The Dot and Cross Products

Two common operations involving vectors are the **dot product** and the **cross product**. Let two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be given.

- **The Dot Product**

The dot product of **u** and **v** is written $\mathbf{u} \cdot \mathbf{v}$ and is defined two ways:

1. $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$.
2. $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$, where θ is the angle formed by **u** and **v.**

The two definitions are the same. They are related to one another by the Law of Cosines. The first method of calculation is easier because it is the sum of the products of corresponding components. The second method of calculation can be used if we know the angle θ formed by **u** and **v**.

It is important to note that the dot product always results in a scalar value. Furthermore, the dot symbol "·" always refers to a dot product of two vectors, not traditional multiplication of two scalars as we have previously known. To avoid confusion, pay attention to the context in which the dot symbol is used. In this book, the product of two scalars *x* and *y* will be written as *xy*, and the scalar multiple *k* of a vector **v** will be written $k\mathbf{v}$. Thus, statements like $k \cdot \mathbf{v}$ are syntactically incorrect and do not have any meaning.

Example: Find $\mathbf{u} \cdot \mathbf{v}$, where $\mathbf{u} = \langle 3, -4, 1 \rangle$ and $\mathbf{v} = \langle 5, 2, -6 \rangle$, then find the angle θ formed by **u** and **v**.

Solution: Using the first method of calculation, we have

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (3)(5) + (-4)(2) + (1)(-6) \\ &= 15 + (-8) + (-6) \\ &= 1. \end{aligned}$$

To find θ, we use the second method of calculation and solve for θ, using a calculator in degree mode for the last step.

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{26}\sqrt{65}}\right) \approx 88.61°.$$

- **Some Properties of the Dot Product**

The dot product of two vectors **u** and **v** has the following properties:

1) The dot product is commutative. That is, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

2) $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$. That is, the dot product of a vector with itself is the square of the magnitude of the vector. This formula relates the dot product of a vector with the vector's magnitude.

3) The dot product of the zero vector $\mathbf{0}$ with any other vector results in the scalar value 0. That is, $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$. It is possible that two non-zero vectors may results in a dot product of 0. This is discussed below.

4) The sign of the dot product indicates whether the angle between the two vectors is acute, obtuse, or zero. Assume that **u** and **v** are two non-zero vectors.

    - If $\mathbf{u} \cdot \mathbf{v} > 0$, that is, $\mathbf{u} \cdot \mathbf{v}$ is positive, then the angle formed by the vectors is acute.
    - If $\mathbf{u} \cdot \mathbf{v} < 0$, that is, $\mathbf{u} \cdot \mathbf{v}$ is negative, then the angle formed by the vectors is obtuse.
    - If $\mathbf{u} \cdot \mathbf{v} = 0$, that is, $\mathbf{u} \cdot \mathbf{v}$ is zero, then the angle formed by the vectors is 90 degrees (or $\pi/2$ radians). In this case, the vectors are perpendicular to one another. Two vectors that have this property are said to be **orthogonal**.

    Speaking in broadest terms, if the dot product of two non-zero vectors is positive, then the two vectors point in the same general direction, meaning less than 90 degrees. If the dot product is negative, then the two vectors point in opposite directions, or above 90 and less than or equal to 180 degrees. Later, when we discuss line and surface integrals, this notion of pointing in the "same" or "opposite" direction will have significant meaning in understanding the effect of a flow on a particle or through a porous membrane.

    The actual numerical value of the dot product does not indicate the size of the angle. At this stage, we are only interested in the sign of the dot product, not necessarily its numerical value. Later, we can place conditions on the vectors so that the numerical value of the dot product has meaning.

- **Orthogonal Projections**

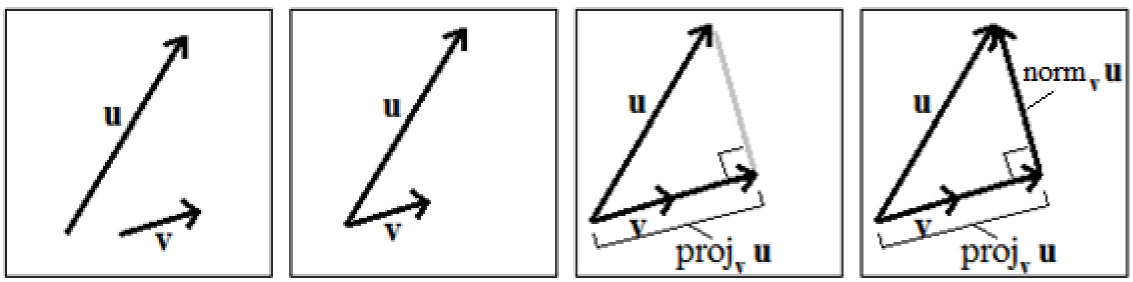
The **orthogonal projection** (or simply, the **projection**) of one vector onto another is facilitated by the dot product. For example, the projection of **u** onto **v** is given by:

