\operatorname{sech} v)^{-1} dv$$

$$= +2\int (-\operatorname{sech}^2 v + 1)\operatorname{sech}^{-2} v \cdot \operatorname{sech} v \tanh v \, dv$$

$$= 2\int (\operatorname{sech}^{-2} v - 1)\operatorname{sech} v \tanh v \, dv = 2\left[-\frac{(\operatorname{sech} v)^{-1}}{-1} + \operatorname{sech} v\right] + C$$

$$= 2\left[\cosh v + \operatorname{sech} v\right] + C = 2\left[\frac{2}{\sqrt{4-x^2}} + \frac{\sqrt{4-x^2}}{2}\right] + C.$$

Exercises To Solve [NO. 4]

(1) $\int_0^3 \frac{dx}{\sqrt{x^2+9}}$ (2) $\int_2^3 \sqrt{x^2-4}\, dx$ (3) $\int \frac{x^2 dx}{(9+x^2)^{1/2}}$

(4) $\int_2^4 \frac{\sqrt{x^2-4}}{x^2}\, dx$ (5) $\int (3+x^2)^{3/2} dx$ (6) $\int x^2 (5+x^2)^{1/2} dx$

(7) $\int \frac{x^3\, dx}{(5-x^2)^{3/2}}$ (8) $\int x^2\sqrt{5-x^2}\, dx$ (9) $\int \csc x\, dx$

(10) $\int \operatorname{sech} x\, dx$ (11) $\int \sec^3 x\, dx$ (12) $\int \csc^3 x\, dx$.

Method [5] Integrals Involving Quadratic Functions

If the integral involve a quadratic function x^2+ax+b, we reduced it to the form u^2+B by completing the square as follows :-

$$x^2+ax+b = x^2+ax+\frac{a^2}{4}+b-\frac{a^2}{4} = \left(x+\frac{a}{2}\right)^2+\left(b-\frac{a^2}{4}\right)$$
$$= u^2+B$$

where $u = x+\frac{a}{2}$ and $B = b-\frac{a^2}{4}$. Then the solution can be found by method [3] or [4].

EX.(1) Evaluate $I = \int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{-(x^2-2x+1-1)}} = \int \frac{dx}{\sqrt{-[(x-1)^2-1]}}$

$$= \int \frac{dx}{\sqrt{1-(x-1)^2}}$$

Let $u = x-1 \Rightarrow du = dx \Rightarrow I = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$

$$= \sin^{-1}(x-1) + C.$$

EX.(2) $I=\int \frac{(4x+5)\,dx}{(x^2-2x+2)} = \int \frac{(4x+5)\,dx}{(x^2-2x+1+1)^{3/2}}$

$$=\int \frac{(4x+5)\,dx}{[(x-1)^2+1]^{3/2}}$$

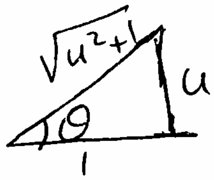
$u = x-1 \Rightarrow du = dx$

$x = u+1$

$$I=\int \frac{[4(u+1)+5]\,du}{(u^2+1)^{3/2}} = \int \frac{4u+9}{(u^2+1)^{3/2}}\,du = \int \frac{4u\,du}{(u^2+1)^{3/2}} + \int \frac{9\,du}{(u^2+1)^{3/2}}$$

$$=2\int 2u(u^2+1)^{-3/2}du + 9\int \frac{du}{(u^2+1)^{3/2}} = 2\,\frac{(u^2+1)^{-1/2}}{-1/2} + 9\int \frac{du}{(u^2+1)^{3/2}}$$

Consider $I_1 = \int \frac{du}{(u^2+1)^{3/2}}$



$u=\tan\theta \Rightarrow du = \sec^2\theta\,d\theta$

$$I_1 = \int \frac{\sec^2\theta\,d\theta}{[\sec^2\theta]^{3/2}} = \int \cos\theta\,d\theta = \sin\theta + C = \frac{u}{\sqrt{u^2+1}} + C$$

$$\therefore\ I = -\frac{4}{\sqrt{u^2+1}} + \frac{9u}{\sqrt{u^2+1}} + C = \frac{9u-4}{\sqrt{u^2+1}} + C = \frac{9(x-1)-4}{\sqrt{(x-1)^2+1}} + C$$

$$= \frac{9x-13}{(x^2-2x+2)^{1/2}} + C.$$

Exercises To Solve [No.5]

① $\int_1^2 \frac{dx}{x^2+2x+5}$ ② $\int_1^2 \frac{3\,dx}{9x^2-6x+5}$ ③ $\int_{-1}^0 \frac{dx}{\sqrt{3-2x-x^2}}$

④ $\int \frac{dx}{(x-1)\sqrt{x^2-2x-3}}$ ⑤ $\lim_{a\to 5} \int_a^{-4} \frac{dx}{\sqrt{-x^2-8x-15}}$ ⑥ $\int \frac{\cos x\ dx}{\sin^2 x+2\sin x+5}$.