الجامعة المستنصرية
كلية التربية الاساسية
قسم الرياضيات

محاضرات الجبر الخطي

المرحلة الثانية

اعداد

الاستاذة منتهى عبد الرزاق حسن

المفردات

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Definition **:**Fi**eld** :a set F with two operation (+)& (x) such that

(1)1-for any a,b∈F,then a+b∈F (**closure**)

2- a+b= b+ a (**commutative)**

3- if c∈F,then a+(b +c)=(a +b)+c (**associative)**

4- There is an element 0∈ $F\ such\ that\ \ if\ a\ \in F, then\ a+0=a$(**identity element)**

5- $\forall\ \boldsymbol{a} \in \boldsymbol{F} \exists -\boldsymbol{a} \in \boldsymbol{F}$ s.t a+(-a)=0 (**inverse)**

(2) 1-for any a, b∈F,then a ×b $\in F$ **(CLOSURE)**

2- a × b=b× a (**commutative)**

3- If c $\in F, then\ \ aX\ (b \times c) = (a \times b)\ \times\ c$ (**associative)**

4- there is an element $1\ \in F\ s.t\ if\ a\ \in F, then\ a\ \times 1 = a$ (**identity element)**

5- $\forall\ a \neq 0\ \in F\ \exists a^{-1} \in F\ s.t\ a\ \times a^{-1} = 1$

(3) $\forall a, b, c\ \in F, then\ a\ \times (b+c) = a \times b + a \times c$

Definition 2:Field

1- (F,+) is commutative group
2- (F\{0}.×) $is\ commutative\ group$
3- $\times\ distribution\ on\ +$

الفصل الاول

تعريف المتجه

في الفيزياء تظهر كميات مثل درجة الحرارة والانطلاق والكتلة والمساحة والحجم تتحدد بمقدار فقط اي (اعداد) لذلك تسمى كل واحدة منها قياسية فقط . وهناك كميات تتحدد بعدد واتجاه مثل حركة الرياح نقول 20كم\ساعة باتجاه الشمال الشرقي .

__Definition : Vectors__

The set of all orders n-tuples $(x_1, x_2, \dots x_n)$ $where$ $x_1, x_2, \dots x_n$ are all real numbers. An n-tuple is called vector and the numbers $x_1, x_2, .. x_n$ $are\ called\ components\ of\ the\ vectors$

Examples: x∈ $\boldsymbol{R}$, $(\boldsymbol{x_1}, \boldsymbol{x_2}) \in \boldsymbol{R^2}, \dots, (\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}) \in \boldsymbol{R^n}$

__Definition of addition on the vecters__

Let X=$(x_1, x_2, .., x_n), Y = (y_1, y_1, \dots, y_n), then$

X+Y=$(x_1, x_2, \dots x_n) + (y_1, y_2, \dots y_n) = (x_1 + y_1, \dots, x_n + y_n)$

__Def: Scalar multiplication:__

Let r ∈ $R, X = (x_1, x_2, \dots, x_n), then\ r(x_1, x_2, \dots, x_n) = (rx_1, rx_2, \dots, rx_n)$

__Def.: equal two vectors__

Let X=$(x_1, x_2, \dots, x_n), Y = (y_1, y_2, \dots, y_n), then\ X = Y\ iff\ x_1 = y_1, x_2 = y_2, \dots, x_n = y_n$

__Examples :__

Let V=(-1,0,2,3,-6),W=(1,0,-2,-3,6),U=(3,7,-6,2,-1)&Z=(0,0,0,0,0,)

Calculate U+V,V+U,3U,,3U+7V,V+W,0U

Def.: Zero vector in R^n

Is the vector in R^n and is denoted by 0_n =(0,0,…,0)

Def. The negative of X in R^n :

Denoted by –X=($-x_1, -x_2, \dots, -x_n$)

Example:

Let X=(1,2,-3,4) .Find –X and calculate X+(-X)

Theorem: let X=($x_1, x_2, \dots, x_n$), $Y = (y_1, y_2, \dots, y_n)$&$Z = (z_1, z_2, .., z_n)$ $be\ vectors\ in\ R^n$ & let r,s∈ $R, then\ prove\ that$

1-X=Y=Y=X

2-(X+Y)+Z=X+(Y+Z)

3-X+0=X

4-X+(-X)=0

5-(r .s)X=r(S x)

6-(r +s)X=r X +s. X

7-r(X+Y)= r.X +r . Y

8-1.X=X

Examples 1:LET X=(1,-1,2),Y=(0,3,6),Z=(-5,0,2) ,then satisfy (X+Y)+Z=X+(Y+Z)

Example 2 :let r=3,s=-5, X=(4,-2,6),then satisfy (rs)X=r(SX)

Example 3 find 3(-5,2,3)

Length of vectors :

Let X=$(x\ ,y)$ by Pythagorean theorem

$\| X \| = \sqrt{x^2 + y^2}$

If X=$(x_1, x_2, \dots, x_n)$, $then$ $\| X \| = \sqrt{x_1^2} + x_2^2 + \dots + x_n^2$

3-IF P=$(x_1, y_1), Q = (x_2, y_2), then\ the\ distance\ between\ P\&Q\ in\ R^2$

$\| PQ \| = \sqrt{(x_2 - x_1)^2} + (y_2 - y_1)^2$

Example

Let P=(2,3),Q=(3,-4) find $\| PQ \|$

Geometric representation of vectors تمثيل المتجه هندسياً

يعرف التمثيل الهندسي للمتجه بانه قطعة المستقيم الموجه له نقطة بداية ونقطة نهاية حيث يمثل طول القطعة المستقيمة بمقدار المتجه ويرمز رأس السهم الى اتجاه المتجه .

اتجاه المتجه : هو الزاوية التي يصنعها المتجه مع محور X.

Examples:

1-let $\vec{V} = (3,4)\ \&\vec{W} = (-8,7,4)$

1-find $\| v \|\ \&\ \| W \|$

2-find the distance a- $P_1P_2\ for\ P_1 = (2,3), P_2 = (4,6)$

b- $P_1 = (8,-4,2)\ \ P_2 = (-6,-1,0)$

3-find
$\| v \|$; $v = (-1,2,5) and\ the\ distance\ between\ P_1 = (2,-3,-5), P_2 = (5,2,-1)$

4-Find the point geometrical

A=(3,4,-2) ,B=(-2,4,-5) C=(-3,-4,2)

Def. scalar multiplication geometrically:

Let
$\vec{V} \neq \vec{0}, k \in$
$R, then\ \overrightarrow{kV}\ is\ the\ vector\ length\ greatest\ then\ \vec{V}\ by\ k\ times\ and\ have\ the\ same$

Direction of $\vec{V}\ if\ k > 0\ .\ if\ k < 0$ have opposite direction.

Dot product (scalar product)(Euclidean inner product)

الضرب النقطي (القياسي)

If u ,v are non-zero vectors in R^2 OR R^3 and θ is the angle between u and v,then we define (Dot product)as follows.

u . v=$\| \boldsymbol{u} \| . \| \boldsymbol{v} \| \boldsymbol{\cos\theta}$;0$\leq \boldsymbol{\theta} \leq \boldsymbol{\pi}$

if u,v ϵR^n is defined by **u .v =$\boldsymbol{u_1 v_1 + u_2 v_2 + \cdots + u_n v_n}$,where**

u=$(\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n})$ & $\boldsymbol{v} = (\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n})$

Example :let u=(0,0,1)& v=(0,2,2) find u . v and θ =45

U . v=$\| u \| . \| v \|\ cos\theta$

$\| u \|= \sqrt{u_1^2 + u_2^2} + u_3^2$ =$\sqrt{0+0+1=}$=1

$\| v \|= \sqrt{0+4+4}$=$\sqrt{2}$

U . v=1.2.$\sqrt{2}$ cos 45=2

Theorem: if u,v,w are vectors in R^2 or R^3 or R^n and k is scalar then

1. u . u =$\| u \|^2$; u . u=$u_1^2 + u_2^2 + \cdots + u_n^2$
2. if u, v are non zero vectors ,then u . v> 0 iff $\theta < 90$
3. u . v< 0 iff $\theta > 90$
4. u . v=90 iff $\theta = 90$
5. u . v = v .u
6. u . (v+ w)= u . v +u . w
7. k(u . v)=(k u) . v =u . (k v)

proof: 1- u . u =$u_1u_1 + u_2u_2 + \cdots u_nu_n$=$u_1^2 + u_2^2 + \cdots + u_n^2$

