Dot and Cross Product

October 31, 2007 - Happy Halloween!

Dot Product (Inner product)

Definition: Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in $\mathbb{R}^n$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the scalar $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n$$

Properties of the Dot Product

If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors in $\mathbb{R}^n$ and c is a scalar then

1) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4) $(c\ \mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\ \mathbf{b})$
5) $\mathbf{0} \cdot \mathbf{a} = 0$.

The angle between two vectors

Theorem: If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

NOTE: Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Projections

Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$:

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$

Vector projection of $\mathbf{b}$ onto $\mathbf{a}$:

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$$

Cross Product

Definition: If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

NOTE: The cross product is only defined for vectors in $\mathbb{R}^3$.

Angles and the Cross Product

Theorem: The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Theorem: If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

Corollary: Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

Properties of the Cross Product

If $\mathbf{a}$ and $\mathbf{b}$ and $\mathbf{c}$ are vectors and c is a scalar, then

1) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

2) $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$

3) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

4) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

5) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

6) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Area and volume the cross product

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

The volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is the absolute value of the scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$