Last lecture: Examples and the column space of a matrix
Suppose that A is an $n \times m$ matrix.

Definition The column space of A is the vector subspace $\mathrm{Col}(A)$ of $\mathbb{R}^n$ which is spanned by the columns of A.

That is, if $A = [a_1, a_2, \ldots, a_m]$ then $\mathrm{Col}(A) = \mathrm{Span}\,(a_1, a_2, \ldots, a_m)$.

Linear dependence and independence (chapter. 4)

- If V is *any* vector space then $V = \mathrm{Span}(V)$.
- Clearly, we can find smaller sets of vectors which span V.
- This lecture we will use the notions of linear independence and linear dependence to find the smallest sets of vectors which span V.
- It turns out that there are many "smallest sets" of vectors which span V, and that the number of vectors in these sets is always the same.

  This number is the dimension of V.

Linear dependence—motivation Let lecture we saw that the two sets of vectors $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \\ 13 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ do not span $\mathbb{R}^3$.

- The problem is that
  $\begin{bmatrix} 5 \\ 9 \\ 13 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$ and
  $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$.

- Therefore,
  $$\text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \\ 13 \end{bmatrix} \right) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \right)$$
  and
  $$\text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \right).$$

- Notice that we can rewrite the two equations above in the following form:
  $2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} - \begin{bmatrix} 5 \\ 9 \\ 13 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and
  $3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

This is the key observation about spanning sets.

Definition

Suppose that V is a vector space and that $x_1, x_2, \ldots, x_k$ are vectors in V.

The set of vectors $\{x_1, x_2, \ldots, x_k\}$ is linearly dependent

if

$$r_1x_1 + r_2x_2 + \cdots + r_kx_k = 0$$

for some $r_1, r_2, \ldots, r_k \in \mathbb{R}$ where at least one of $r_1, r_2, \ldots, r_k$ is non–zero.

Example

$$2\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}3\\5\\7\end{bmatrix} - \begin{bmatrix}5\\9\\13\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$ and

$$3\begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}3\\5\\7\end{bmatrix} - \begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

So the two sets of vectors $\left\{\begin{bmatrix}5\\9\\13\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\5\\7\end{bmatrix}\right\}$ and $\left\{\begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\5\\7\end{bmatrix}\right\}$ are linearly dependent.

Question Suppose that $x, y \in V$. When are x and y linearly dependent?

Question What do linearly dependent vectors look like in $\mathbb{R}^2$ and $\mathbb{R}^3$?

Example

Let $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $y = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and $z = \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix}$. Is $\{x_1, x_2, x_3\}$ linearly dependent?

We have to determine whether or not we can find real numbers r, s, t, which are not all zero, such that $rx + sy + tz = 0$.

To find all possible r, s, t we have to solve the augmented matrix equation:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 2 & 2 & 4 & 0 \\ 3 & 1 & 8 & 0 \end{array}\right] \xrightarrow[R_3:=R_3-3R_1]{R_2:=R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & -8 & 8 & 0 \end{array}\right]$$

$$\xrightarrow{R_3:=R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So this set of equations has a non–zero solution.
Therefore, $\{x, y, z\}$ is a linearly dependent set of vectors.

To be explicit, $3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Linear dependence–Example II

Example Consider the polynomials $p(x) = 1+3x+2x^2$, $q(x) = 3 + x + 2x^2$ and $r(x) = 2x + x^2$ in $\mathbb{P}_2$.
Is $\{p(x), q(x), r(x)\}$ linearly dependent?

We have to decide whether we can find real numbers r, s, t, which are not all zero, such that
$rp(x) + sq(x) + tr(x) = 0$.
That is:

$$0 = r(1 + 3x + 2x^2) + s(3 + x + 2x^2) + t(2x + x^2)$$
$$= (r+3s)+(3r+s+2t)x+(2r+2s+t)x^2.$$

This corresponds to solving the following system of linear equations

$$\begin{array}{cccc} r & +3s & & = 0 \\ 3r & +s & +2t & = 0 \\ 2r & +2s & +t & = 0 \end{array}$$

We compute:

$$\begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \xrightarrow[R_3:=R_3-2R_1]{R_2:=R_2-3R_1} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -8 & 2 \\ 0 & -4 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2:=R_2-R_3} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

Hence, $\{p(x), q(x), r(x)\}$ is linearly dependent.

Linear independence

In fact, we do not care so much about linear dependence as about its *opposite* linear independence:

Definition

Suppose that V is a vector space.

The set of vectors $\{x_1, x_2, \ldots, x_k\}$ in V is linearly independent if the only scalars $r_1, r_2, \ldots, r_k \in \mathbb{R}$ such that

$$r_1 x_1 + r_2 x_2 + \cdots + r_k x_k = 0$$

are $r_1 = r_2 = \cdots = r_k = 0$.

(That is, $\{x_1, \ldots, x_k\}$ is not linearly dependent!)

- If $\{x_1, x_2, \ldots, x_k\}$ are linearly independent then it is not possible to write any of these vectors as a linear combination of the remaining vectors.

  For example, if $x_1 = r_2 x_2 + r_3 x_3 + \cdots + r_k x_k$ then

  $$-x_1 + r_2 x_2 + r_3 x_3 + \cdots + r_k x_k = 0$$

  $\Longrightarrow$ all of these coefficients must be zero!!??!!

Linear independence—examples

The following sets of vectors are all linearly independent:

- $\left\{ [1] \right\}$ is a linearly independent subset of $\mathbb{R}$.
- $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a linearly independent subset of $\mathbb{R}^2$.
- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a linearly independent subset of $\mathbb{R}^3$.
- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a linearly independent subset of $\mathbb{R}^4$.
- $\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$ is a linearly independent subset of $\mathbb{R}^m$.
- $\{1\}$ is a linearly independent subset of $\mathbb{P}_0$.
- $\{1, x\}$ is a linearly independent subset of $\mathbb{P}_1$.
- $\{1, x, x^2\}$ is a linearly independent subset of $\mathbb{P}_2$.