2-u.v=$\| \boldsymbol{u} \|.\| \boldsymbol{v} \|$ $\boldsymbol{c}os\ 90$ =0 ,cos $\theta > 0$ $in\ the\ first\ and\ fourth\ quarterts$

Cos $\boldsymbol{\theta} < 0$ $in\ the\ second\ \ and\ third\ quarters$

6- u(v+ w)=u .v +u . w

Let u .(v+ w)=$u_1(v_1 + w_1) + u_2(v_2 + w_2) + \cdots + u_n(v_n + w_n)$

=$(u_1v_1 + u_2v_2 + \cdots + u_nv_n) + (u_1w_1 + \cdots + u_nw_n)$=u.v+u .w

7- k(u .v)=(k u).v

K(u .v)=k$(u_1v_1 + u_2v_2 + \cdots + u_nv_n)$

=(k$u_1)v_1 + (ku_2)v_2 + \cdots (ku_n)v_n$=(ku).v

Examples:

1- find (3u+2v).(4u+v)

2- if u=(1,3,-2,7),v=(0,7,2,2),find $\| u \|, \| v \|, d(u,v)$

Cross Product:

If U,V are non-zero vectors in R^3,U=$(u_1,u_2,u_3),V=(v_1,v_2,v_3)$,then the cross product of U &V is the

U $\times V=(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}$,-$\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}$,$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$)$\in R^3$

=$(u_2v_3-u_3v_2,-(u_1v_3-u_3v_1),u_1v_2-u_2v_1)$

Which mean U$\times V=\begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ =i$(u_2v_3-u_3v_2)-j(u_1v_3-u_3v_1)+k(u_1v_2-u_2v_1)$

NOTE: the cross product is define only in R^3

Theorem :**the vector U $\times V$ is orthogonal to both U &V**

Theorem: **if θ *is the angle between* U &V *then***

$\parallel U \times V \parallel=\parallel U \parallel \parallel V \parallel$ *sine* θ

***Corollary*: the two non-zero vectors U&V are parallel if and only if U $\times V$ =0**

Example:

Let U=(1,2,-2), V=(3,0,1)find U$\times V$

U$\times V=(\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix}$,-$\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}$,$\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}$ =(2,-7,-6)

Example

Find U $\times V$, $where\ U = (3, -4,1)$ & V=(5,2,-6), then find the angle a form U &V.

Theorem : **if u, v &w are vectors in R^3 and k is scalar then:**

1- $U \times V = -(V \times U)$
2- $U \times (V + W) = U \times V + U \times W$
3- $(U+V) \times W = U \times W + V \times W$
4- $k(U \times V) = (kU) \times V) = U \times (kV)$
5- $U \times 0 = 0 \times U = 0$
6- $U \times U = 0$

Proofe:1- $U \times V = (u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1)$

$V \times U = (v_2u_3 - v_3u_2, -(v_1u_3 - v_3u_1), v_1u_2 - v_2u_1)$

Area the cross product

The length of the cross product $U \times V$ is equal to the area of the parallelogram determined by U&V

Area of the parallelogram $= \| U \times V \| = \| U \| \| V \| \sin\theta$

Area of the triangle $= \frac{1}{2} \| U \times V \|$

Volume of parallelogram $= \| A . (B \times C) \|$

Where A=height ,B=length,c= width

Example

Find the area of the triangle determined by $p_1 = (2,2,0), p_2 = (-1,0,2), p_3 = (0,4,3)$

Sol. $P_1P_2 = U = (-1,0,2) - (2,2,0) = (-3,-2,2)$

$P_1P_3 = V = (0,4,3) - (2,2,0) = (-2,2,3)$

$U \times V = (-10,5,,10), \| U \times V \| = \sqrt{(-10)^2 + 5^2 + 10^2} = \sqrt{225} = 15$

$\text{Area} = \frac{1}{2} \| U \times V \| = \frac{1}{2} 15 = 7.5$

NOTE: **1-U.V is constant**

2-U$\times$ $\boldsymbol{V\ is\ a\ vector}$

3-the vector U$\times$ $\boldsymbol{V}$ is perpendicularly to both U&V

4-The vector i=(1,0,0),j=(0,1,0), k=(0,0,1)

EXAMPLE:

Find the volume of the parallelogram that generated by A=(2,1,3), B=(-1,3,2), C=(1,1,-3)

Sol. V=$\| A.(B\times C) \|$

B$\times$ $C = \begin{vmatrix} i & j & k \\ -1 & 3 & 2 \\ 1 & 1 & -3 \end{vmatrix}$ =-11i-j-4k

A. (-11,-1,-4)=-22-2-12=-36

$\| A.E \| = \| -36 \| = 36$

EXAMPLE

Find the volume of parallelepiped ,where

A=(2,1,3),B=(1,2 -,0),C=(3,0,1)

Definition of Vector space:

A vector space over F is a set V together with two operation additive and multiplication such that satisfying each of the following:

1-**(addition is closed) u+ v$\in$ $\boldsymbol{V\ for\ all\ u,v\ \in V}$**

2-(addition is commutative) u+ v=v+ u for all u,v $\in$ $\boldsymbol{V}$

3-(addition is associative) u+(v+ w)=(u+ v)+w for all u,v,w $\in$ $\boldsymbol{V}$.

4-(additive identity) there exist an element 0 ∈ V called ZERO VECTOR such that u+0=0+u=u for all u ∈ $\boldsymbol{U}$.

5-(additive inverse) for every v∈ $\boldsymbol{V\ there\ exist\ an\ element\ -u\ \in V\ ,such\ that\ u+(-u)=0=-u+u}$

(V,+) is a commutative group

6- (. is closure) $\boldsymbol{\alpha v\in V\ for\ all\ v\in V,\alpha\in F}$

7- (. Is associative) ($\boldsymbol{\alpha\beta}$)v=$\boldsymbol{\alpha(\beta v) for\ all\ v\in V,\alpha,\beta\in F}$

8- (. Is distribution on +) $\boldsymbol{\alpha(u+v)=\alpha u+\alpha v\ for\ all\ u,v\in V\ and\ \alpha\epsilon F}$

9- ($\boldsymbol{\alpha+\beta)u=\alpha u+\beta u\ for\ all\ u\ \in V,\alpha,\beta\in F}$

10 - for all u∈ $\boldsymbol{V\ there\ exist\ an\ element\ 1\ \in F}$**,called the multiplicative identity of u,;1.u=u.1=u**

Example: explain that (R,+)form a vector space over R . H.W

2- is (Z,+,.) form a vector space over R

Sol. Let ∝= $\frac{1}{3}\in R,2\in Z\ SO\ \frac{1}{3}.2\notin Z$ **,so (Z,+,.)** is not vector space over R

3-let V={$(x_1,x_2);x_1,x_2\in R,x_2=\frac{1}{2}x_1+1\}$ is (V,+,.)form a vector space over R^2.