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$$

Viewing **u** as the hypotenuse of a triangle and its projection onto **v** as the adjacent leg, then the opposite leg is called the **normal to the projection** of **u** onto **v**, written $\text{norm}_{\mathbf{v}}\,\mathbf{u}$, with the relationship that

$$\text{proj}_{\mathbf{v}}\mathbf{u} + \text{norm}_{\mathbf{v}}\mathbf{u} = \mathbf{u}$$

Here is a pictorial way to view a vector **u** projected onto **v**:



Two vectors **u** and **v** are given. | If it helps, picture **u** and **v** as having a common foot. | The projection of **u** onto **v** forms a vector parallel to **v**. | The legs $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ and $\text{norm}_{\mathbf{v}}\,\mathbf{u}$ form the adjacent and opposite legs of a right triangle, with **u** as the hypotenuse.

Remember, the statement

$$\text{proj}_{\mathbf{v}}\mathbf{u} + \text{norm}_{\mathbf{v}}\mathbf{u} = \mathbf{u}$$

is a sum of two vectors. The magnitudes of the vectors are related by the Pythagorean Formula:

$$|\text{proj}_{\mathbf{v}}\mathbf{u}|^2 + |\text{norm}_{\mathbf{v}}\mathbf{u}|^2 = |\mathbf{u}|^2.$$

- **The Cross Product**

The cross product of $\mathbf{u}$ and $\mathbf{v}$ is defined and best memorized as the expansion of a 3 by 3 determinant:

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}.$$

The cross product of $\mathbf{u}$ and $\mathbf{v}$ is a vector, with the property that it is orthogonal to the two vectors $\mathbf{u}$ and $\mathbf{v}$. Thus, if we take the dot product of $\mathbf{u}\times\mathbf{v}$ with $\mathbf{u}$ and then $\mathbf{u}\times\mathbf{v}$ with $\mathbf{v}$, we get zero both times:

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} = 0, \text{ and } (\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = 0.$$

This check should always be performed to ensure that the cross product is correct.

Example: Find $\mathbf{u}\times\mathbf{v}$, where $\mathbf{u} = \langle 3,-4,1\rangle$ and $\mathbf{v} = \langle 5,2,-6\rangle$.

Solution: We have

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 1 \\ 5 & 2 & -6 \end{vmatrix} = \begin{vmatrix} -4 & 1 \\ 2 & -6 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ 5 & -6 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -4 \\ 5 & 2 \end{vmatrix}\mathbf{k}$$

$$= 22\mathbf{i} - (-23)\mathbf{j} + 26\mathbf{k}, \text{ or } \langle 22, 23, 26\rangle.$$

Now we check:

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} &= \langle 22,23,26\rangle\cdot\langle 3,-4,1\rangle \\ &= (22)(3) + (23)(-4) + (26)(1) = 66 - 92 + 26 = 0.\end{aligned}$$

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} &= \langle 22,23,26\rangle\cdot\langle 5,2,-6\rangle \\ &= (22)(5) + (23)(2) + (26)(-6) = 110 + 46 - 156 = 0.\end{aligned}$$

Since both cases produce 0, we are confident that the cross product is correct.

- **Some Properties of the Cross Product**

The cross product of two vectors $\mathbf{u}$ and $\mathbf{v}$ has the following properties:

1) Reversing the order of $\mathbf{u}$ and $\mathbf{v}$ results in a negated cross product. That is, $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$.

   Visually, think of $\mathbf{u}$ and $\mathbf{v}$ as lying in a common plane. Their cross product $\mathbf{u} \times \mathbf{v}$ is a vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, so it is orthogonal to the plane in which $\mathbf{u}$ and $\mathbf{v}$ lie. If we cross $\mathbf{v}$ and $\mathbf{u}$, we get $-(\mathbf{u} \times \mathbf{v})$, which is also a vector orthogonal to the plane in which $\mathbf{u}$ and $\mathbf{v}$ lie.

2) The magnitude of the cross product is $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$. This is equal to the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$. Half of this value is the area of a triangle formed by $\mathbf{u}$ and $\mathbf{v}$.

3) If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ (the zero vector), then $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors.