- $\{1, x, x^2, \ldots, x^n\}$ is a linearly independent subset of $\mathbb{P}_n$.

Linear independence—example 2

Example

Let $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $y = \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix}$ and $z = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$.

Is the set $\{x_1, x_2, x_3\}$ linearly independent?

We have to determine whether or not we can find real numbers r, s, t, which are not all zero, such that $rx + sy + tz = 0$.

Once again, to find all possible r, s, t we have to solve the augmented matrix equation:

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 9 & -1 & 0 \end{array}\right] \xrightarrow[R_3 := R_3 - 3R_1]{R_2 := R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 0 & -16 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 := -\frac{1}{16} R_3]{R_2 := -\frac{1}{4} R_2} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Hence, $rx + sy + tz = 0$ only if $r = s = t = 0$.

Therefore, $\{x_1, x_2, x_3\}$ is a linearly independent subset of $\mathbb{R}^3$.

Linear independence—example 3

Example

Let $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $x_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$ and $x_4 = \begin{bmatrix} 3 \\ 5 \\ 5 \\ 7 \end{bmatrix}$.

Is $\{x_1, x_2, x_3, x_4\}$ linear dependent or linearly independent?

Again, we have to solve the corresponding system of linear equations:

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 1 & 5 \\ 1 & 4 & 2 & 7 \end{bmatrix} \xrightarrow[R_4 = R_4 - R_1]{\substack{R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1}} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow[R_4 = R_4 - 3R_2]{R_3 = R_3 - 2R_2} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$$\xrightarrow{R_4 = R_4 - R_3} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, after much work, we see that $\{x_1, x_2, x_3, x_4\}$ is linearly dependent.

Linear independence—example 4

Example

Let $X = \{\sin x, \cos x\} \subset \mathbb{F}$.

Is X linearly dependent or linearly independent?

Suppose that $s \sin x + t \cos x = 0$.

Notice that this equation holds for all $x \in \mathbb{R}$, so

$$\begin{aligned} x = 0: &\qquad s \cdot 0 + t \cdot 1 &= 0 \\ x = \tfrac{\pi}{2}: &\qquad s \cdot 1 + t \cdot 0 &= 0 \end{aligned}$$

Therefore, we must have $s = 0 = t$.

Hence, $\{\sin x, \cos x\}$ is linearly independent.

What happens if we tweak this example by a little bit?

Example Is $\{\cos x, \sin x, x\}$ is linearly independent?

If $s \cos x + t \sin x + r = 0$ then

$$\begin{aligned} x = 0: &\qquad s \cdot 0 + t \cdot 1 + r \cdot 0 &= 0 \\ x = \tfrac{\pi}{2}: &\qquad s \cdot 1 + t \cdot 0 + r \cdot \tfrac{\pi}{2} &= 0 \\ x = \tfrac{\pi}{4}: &\qquad s \cdot \tfrac{1}{\sqrt{2}} + t \cdot \tfrac{1}{\sqrt{2}} + r \cdot \tfrac{\pi}{4} &= 0 \end{aligned}$$

Therefore, $\{\cos x, \sin x, x\}$ is linearly independent.

Linear independence—last example

Example

Show that $X = \{e^x, e^{2x}, e^{3x}\}$ is a linearly independent subset of $\mathbb{F}$.

Suppose that $re^x + se^{2x} + te^{3x} = 0$.

Then:

$$\begin{array}{lll} x = 0 & r + s + t & = 0, \\ x = 1 & re + se^2 + te^3 & = 0, \\ x = 2 & re^2 + se^4 + te^6 & = 0, \end{array}$$

So we have to solve the matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 \\ e & e^2 & e^3 \\ e^2 & e^4 & e^6 \end{bmatrix} \xrightarrow[R_3:=\frac{1}{e^2}R_3]{R_2:=\frac{1}{e}R_2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e & e^2 \\ 1 & e^2 & e^4 \end{bmatrix}$$

$$\xrightarrow[R_3:=R_3-R_1]{R_2:=R_2-R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & e-1 & e^2-1 \\ 0 & e^2-1 & e^4-1 \end{bmatrix}$$

$$\xrightarrow[R_3:=\frac{1}{e^2-1}R_3]{R_2:=\frac{1}{e-1}R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & e+1 \\ 0 & 1 & e^2+1 \end{bmatrix}$$

$$\xrightarrow{R_3:=R_3-R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & e+1 \\ 0 & 0 & e^2-e \end{bmatrix}$$

Therefore, $\{e^x, e^{2x}, e^{3x}\}$ is a set of linearly independent functions in the vector space $\mathbb{F}$.

The Basis of a Vector Space:
We now combine the ideas of spanning sets and linear independence.

Definition Suppose that V is a vector space.
A basis of V is a set of vectors $\{x_1, x_2, \ldots, x_k\}$ in V such that

- $V = \text{Span}(x_1, x_2, \ldots, x_k)$ and
- $\{x_1, x_2, \ldots, x_k\}$ is linearly independent.

Examples

- $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right.$ is a basis of $\mathbb{R}^2$.
- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.
- $\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^m$.
- $\{1, x, x^2\}$ is a basis of $\mathbb{P}_2$.
- $\{1, x, x^2, \ldots, x^n\}$ is a basis of $\mathbb{P}_n$.
- In general, if W is a vector subspace of V then the challenge is to find a basis for W.