***SOL*.** let X=$(x_1,x_2)\ such\ that\ x_2=\frac{1}{2}x_1+1$

Y=$\boldsymbol{(y_1,y_2);y_2=\frac{1}{2}y_1+1}$

X+Y=$\boldsymbol{(x_1,x_2)+(y_1,y_2)=(x_1+y_1,x_2+y_2);}$

To prove $\boldsymbol{x_2+y_2=\frac{1}{2}(x_1+y_1)+1}$

$x_1+y_1=\frac{1}{2}x_1+1+\frac{1}{2}y_2+1=\frac{1}{2}(x_1+y_1)+2$

$so(V,+,.)IS\ NOT\ V.S\ OVER\ R^2$

Example: **let $\mathbb{C}$ be the set of complex number .Define addition in $\mathbb{C}$**

by $(\mathbf{a}+\mathbf{bi})+(\mathbf{c}+\mathbf{di})=(\mathbf{a}+\mathbf{c})+\mathbf{i}(\mathbf{b}+\mathbf{d})$ $\boldsymbol{for\ all\ a,b,c,d} \in \boldsymbol{R}$

and we define the scalar multiplication by $\boldsymbol{\alpha}(\mathbf{a}+\mathbf{bi})$
$=\boldsymbol{\alpha}\mathbf{a}+\boldsymbol{\alpha}\mathbf{bi}$ for all $\boldsymbol{\alpha} \in \mathbf{R}\ \&\forall \mathbf{a},\mathbf{b} \in \mathbf{R}$

1- The addition is closed u +v=(a +c)+i(b +d) since a ,b ,c, d are real numbers
2- The addition is commutative
3- U+V=(a +bi)+c +di=(a +c)+i(b +d)=(c +a)+i(d +b) because addition are commutative on R
4- =(c +di)+(a + i b)=v +u
5- The addition is associative since +is associative on R (H.W)
6- The additive identity :for all u=a+ib ∈ $\mathbb{C}$ we have (a+bi)+(0+i0)=a+ib
7- The additive inverse. For all u=a +bi ∈ $\mathbb{C}$ we have (a+ib)+(-a+(-b)i)=0+0i
8- Let u=a +bi ,v=c +di ∈ $\mathbb{C}$, $\alpha \in R$

A-$\alpha u=\alpha(a+b\,i)=\alpha\,a+i\,\alpha b \in \mathbb{C}$

B-$\alpha(u+v)=\alpha[(a+i\,b)+(c+i\,d)=\alpha(a+c+I\alpha(b+d)$ because Multiplication distributes over additionin R

Also the right hand

$\alpha u+\alpha v=\alpha(a+c)+i\alpha\ (b+d)$

c- $(\alpha+\beta)\ u==(\alpha+\beta)(a+b\ i)=(\alpha a+\beta a)+i(\alpha+\beta)b=$

$\alpha a+i\ \alpha b)+(\beta a+i\ \beta b)=\alpha u+\beta u$

D-$(\alpha\beta)u=(\alpha\beta)(a+i\ b)=\alpha(\beta a+i\ \beta b)=\alpha(\beta u)$

9-the multiplication identity :1.u=u ;u=a+ib

So$(\mathbb{C},+,.)$ $is\ vector\ space$

Example:

Let $M_{2\times2}(R) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}$;a,b,c,d ∈ $R\}$.

IS $(M_{2\times2}, +, .)\ vector\ space.$ **H.W**

Standard vectors in R^n, R^n

$e_1 = (1,0) \rightarrow (1,0,0,..,0)$

$e_2 = (0,1) \rightarrow (0,1,0, \dots ,0)$

$e_3 = (0,0,1,0, \dots ,0)$

$e_n = (0,0,..,1)$

Proposition(1) **:every vector space has a unique addition identity**

Proof: suppose 0 ,$0^\backslash$ are two additive identity

0+$o^\backslash = 0^\backslash$(since 0 is identity)

0+$0^\backslash = 0\ (since\ 0^\backslash\ is\ identity\ )$

0=0+$0^\backslash = 0^\backslash\ hence\ 0 = 0^\backslash$

Proposition (2): **every v∈ V has a unique additive invers**

Proof: suppose that there are two additive invers of v ,let w,$w^\backslash$

V+w=0&v+$w^\backslash = 0, so\ w = w + 0 = w + \left(v + w^\backslash\right) = (w + v) + w^\backslash = 0 + w^\backslash = w^\backslash$

Proposition (3) **:0 v=0 for all v ∈ V**

Proof:0.v=(0+0).v=o.v+0.v

0.v-0.v=0.v+0.v)+(-0.v)

0=0.v+(0.v+(-0.v)=0=0.v+0,so 0=0.v

Proposition (4) **: (-1).v=-v for every v $\in \boldsymbol{V}$**

(-1)V+V=(-1)V+1.V

=(-1+1)V=0.V=0

(-1)V+V=0

(-1)V+V+(-V)=0+(-V)

(-1)V+(V-V)=-V

(-1)V+0=-V

(-1)V=-V

H.W **IF k .u=0 ,then k-0 or u=0**

Proposition (5) for all u ,v ,w $\in \boldsymbol{V}$

If u+v=u+w , then v=w **H.W**

Proposition (6) **:for all u,v $\in \boldsymbol{V}$, $\boldsymbol{the\ equation\ u+x=v\ has\ a\ unique\ solutio\ x=v-u \in V}$**

SOL: u+ x=v

-u+ u+ x=-u +v

(-u+ u)+x=v-u

0+x=v-u implies x=v-u

Suppose there exist another solution x' s.t u+$x^{\backslash} = v\ to\ prove\ x = x^{\backslash}$

X=v-u

=u+$x^{\backslash}$-u

=$x^{\backslash}$+u-u

$x^{\backslash} + 0 = x^{\backslash}$

$the\ solusion\ is\ unique$

Proposition (7) :for all u $\in V, we\ have — (-u) = u$

Sol: u+(-u)=0

(-u)+(-(-u))=0

(-u)+(-(-u))=-u +u

u+(-u) +(-(-u))=u +(-u)+u

[u+(-u)]+(-(-u))=u +[-u+u]

0+(-(-u))=u+0

-(-u))=u

Subspace

Let (V,+,.) is a vector space over k

Definition **:a non-empty subset U of V is called a subspace of V if (U,+,.)is a vector space over k**

Proposition 2.2:anon-empty subset U of a vector space V over k is a subspace of V if and only if the following condition are satisfied

1- 0$\in U$
2- 2-for all u ,v $\in U$, $we\ have\ u + v \in U$
3- For all u $\in U\ and\ \alpha \in K, we\ have\ \alpha u \in U$

Remark: Every vector space V has two subspaces, namely V and {0}, which are **called trivial subspaces**. Any other subspace of V is called a **proper subspace**

Examples:

1- Let **U**=$\{(x_1,x_2,x_3) \in F^3; x_1+2x_2=0\}$ *is U subspace of* F^3

SOL:

1- 0=(0,0,0) $\in U$,since 0+2.0=0

$$2-let\ \boldsymbol{x}=(x_1,x_2,x_3), y=(y_1,y_2,y_3) \in U; x_1+2x_2=0\&y_1+2y_2 = 0, then\ x+y=(x_1,x_2,x_3)+(y_1,y_2,y_3) = (x_1+y_1,x_2+y_2,x_3+y_3)$$

$to\ prove\ x_1+y_1+2(x_2+y_2)=0$

$(x_1+2x_2)+(y_1+2y_2)=0+0=0$

x +y$\in U$

3-let $\alpha \in F, x=(x_1,x_2,x_3) \in U; x_1+2x_2=0$

$\alpha x=\alpha(x_1,x_2,x_3)=(\alpha x_1,\alpha x\quad,\alpha x_3);$

$\alpha x_1+2\alpha x_2=\alpha(x_1+2x_2)=\alpha.0=0, so\ \alpha x \in U$

(U,+,.)is a subspace of F^3

2-let W={(a,b,0);a,b$\in R\} \subset R^3 is\ W\ subspace\ of\ R^3$

<u>***Proposition** : **let** $\boldsymbol{k(-u)=(-k)u=-ku}$*</u>

Proof : (-k) u+ k u=0 so (-k +k)u=0

(-k) u +k u-k u=0-ku

(-k)u=-k u

Proposition: if $W_1 \& W_2 are\ subspace\ of\ V, then\ W_1 \cap W_2 is\ subspace\ of\ V$

Proof: let $\boldsymbol{w_1 \neq \emptyset \ \& \ w_2 \neq \emptyset \ so \ w_1 \cap w_2 \neq \emptyset}$

a) **since** $\boldsymbol{w_1 \& w_2\ are\ subspace\ of\ V\ , then 0 \in w_1 \&\ 0 \in w_2, hence\ 0 \in w_1 \cap w_2}$

b) let u,v$\in W_1 \cap W_2, henu, v, u \in W_1 \& u, v \in W_2, then\ u + v \in W_1 \& u + v \in W_2, then\ u + v \in W_1 \cap W_2$

c)let $\boldsymbol{\alpha} \in K \& u \in W_1 \cap W_2, then\ since\ W_1 \& W_2\ are\ subspace\ of\ V, then\ \alpha u \in W_1 \&\ \alpha u \in W_2, hence\ \alpha u \in W_1 \cap W_2$