- **Physical Interpretations of the Dot and Cross Products**

The dot product is often used to find the **work**, *W*, performed by a force $\mathbf{F}$ (in Newtons, N) acting on an object, moving it a distance $\mathbf{D}$ in meters. That is, $W = \mathbf{F} \cdot \mathbf{D}$. Note that work *W* is a scalar. Often, the force is applied at an angle to the direction that the object is moving. Furthermore, the force and distance are stated as scalar values and the angle θ is given, so that when finding $\mathbf{F} \cdot \mathbf{D}$, we often use the formula $|\mathbf{F}||\mathbf{D}| \cos\theta$.

The cross product is used to find the **torque**, denoted τ (the Greek letter tau), formed by the combined action of two vectors $\mathbf{u}$ and $\mathbf{v}$. We can think of a force $\mathbf{u}$ “pushing” against a vector $\mathbf{v}$, where $\mathbf{v}$’s foot acts as a pivot, much like the hinges of a door. In a broad sense, vectors $\mathbf{u}$ and $\mathbf{v}$ combine to form a “twisting” action at a point. This twisting is torque and is calculated by the cross product.

For example, if two vectors are parallel, then their cross product is $\mathbf{0}$. No torque is being applied in this case. Imagine a door is represented by a vector $\mathbf{v}$, with its foot being the hinge of the door. Would you open or close the door by applying a force parallel to $\mathbf{v}$? It makes more sense to apply a force orthogonal to the door to achieve torque.

- **Practice**

Let $\mathbf{u} = \langle 1,4,-5\rangle$, $\mathbf{v} = \langle -4,3,-1\rangle$ and $\mathbf{w} = \langle 7,-2,-3\rangle$. Classify each expression below as a vector, a scalar, or not defined. If it's a vector or scalar, find the result. If it's not defined, explain why. Note: the dot symbol always refers to the dot product.

1. $\mathbf{u} \cdot \mathbf{w}$
2. $(\mathbf{u} \cdot \mathbf{w})\mathbf{v}$
3. $(\mathbf{u} \cdot \mathbf{w}) \cdot \mathbf{v}$
4. $(\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v}$
5. $(\mathbf{u} \times \mathbf{w}) \times \mathbf{v}$
6. $(\mathbf{u} \times \mathbf{w})\mathbf{v}$
7. $(\mathbf{u} \cdot \mathbf{w}) \times \mathbf{v}$
8. $(\mathbf{u} \times \mathbf{w}) \times \mathbf{u}$
9. $(\mathbf{u} \times \mathbf{w}) \cdot (\mathbf{v} \times \mathbf{w})$
10. $(\mathbf{u} \times \mathbf{w}) \times (\mathbf{v} \times \mathbf{w})$
11. $(\mathbf{u} \cdot \mathbf{w}) \times (\mathbf{v} \cdot \mathbf{w})$
12. $(\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{w})$

13. You start walking from the origin in the direction of $\langle 3,1\rangle$, with the intention of ending at point $(7,1)$. You are allowed one right-angle turn. Find (a) the point at which you make this turn, (b) how far you walked in the $\langle 3,1\rangle$ direction, and (c) how far you walked orthogonal to the $\langle 3,1\rangle$ direction.
14. Given three points $A = (1,3,1), B = (2,5,-3), C = (-4,1,8)$. Find (a) the angle in degrees at vertex A, and (b) find the area within the triangle formed by the three points.

Answers:

1. 14
2. $\langle -52, 39, -13\rangle$
3. Not defined. $\mathbf{u} \cdot \mathbf{w}$ form a scalar, but then the dot product of a scalar with a vector is not defined.
4. 22
5. $\langle 122,98,-194\rangle$
6. Not defined. The expression $(\mathbf{u} \times \mathbf{w})\mathbf{v}$ indicates a scalar multiple of $\mathbf{v}$, but $\mathbf{u} \times \mathbf{w}$ is a vector, not a scalar.
7. Not defined. The expression $\mathbf{u} \cdot \mathbf{w}$ forms a scalar, but then the cross product of a scalar with a vector is not defined.
8. $\langle 280, -140, -56\rangle$
9. 1240
10. $\langle -154, 44, 66\rangle$
11. Not defined. The operations inside the parentheses form scalars, but the cross product of two scalars is not defined.
12. –434
13. (a) (6.6, 2.2); (b) $\sqrt{48.4} \approx 6.957$; (c) $\sqrt{1.6} \approx 1.2649$
14. (a) $\theta = \cos^{-1}(-37/\sqrt{1638}) \approx 156.09°$; (b) $\frac{1}{2}\sqrt{269} \approx 8.2$