***Proposition*: let** $\boldsymbol{W_1, W_2, \dots, W_3 are\ subspace\ of\ \ vector\ space\ V\ over\ field\ K\ , then}$

1-$\boldsymbol{W_1 \cap W_2 \cap \dots \cap W_N\ is\ a\ subspace}$

2-$\boldsymbol{W_1 + W_2 + \cdots + W_N\ is\ a\ subspace\ of\ V}$

Inner product space:

Definition: an inner product is a vector space with addition structure .More specifically each pair of vectors is associated with a scalar quantity known as of the inner product of the vectors .The inner product is denoted as $< , >$

Def.2 an inner product on V is a map :V× $\boldsymbol{V \to F}$

Remark :the inner product is a generalizes the concept of dot product Euclidean space

The inner product is defined by the following axioms

1-linearity in first slot addition :for all u ,v, w∈ V

$< u+v, w \geq < u, w > + < v, w >$

2-homogenous:$< \alpha u, v >= \alpha < u, v >$

3-positivity:$< u, u > \geq 0 \ \ \forall u \in V$

4-Positive definition:$< u, u \geq 0 \ iff \ u = 0$

5-conjugate symmetry:$< u, v > = \overline{< v, u >}$ **for all u,v ∈** V

Example: **x +i y=u ,so** $\bar{u} = x - iy$

LEMMA:1-theinner product is anty- linear

2-the seconed slot, that is

a-$< u, v+w > = < u, v > + < u, w >$
b-$< u, \alpha v >= \bar{\alpha} < u, v > \ for\ all\ u, v, w \in V$
Proof : 1-$< u, v+w > = \overline{< v+w, u >} = \overline{< v, u >} + \overline{< w, u >} =$

$$= < u, v > + < u, w >$$

1-$< u, \alpha v > = \overline{< \alpha v, u >} = \bar{\alpha} < u, v >$
NOTE:$< v, 0 > = < 0, v > = 0$

Proof: **suppose** $< v, 0 > = a \ \neq 0\ , so\ \alpha < v, 0 > = < v, \alpha . 0 >$

$= < v, 0 > \ \ , so\ that\ \alpha a = a$

$which\ is\ a\ controduction\ if\ a\ \neq 0\ , so\ a = 0$

$Example: if\ U = (u_1, u_2), V = (v_1, v_2), then$

$< U, V > = 3u_1v_1 + 2u_2v_2 . show\ that\ < U, V >> is\ inner\ product\ on\ R^2$

$sol.$ Let W=$(w_1, w_2), U + V = (u_1 + v_1, u_2 + v_2)$

$< U + V, W \geq 3(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2$

$= 3u_1w_1 + 3v_1w_1 + 2u_2w_2 + 2v_2w_2$

$= 3u_1w_1 + 2u_2w_2 + 3v_1w_1 + 2v_2w_2$

$=< U, W > +< V, W >$

$\mathbf{2-< KU, V > = 3(ku_1)v_1 + 2(ku_2)v_2 = k \ < u, v >}$

$\mathbf{3}-< u, u > = 3u_1u_1 + 2u_2u_2 =$

$= 3u_1^2 + 2u_2^2 \geq 0$

$< \boldsymbol{u}, \boldsymbol{u} \geq \mathbf{3}u_1^2 + 2u_2^2 = 0 \ iff \ u_1 = u_2 = 0$

$\mathbf{5-< u, v > = 3u_1v_1 + 2u_2v_2}$

$= 3v_1u_1 + 2v_2u_2 =< v, u >$

$\boldsymbol{so} \ < \boldsymbol{U}, \boldsymbol{V} > \boldsymbol{i}s \ inner \ product$

Example:

$\boldsymbol{if} \ \boldsymbol{U} = \begin{pmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} \\ \boldsymbol{u_3} & \boldsymbol{u_4} \end{pmatrix}$ V=$\begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$,then $< U, V \geq u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4 \ is \ inner \ product \ on \ M_{2\times 2}$

$\boldsymbol{Induced \ by \ norm}$

Definition: let V be a vector space over F.a map

$\|.\|: \boldsymbol{V} \rightarrow \boldsymbol{R}$ $, \boldsymbol{which \ mean}$ $\|.\| \ : \boldsymbol{V} \rightarrow \| \boldsymbol{V} \| = \sqrt{< \boldsymbol{V}, \boldsymbol{V} >} \ \ \forall \boldsymbol{v} \in \boldsymbol{V}$

$\boldsymbol{In \ norm \ on \ V}$ $, \boldsymbol{three \ codition \ are \ satisfied}$

1-positive definition. $\| \boldsymbol{u} \| = \mathbf{0} \ \boldsymbol{iff} \ \boldsymbol{u} = \mathbf{0}$

2-positive homogeneity .$\| \boldsymbol{a} \ \boldsymbol{u} \| = \| \boldsymbol{a} \|.\| \boldsymbol{u} \| \ \boldsymbol{for \ all} \ \boldsymbol{a} \in \boldsymbol{F}, \boldsymbol{u} \ \in \boldsymbol{V}$

3-Triangle inequality:$\| \boldsymbol{u} + \boldsymbol{v} \| \leq \| \boldsymbol{u} \| + \| \boldsymbol{v} \| \ \boldsymbol{for \ all} \ \boldsymbol{u}, \boldsymbol{v} \ \in \boldsymbol{V}$

Remark:$\| \boldsymbol{u} \| \geq \mathbf{0}$

Proof: u-u=0,so $\| u-u \|=0, implies\ \| u \|+\| -u \| \geq 0. thus$

$\| u \|+\| u \| \geq 0, so\ that\ 2 \| u \| \geq 0, therefore\ \| u \| \geq 0$

$2-if\ we\ take\ V=R^n$ **,then the norm define by the usual dot product i.e U=**$(u_1,u_2,\ldots u_n), then\ \| u \|=\sqrt{u_1+u_2+\cdots u_n}$

Example

Let f(V)=$\sqrt{<V,V>}$**, Is f(V) satisfies the axioms of a norm**

<u>*Sol:*</u> 1-since $<V,V> \geq 0 so \sqrt{<v,v> \geq 0}$ **,**$<V,V>=0\ IFF\ V=0$

IT FOLLOW THAT f(V)≥ 0

2-$\| \alpha V \|=\| \alpha \| \| V \|$

F$(\alpha V)=\sqrt{<\alpha V,\alpha V>}$ **=**$\sqrt{\alpha^2<V,V>}$ **=**$\| \alpha \| \sqrt{<V,V>}$ **=**$\| \alpha \| \| V \|$

3-$\| V+U \|=\| V \|+\| U \|$

F(V+U)=$\sqrt{<V+U,V+U>}$ **=**$\sqrt{<V,V>+2<V,U>+<U,U>}$

$\leq \sqrt{<V,V>+<U,U>}$

$\leq \sqrt{<V,V>}$ **+**$\sqrt{<U,U>}$

$\leq F(V)+F(U)$

$\therefore F(V) is\ a\ norm$

example2

(R,$\|.\|) is\ a\ norm$

Orthogonal

Definition: two vectors u ,v $\in V$ ***are orthogonal if*** $\boldsymbol{< u, v >= 0}$

Denoted by u$\perp \boldsymbol{v}$

Theorem:

If u ,v $\in \boldsymbol{V, with\ u \perp v, then}\ \| \boldsymbol{u + v} \|^2 = \| \boldsymbol{u} \|^2 + \| \boldsymbol{v} \|^2$

Proof: **suppose u**$\perp \boldsymbol{v}$ **,** $\boldsymbol{since} \| \boldsymbol{u + v} \| = \sqrt{\boldsymbol{< u + v, u + v >}}$

$\| u + v \|^2 = < u + v, u + v >$

$= < u, u + v > + < v, u + v >$

$= < u, u > + < u, v > + < v, u > + < v, v >$

So $< u, v >, < v, u > = 0$ $since\ u \perp v$

$= < u, u > + < v, v > = \| u \|^2 + \| v \|^2$

Theorem: caushy –schwarz inequality)

Given any u,v$\in \boldsymbol{V, we\ have} \ \| \boldsymbol{< u, v >} \| \leq \| \boldsymbol{u} \| . \| \boldsymbol{v} \|$

Or $\boldsymbol{< u, v >^2 \leq < u, u > . < v, v >}$

Example :hold the caushy –schwarz inequality if u=(2,1),

v=(1,-3),then $\boldsymbol{< u, v \geq 6u_1v_1 + 2u_2v_2}$

$$\boldsymbol{< u, v \geq 6(2.1) + 2(1.(-3) = 6}$$

$$\boldsymbol{< u, u > = 6.4 + 2.1 = 26}$$

$$\boldsymbol{< v, v > = 6 + 2.9 = 24}$$

$$\boldsymbol{6^2 \leq 26 . 24, so 36 \leq 624}$$

Theorem (Triangle inequality)

For all u ,v$\in V, \boldsymbol{then}\ \ \| \boldsymbol{u}+\boldsymbol{v} \| \leq \| \boldsymbol{u} \| + \| \boldsymbol{v} \|$

Proof: By a straight forward ,calculation, we obtain

$\| u+v \|^{2} = < u+v, u+v > = < u,u > + < u,v > + < v,u > +$

$< v,v >$

$= < u,u > \ + < v,v > \ + < u,v > + \overline{< u,v >}$

$= \| u \|^2 + \| v \|^2 + 2R < u,v >$ since Re$< u,v > \leq \| < u,v > \|$ by using the caushy –shuarz we have

$\| u+v \|^2 \leq \| u \|^2 + \| v \|^2 + 2 \| u \| . \| v \|$

$\leq (\| u \| + \| v \|)^2$

Taking the square root of both sides , we obtain the triangle inquality

3-Linear combinations and span:

Definition ::we say that the vector x is linear combination of vectors $\boldsymbol{x_1, x_2, \ldots, x_n}, \boldsymbol{if\ we\ can\ write\ \ x\ as\ the\ form}$

X=$\boldsymbol{c_1x_1 + c_2x_2 + \cdots + c_nx_n; c_1, c_2, .., c_n \in R}$

Example :let $\boldsymbol{x_1 = (1,2,1,-1), x_2 = (1,0,2,-3), x_3 = (1,1,0,-2) in\ R^4}$ **.show that x=(2,1,0,-2) is a linear combination of** $\boldsymbol{x_1, x_2, x_3}$

Sol. let $\exists \boldsymbol{c_1}, c_2, c_3 \in R; X = c_1x_1 + c_2x_2 + c_3x_3$

(2,1,5,-5)=$\boldsymbol{c_1(1,2,1,-1) + c_2(1,0,2,-3) + c_3(1,1,0,-2)}$

$c_1 + c_2 + c_3 = 2$ (1)

$2c_1 + c_3 = 1$ (2)

$c_1 + 2c_2 = 5$ (3)

$-c_1 - 3c_2 - 2c_3 = -5$ (4)

$from\ (1) and\ (4)\ we\ have\ -2c_2 - c_3 = -3, so$

$c_3 = 3 - 2c_2$ (5) if we compensate (5) in(2) we have

$2c_1 - 2c_2 = -2$ (6) and by addition(6)with (3) we have $c_1 = 1$

Compensate to obtain $c_3 = -1, c_2 = 2$, so X is a linear combination of x_1, x_2, x_3

Example (2):let u=(1,2,-1),v=(6,4,2) in R^3,show that w=(9,2,7) is a linear combination of u &v and $w^\backslash\ is\ not\ linear\ combination\ of\ u\&v$

Theorem: **let V be a vector space and** $\boldsymbol{x_1, x_2, \dots,} \in$ $\boldsymbol{V, let\ W\ be\ set\ of\ all\ linear\ combination\ of\ x_1, x_2, \dots, x_n, i.e\ W =}$ $\boldsymbol{\{c_1x_1 + c_2x_2 + \dots + c_nx_n; c_1, c_2, \dots c_n \in R\}, then\ W is\ a\ subspace\ of\ V}$

Proof 1:0=0.$x_1 + 0.x_2 + \dots + 0.x_n$,so 0$\in W \neq \emptyset$

2- Let u$\in W, \exists c_1, \dots, c_n \in R\ s.t\ u = c_1x_1 + c_2x_2 + \dots + c_nx_n$
Let v $\in W; \exists d_1, d_2, \dots, d_n \in R\ s.t\ v = d_1x_1 + d_2x_2 + \dots + d_nx_n$

U+ v = $(c_1 + d_1)x_1 + (c_2 + d_2)x_2 + \dots + (c_n + d_n)x_n s.t$

$e_i = c_i + d_i; i = 1,2, \dots, n, so\ u + v \in W$

3- let u $\in W, \exists\ c_1, c_2, \dots, c_n \in R\ s.t\ u = c_1x_1 + c_2x_2 + \dots + c_nx_{n,}\ let\ \alpha \in R; \alpha u = (\alpha c_1)x_1 + (\alpha c_2)x_2 + \dots + (\alpha c_n)x_n$

$=k_1x_1 + k_2x_2 + \dots + k_nx_n$

so $\alpha u \in W\ \therefore W\ is\ subspace\ of\ V$

***Spanned(generate)*:Let S =$\{x_1, x_2, \dots, x_n\}$ be a set of vector space ,then we say that S spans V or S generates V ,if every vector space V can be written as a linear combination of vectors of which mean $\forall x \in V; X = c_1x_1 + c_2x_2 + \cdots + c_nx_n$**

Example: **let V=$R^3, S = \{v_1, v_2, v_3\} s.t v_1 = (1,2,1), v_2 = (1,0,2), v_3 = (1,1,0)$, *is S span* R^3**

Sol. Let (a ,b, c)∈ R^3

(a, b, c)=$c_1v_1 + c_2v_2 + c_3v_3$

=$c_1(1,2,1) + c_2(1,0,2) + c_3(1,1,0)$

a=$c_1 + c_2 + c_3$ 1

b=$2c_1 + c_3$ 2

c=$c_1 + 2c_2$ 3

so by solve the equations we have

$C_1=\frac{-2a+2b+c}{3}$, $C_2=\frac{a-b+c}{3}$, $C_3=\frac{4a-b-2c}{3}$

$so\ \boldsymbol{S}\ span\ R^3$

$\boldsymbol{Example\ 2}$:explain that S={(1,0,0),(0,1,0),(0,0,1)}span R^3

Remarks ***|A|=0,then the vectors does not span the vector(A transaction matrix مصفوفة المعاملات***

Linearly independence and linearly dependent:

Definition **:let V be a vector over a field K .Asubset** $\{v_1, v_2, \dots., v_n\}$ ***in V is linearly depent over K if there exist scalar***

$\alpha_1, \alpha_2, \dots, \alpha_n \in K.$ *(not all zero)such that*

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$

Definition :let Vbe a vector space over a field K .A subset $\{v_1, v_2, \dots, v_n\}$ ***in V is linearly indepent over K if***

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0,$ ***then*** $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

Example: let $x_1 = (1,0,1,2), x_2 = (0,1,1,2), x_3 = (1,1,1,3) \subseteq R^n$ **.is S linearly independent or not .**

Sol. $c_1x_1 + c_2x_2 + c_3x_3$**=0**

$$c_1(1,0,1,2) + c_2(0,1,1,2) + c_3(1,1,1,3) = (0,0,0,0)$$

$$c_1 + c_3 = 0 \qquad 1$$

$$c_2 + c_3 = 0 \qquad 2$$

$$c_1 + c_2 + c_3 = 0 \qquad 3$$

$$2c_1 + 2c_2 + 3c_3 = 0 \qquad 4$$

By multiply 3 by (2)and subtraction to 4 ,we have

C_3=0,we compensate her to 1,we have

C_1=0, and we compensate her to 2, we have

C_2=0. So C_1=c2=c3 =0, implies x_1, x_2, x_3 is linearly indepent.

Example:

x_1 **= (1,2,-1),** x_2**= (1,-2,1) ,** x_3**= (-3,2,1),** x_4**= (2,0,0) is linearly indepent or not.**

H.W

Theorem : let S be a set of vectors in vector space V ,then S is linearly dependent if and only if the vectors of S is linear combination of the other vectors of S.

Example : let f(x)=$x^2, g(x) = x, h(x) = 1$ and $j(x) = (x + 1)^2$,explain that S={f,g,h,j} is linearly indepent

Sol. By theorem $\exists c_1, c_2, c_3 \in R$

J(x)=$c_1 f(x) + c_2 g(x) + c_3 h(x)$

$(x + 1)^2 = c_1 x^2 + c_2 x + c_3$

$x^2 + 2x + 1 = c_1 x^2 + c_2 x + c_3$

$c_1 = 1, c_2 = 2, c_3 = 1$

{f, g, h ,j} is linearly dependent

Remark: if |A|=0 فأن vectors linearly dependent

If $|A| \neq 0$, *the vectoers linearly indepent*

Basis :

we say that a set of vectors S=$\{x_1, x_2, \dots, x_n\}$ of a *vector space V is a basis for this vector space*

if the following condition hold

1-S is span V

2-S is linearly independent

Examples:1- let $S_1 = \{(1,0), (1,1)\}$

2-$S_2 = \{(-2,1), (2,3)\}$

Explain that S_1 *and* S_2 *form a basis for* R^2

1-let X=(a,b) $\in R^2, \exists c_1, c_2 \in R; X = c_1 x_1 + c_2 x_2$

(a,b)=$c_1(1,0)+c_2(1,1)$

$(c_1,0)+(c_2,c_2)$

a=$c_1+c_2 \ldots \ldots..1$

b=$c_2 \ldots \ldots \ldots \ldots.2$,so we have a-b=$c_1$

$\therefore S_1\ span\ R^2$

$C_1X_1+C_2X_2+0$

$(c_1,0)+(c_1,c_2)=(0,0)$

$c_1+c_2=0$

$c_2=0$

$\therefore c_1=c_2=0$, soS_1 $is\ linearly\ independent$

$S_1\ is\ basis\ for\ R^2$

Standard basis (unit vector) or natural basis :

1-S=$\{e_1=(1,0),e_2=(0,1)\}\ basis\ for\ R^2$

2-S={i=(1,0,0),j=(0,1,0),k=(0,0,1)} basis for R^3

3S=$\{e_1=(1,0,0,\ldots,0),e_2=(0,1,0,\ldots,0),\ldots,e_n=(0,0,\ldots,1)\}\ basis\ for\ R^n$

3-$\{E_1=\begin{pmatrix}1&0\\0&0\end{pmatrix},E_2=\begin{pmatrix}0&1\\0&0\end{pmatrix},E_3=\begin{pmatrix}0&0\\1&0\end{pmatrix}\ E_4=\begin{pmatrix}0&0\\0&1\end{pmatrix}\}\ basis\ for\ M_{2\times2}$

4-S={1,t,t^2}is stander basis for P_2 ; $P_2(t)=a_0+a_1t+a_2t^2$

5-S={1,t,$t^2,\ldots,t^n\}\ is\ stander\ basis\ for\ P_n$

Theorem

If S =$\{X_1,X_2,\ldots,X_N\}$ is a basis for vector space V ,then every vector space in V can be write in only one form as a linear combination of vectors in S.

Proof: since S is basis of V ,then S is span V,SO for each x $\in V$ $\exists c_1, c_2, .., c_n \in R$; X=$c_1x_1 + c_2x_2 + \cdots + c_nx_n$ … … .1

Suppose that $\exists d_1, d_2, \dots, d_n \in R$; $X = d_1x_1 + d_2x_2 + \cdots + d_nx_n$ … … 2

By subtract 2 from 1 we obtain

0=$(c_1 - d_1)x_1 + \cdots + (c_n - d_n)x_n$ *, so S is linearly indepeandent*

$$c_1 - d_1 = 0 \to c_1 = d_1$$

$$c_2 - d_2 = 0 \to c_2 = d_2$$

$$c_n - d_n = 0 \to c_n = d_n$$

Every vector in V can be written in only one form as a linear combination of S.

Theorem: **if S=$\{x_1, x_2, \dots, x_n\}$ *form abasis for vector space V and***

T=$\{y_1, y_2, \dots, y_m\}$ *form a set is linealy indepented , then* $n \geq m$

Proposition: **If**
S=$\{x_1, x_2, \dots, x_n\}$ *and* $T =$
$\{y_1, y_2, \dots, y_m\}$ *form basis for vector space V , then* $n = m$

Proof: since T is basis ,so T is linearly independent

&S is basis then by theorem n$\geq m$ … … … 1

And S is linearly independent & T is basis by theorem m$\geq n$ … … 2

From 1&2 we have n=m

Theorem :if S=$\{v_1, v_2, \dots, v_n\}$ is basis for V ,then any set contain (n) vectors is linearly dependent.

Example :is S basis of R^2; S={(1,1),(1,2),(0,1)}$\in R^2$

Dimension:

The number of vectors in a basis for V is called the dimension of V, denoted by Dim(V) or dim(V).

Example : 1- dim R=1 since natural basis ,S={1}

2-dimR^2 =2 ,,since S={(1,0),(0,1)}

3-dim R^3=3,since S={I, j, k}

4-dim (ℂ)=2, since ℂ =a+ib

5-dimR^n n, since S={$e_1, e_2, \dots, e_n$}

6-dim $M_{n\times n} = n \times n$

7-dim $P_2 = 3, since\ S = \{1, t, t^2\}$

8-dim $P_{n+1} = n + 1, since\ S = \{1, t, t^2, \dots, t^n\}$

Theorem: let V be a finite dim.v.s. with dim(V)=n, then

1-If U$\subseteq V\ is\ subspace\ of\ V, then\ dim(U) \subseteq dim(V)$
2-If V=span $\{V_1, V_2, \dots, V_n\}, then\ \{V_1, \dots, V_n\}\ basis\ for\ V$
3-If $V_1, \dots, V_n\}is\ linearly\ independent\ in\ V\ , then\ \{V_1, \dots, V_n\}\ is\ basis$
Of V

H.W: find dim of S={(1,2),(2,3)}$\in R^2$

EX: let W={(a, b, c, d);c=a-b, d=a+ b}

Show that W is a subspace of R^4 ? find dim(W)?

To find the basis of W?

Let a=1,b=0,c=1,d=1,then $X_1 = (1, 0, 1, 1)$

Let a=0,b=1,c=-1,d=1,then $X_2 = (0, 1, -1, 1)$

S={X_1, X_2},let (a,b,c,d)$\in W \exists c_1, c_2 \in R; (a, b, c, d) = c_1X_1 + c_2X_2$

$=(c_1, 0, c_1, c_1) + (0, c_2, -c_2, c_2) = (c_1, c_2, c_1 - c_2, c_1 + c_2)$

$c_1 = a, c_2 = b, c_1 - c_2 = c, c_1 + c_2 = d$,so S span W

let $c_1, c_2 \in R; (0, 0, 0, 0,) = c_1X_1 + c_2X_2$

$$=(c_1, c_2, c_1 - c_2, c_1 + c_2)$$

$$c_1 = c_2 = 0$$

So S is linearly independent ,dim(S)=2

Row rank ,vertical rank and the relationship between them.

تعريف : ليكن لدينا مصفوفة(المرتبة الصفية)عدد الصفوف المستقلة خطيا في المصفوفة

المرتبة العمودية :هي عدد الاعمدة المستقلة خطيا

ملاحظة:المرتبة الصفية =المرتبة العمودية +رتبة المصفوفة(Rank of A)

كيفية ايجاد رتبة المصفوفة)

1- نضع المصفوفة في الصيغة المميزة باستخدام عمليات الصفوف الابتدائية
2- الرتبة= عدد الصفوف غير الصفرية في المصفوفة(اذا كانت عليا او سفلى)
4- يسمى الفضاء الجزئي المنشأ من متجهات الصفوف بفضاء الصفوف للمصفوفة *A*
5- يسمى الفضاء الجزئي المنشأ من متجهات الاعمدة بفضاء الاعمدة للمصفوفة

Example :let A=$\begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{pmatrix}$find row vectors and column vectors

Sol. The row vectors :$r_1 = (2, 1, 0), r_2 = (3, -1, 4)$

The column vectors :$e_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

Theorem :the simple operations on rows do not change the row space of a matrix

Theorem: the non-zero row vectors in the characteristic row image of matrix form the basis of the row space of the matrix.

Example :find the basis of the row space of the vectors

V1=(1, -2, 0, 0, 3), V2= (2, -5, -3, -2, 6) , V3=(0 , 5, 15, 10, 0) ,
V4= (2, 6, 18, 8, 6)

$By\ charac\ \ teristic\ row\ we\ have\ W_1 = (1, -2, 0, 0, 3),$

$W_2 = (0, 1, 3, 2, 0), W_3 = (0, 0, 1, 1, 0)$ be basis of row space

$remark$:column space of A=row space of A^T so

$$basis\ of\ column\ space\ of\ A = basis\ of\ \ row\ space\ of\ A^T$$

Theorem: If A is a matrix, then the row space and column space have the same dimension

Definition: dimension of row space and column space of A is called rank (A).

<u>Gram Schmidt algorithm or (Gram – Schmidt orthogonalization procedure)</u>

Definition :let $V_1, V_2, \dots, V_n\ form\ abasis\ \ for\ subspace\ V\ is\ inner\ product$

Space V ,then to convert this basis to orthonormal basis to V ,apply the following steps:

Put $W_1 = V_1$

$$W_2 = V_2 - \frac{<V_2, W_1>}{\|W_1\|^2} . W_1$$

$$W_3 = V_3 - \frac{<V_3, W_1>}{\|W_1\|^2} . W_1 - \frac{<V_3, W_2>}{\|W_2\|^2} . W_2$$

$$W_n = V_n - \frac{<V_n, W_1>}{\|W_1\|^2} . W_1 - - \dots - \frac{<V_n, W_{n-1}>}{\|W_{n-1}\|^2} . W_{n-1}$$

$then\ \{\frac{W_1}{\|W_1\|}, \frac{W_2}{\|W_2\|}, \dots, \frac{W_n}{\|W_n\|}\}$ form orthonormal basis to V

$\{\boldsymbol{W_1}, \boldsymbol{W_2}, \dots, \boldsymbol{W_n}\}$ **is called orthogonal basis**

Examples: **let** **S=**$\{\boldsymbol{V_1 = (1,1,1), V_2 = (0,1,1), V_3 = (0,0,1)}\}$ $\boldsymbol{is\ a\ basis\ for\ R^3\ convert\ this\ basis\ to}$

$\boldsymbol{orthonormal\ basis}$

SOL .Let $W_1 = V_1$

$$W_2 = V_2 - \frac{<V_2,W_1>}{\|W_1\|^2} . W_1$$

$$= (0,1,1) - \tfrac{2}{3}(1,1,1) = (\tfrac{-2}{3}, \tfrac{1}{3}, \tfrac{1}{3}) = (\tfrac{-2}{3}, \tfrac{1}{3}, \tfrac{1}{3})$$

$$W_3 = V_3 - \frac{<V_3,W_1>}{\|W_1\|^2} W_1 - \frac{<V_3,W_2>}{\|W_2\|^2} W_2$$

$$= (0,0,1) - \tfrac{1}{3}(1,1,1) - \tfrac{1}{2}\left(\tfrac{-2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) = (\tfrac{-2}{9}, \tfrac{-4}{9}, \tfrac{2}{9})$$

The orthogonal basis is $\{\boldsymbol{W_1}, \boldsymbol{W_2}, \boldsymbol{W_3}\}$

The orthonormal basis is$\left\{\boldsymbol{\frac{W_1}{\|W_1\|}, \frac{W_2}{\|W_2\|}, \frac{W_3}{\|W_3\|}}\right\}$

H.W: convert this basis S=$\{\boldsymbol{V_1 = (1,1,1,1), V_2 = (1,2,4,5), V_3 = (1,-3,-4,-2)}\}\boldsymbol{of\ R^4\ to\ orthonormal\ basis}$

Definition :orthogonal set

A subset S=$\{\boldsymbol{v_1, v_2, \dots, v_n}\}$ **of a vector space V form an orthogonal set if all vectors in S are orthogonal to each other** $(\boldsymbol{u_i . u_j = 0\ for\ i \neq j}$**In addition if all vectors is an orthogonal set ,if S has length one i. e** $\boldsymbol{\| v_j \| = 1}$**,then S is called an orthonormal set.**

Example :let$P_2, which\ is\ orthonormal\ set$

1-$\boldsymbol{\frac{1}{\sqrt{2}}}X + \boldsymbol{\frac{1}{\sqrt{2}}}X^2$, X^2, is not orthonormal ,since $\boldsymbol{\frac{1}{\sqrt{2}}}. 1 = \boldsymbol{\frac{1}{\sqrt{2}}} \neq 0$

2-$\frac{2}{3}-\frac{2}{3}X+\frac{1}{3}X^2,\ \ \frac{2}{3}+\frac{1}{3}X-\frac{2}{3}X^2$

Pq=$\frac{2}{3}.\frac{2}{3}+\left(-\frac{2}{3}\right).\left(\frac{1}{3}\right)+\frac{1}{3}\left(-\frac{2}{3}\right)$ =0

$\sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}}$ =1 , $\sqrt{\frac{4}{9}+\frac{1}{9}+\frac{4}{9}}$ =1

so S is orthonormal

H.W let X=$\left(\frac{1}{\sqrt{5}},-\frac{1}{\sqrt{5}}\right), Y=\left(\frac{2}{\sqrt{30}},\frac{3}{\sqrt{30}}\right).$ ***show that*** $S=\{X_1,X_2\}$ **be orthogonal if the inner product**

$< U,V \geq 3u_1v_1+2u_2v_2$ **and is not exist if Euclidean inner product.**

Linear transformation on vector space:

Definition :let V and W are two vector spaces over field K .A linear transformation T from V into W is a mapping

T:V→W; 1- T(u+v)=T(u)+T(v)
2-T (αu)= αT(u) for all u,v∈V and α ∈K

If T:V → V

, ***then we say that T is a linear transformation on V***

Example: **show that T:**$R^3 \to R^2$ ***defined by***

T$(a_1,a_2,a_3)=(a_1+a_2,a_2-a_3)$

Is a linear transformation

Sol: **u=**$(a_1,a_2,a_3), v=(b_1,b_2,b_3)\in R^3$

Then u+ v=$(a_1+b_1,a_2+b_2,a_3+b_3)$

T(u+ v)=T$(\boldsymbol{a_1+b_1, a_2+b_2, a_3+b_3})$

=$(\boldsymbol{a_1+b_1+a_2+b_2, a_2+b_2-a_3-b_3})$

=$(\boldsymbol{a_1+a_2, a_2-a_3})+(\boldsymbol{b_1+b_2, b_2-b_3})$

=T(u)+T(v)

2- let $\boldsymbol{\alpha \in K, then\ \alpha u=(\alpha a_1, \alpha a_2, \alpha a_3)}$

T$(\boldsymbol{\alpha u})=(\boldsymbol{\alpha a_1+\alpha a_2, \alpha a_2-\alpha a_3})$

=$\boldsymbol{\alpha}(\boldsymbol{a_1+a_2, a_2-a_3})$

=$\boldsymbol{\alpha T(u)}$

So T is a linear transformation

Exercise :1-Let M $\in \boldsymbol{M_{m\times n}(K)\&N \subseteq M_{m\times n}(K)}$

Define T:$\boldsymbol{M_{m\times n}(K)\to M_{m\times n}(K) by T(A)=M.A.N}$

For all A$\subseteq M_{m\times n}(K)$, show that T is a linear transformation.

2- let T:$R^2\to R^2; T(x,y)=(y, x+2y, 2x-y)$,is T linear transformation.

Properties of linear transformation

Let T:V$\to \boldsymbol{W}$ be linear transformation;

1-T(0)=0 ;0$\in \boldsymbol{V}$

2-T(-v)=-T(v) $\forall \boldsymbol{v \in V}$

3-T(v-w)=T(v)-T(w) $\forall \boldsymbol{v, w \in V}$

Proof:

1-T(0)=T(0.v)=0 T(v)=0

2-T(-v)=T(-1.v)=-T(v)

3-T(v+(-v))=T(v+(-1)w)=T(v)-T(w)

Definition :composition of linear transformation

Let $T_1 : V \to U$ &$T_2 : U \to W$ be linear transformation so

$\boldsymbol{T_2 \circ T_1 : V \to W ; T_2 \circ T_1(v) = T_2(T_1(v); v \in V}$

Theorem :the composition transformation is linear transformation.

Proof :suppose $T_1 : V \to U, T_2 : U \to W$ be linear transformation ,then

1-$T_2 \circ T_1(v+u) = T_2[T_1(v+u)] =$

$T_2[T_1(v) + T_1(u)]$ =$T_2 \circ T_1(v) + T_2 \circ T_1(u)$

$2 - T_2 \circ T_1(kv) = T_2(T_1(kv) = T_2(kT_1(v)$

=k$\boldsymbol{T_2(T_1(v) = kT_2 \circ T_1(v)}$

$T_2 \circ T_1$ is linear transformation

Kernel and Rang of linear transformation

Definition :let T:V→ $\boldsymbol{W\ be\ linear\ transformation}$

Ker (T)={v∈ $V; \boldsymbol{T(v) = 0}$}

Definition rang of linear transformation :is the set of imag by T

Example: if T:V→ W be zero transformation,so

T:V→ $\vec{0}$,so ker T={V} ,rang T={$\vec{0}$}

Example :let S={v_1, v_2, v_3} ∈ R^3;v_1=(1,1,1),$v_2 = (1,1,0), v_3 = (1,0,1)$

Let T:$R^3 \to R^2$ be linear transformation ;T(v_1) $= (1,2)$

T(v_2) $= (3,-1), T(v_3) = (-1,5) find\ T(2,-3,1)$

Sol.:(2,-3,1)=$k_1(1,1,1) + k_2(1,1,0) + k_3(1,0,1)$

$$k_1 + k_2 + k_3 = 2 \ldots\ldots\ldots 1$$

$$k_1 + k_2 = -3 \ldots\ldots\ldots\ldots.. 2$$

$$k_1 + k_3 = 1 \qquad \dots\dots\dots.3$$

so $k_1 = 1, k_2 = -2, k_3 = 3$

Remark: rank T=rank A

2-ker T=solution of AX=0

Theorem :let A be m×n matrix ,so dimension of solution of

AX=0=n-rank A; n=number of column.

Matrices of linear transformation:

مصفوفة التحويل الخطي:

Let $e_1, e_2, \dots, e_n$ be standard basis ,let A be m× n matrix where it column T(e_1), $T(e_2)$, … $T(e_n)$

If T:$R^2 \to R^2$; $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_1 - x_2 \end{bmatrix}$

T(e_1) $= T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

T(e_2) $= T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$; A=$\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

In general :T(e_1) $= \begin{bmatrix} a_{11} \\ a_{21} \\ a_{n1} \end{bmatrix}, T(e_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{n2} \end{bmatrix}, \dots T(e_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ a_{nn} \end{bmatrix}$

Examples: Let the matrix A ;find linear transformation of A,where

T:$R^3 \to R^2$ $where\ A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 4 & 6 \end{bmatrix}$

T($\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + x_3 \\ 3x_1 + 4x_2 + 6x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-2\\4\end{bmatrix} \quad \rightarrow \quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\6\end{bmatrix}$$

2- *Matrices of linear transformation to the basis B,B*

Let V be space dim is n and basis B=$\{u_1, u_2, \dots, u_n\}$

Let W be space dim is m and basis $B^{\backslash}=\{v_1, v_2, \dots, v_n\}$

Let x$\in \vec{V} \rightarrow [x]_n \in R^n \& [T(x)]_B$ matrix ,the matrix of linear transformation is vector in R^m,for each $[x]_B$ $in\ R^n has\ image\ in\ [T(x)]_B\ in\ R^m$

This transformation is linear transformation by the matrix A;A$[x]_{B=}[T(x)_B$,

where A=$\begin{bmatrix}a_{11} & a_{12} & a_{1n}\\ a_{21} & a_{22} & a_{2n}\\ a_{n1} & a_{n2} & a_{nn}\end{bmatrix}$

So the matrix of linear transformation is

A=$\left[T(_{u1)}{}_{B}, T(u_{2)}{}_{B}, \dots, T(u_{n)}{}_{B}\right]$

Example :let T:$P_2 \rightarrow P_4 be\ linear\ transformation$;

T(P(x))=$x^2P(x)$,find matrix transformation T to the two basis B=$\{p_1, p_2, p_3\}$, $B^{\backslash}=\{p_1^{\backslash}, p_2^{\backslash}, p_3^{\backslash}\}$, where

$p_1=1+x^2, p_2=1+2x+3x^2, p_3=4+5x+x^2$

$p_1^{\backslash}=1, p_2^{\backslash}=x, p_3^{\backslash}=x^2, p_3^{\backslash}=x^4$

$sol. T(p_1)=x^2(1+x^2)=x^2+x^4$

$T(p_2)=x^2(1+2x+3x^2)=x^2+2x^3+3x^4$

$T(p_3)=x^2(4+5x+x^2)=4x^2+5x^3+x^4$

وبذلك تكون مصفوفة الاحداثيات

T(p_1) , $T(p_2)$. $T(p_3)$ B هو الاساس لها

$[T(p_{1)}]_{B\backslash}$=$\begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix}$, $[T(p_1)]_{B\backslash} = \begin{bmatrix}0\\0\\1\\2\\3\end{bmatrix}$, $[T(p_3)_{B}$= $\begin{bmatrix}0\\0\\4\\5\\1\end{bmatrix}$

$$\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\1 & 1 & 4\end{bmatrix}$$

$$\begin{bmatrix}0 & 2 & 5\\1 & 3 & 1\\ & & \end{bmatrix}$$

Chapter 3

Eigen values and Eigen vectors

Definition :let A be an n$\times$ n matrix ,if there is a scalar $\lambda \in \mathbb{C}$,if there exists a nonzero vector x ;A x=λx,then we say that λ is an eigenvalue for A ,and x is called an eigen vector for A with eigenvalue λ

Remark

the characteristic equation of a square matrix A is the equation $|A - \lambda I| = 0$

***Theorem*: the eigenvalue of a square matrix A are the solution of the characteristic equation $|A - \lambda I| = 0$**

Examples : find the eigenvalue and the eigenvectors of the matrix A=$\begin{bmatrix}2 & 3\\3 & -6\end{bmatrix}$

We have to find A-λI

A-$\lambda I = \begin{pmatrix}2 & 3\\3 & -6\end{pmatrix} - \lambda\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}2-\lambda & 3\\3 & -6-\lambda\end{pmatrix}$

Now to find the solution of characteristic equation

$|A-\lambda I|=0\,, SO(2-\lambda)(-6-\lambda)-9=0$

$\lambda^2+4\lambda-21=0$

$(\lambda+7)(\lambda-3)=0\,, so\ \lambda=-7\ or\ \lambda=3$

To fined the eigenvector x if $\lambda=-7$,to solve the following system (A-λI))$x=0$

$(\begin{pmatrix}2 & 3\\3 & -6\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix})\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

$\begin{pmatrix}2-(-7) & 3\\3 & -6-(-7)\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

$\begin{pmatrix}9 & 3\\3 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

By multiply the first row $\frac{1}{9}$ we have $\begin{pmatrix}1 & \frac{1}{3}\\3 & 1\end{pmatrix}$

Multiply the first row (-3) and addition to the second row

We have $\begin{pmatrix}1 & \frac{1}{3}\\0 & 0\end{pmatrix}$,now we can solve

$\begin{pmatrix}1 & \frac{1}{3}\\0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

$x_1+\frac{1}{3}x_2=0\,, so\ x_1=-\frac{1}{3}x_2$

$assume\ \ x_2=c_1, gives\ x=\begin{pmatrix}\frac{1}{3}c_1\\c_1\end{pmatrix}=c_1\begin{pmatrix}\frac{1}{3}\\1\end{pmatrix}\ ; c_1\in R$

Similarly ,we show that the eigenvector of $\lambda_2=3\ is$

X=$c_2\begin{pmatrix}3\\1\end{pmatrix}\ ; c_2\in R$

H.W find the eigenvalue and the eigenvectors of the matrix A=$\begin{pmatrix}2 & 3\\4 & 3\end{pmatrix}$