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محاضرات الجبر الخطي

المرحلة الثانية

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Cross Product:

If U, V are non-zero vectors in R^3 , $U = (u_1, u_2, u_3)$, $V = (v_1, v_2, v_3)$, then the cross product of U & V is the

$$U \times V = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \in R^3$$
$$= (u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1)$$

Which mean $U \times V = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = i(u_2 v_3 - u_3 v_2) - j(u_1 v_3 - u_3 v_1) + k(u_1 v_2 - u_2 v_1)$

NOTE: the cross product is define only in R^3

Theorem: the vector $U \times V$ is orthogonal to both U & V

Theorem: if θ is the angle between U & V then

$$\| U \times V \| = \| U \| \| V \| \sin \theta$$

Corollary: the two non-zero vectors U & V are parallel if and only if $U \times V = 0$

Example:

Let $U = (1, 2, -2)$, $V = (3, 0, 1)$ find $U \times V$

$$U \times V = \left(\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \right) = (2, -7, -6)$$

Example

Find $U \times V$, where $U = (3, -4, 1)$ & $V = (5, 2, -6)$, then find the angle a form U & V .

Theorem :if u, v & w are vectors in R^3 and k is scalar then:

1- $U \times V = -(V \times U)$

2- $U \times (V + W) = U \times V + U \times W$

3- $(U+V) \times W = U \times W + V \times W$

4- $k(U \times V) = (kU) \times V = U \times (kV)$

5- $U \times 0 = 0 \times U = 0$

6- $U \times U = 0$

Prooffe:1- $U \times V = (u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1)$

$V \times U = (v_2u_3 - v_3u_2, -(v_1u_3 - v_3u_1), v_1u_2 - v_2u_1)$

Area the cross product

The length of the cross product $U \times V$ is equal to the area of the parallelogram determined by U & V

Area of the parallelogram $= \| U \times V \| = \| U \| \| V \| \sin \theta$

Area of the triangle $= \frac{1}{2} \| U \times V \|^2$

Volume of parallelogram $= \| A \cdot (B \times C) \|^2$

Where A =height, B =length, C = width

Example

Find the area of the triangle determined by $p_1 = (2,2,0), p_2 = (-1,0,2), p_3 = (0,4,3)$

Sol. $P_1P_2 = U = (-1,0,2) - (2,2,0) = (-3,-2,2)$

$P_1P_3 = V = (0,4,3) - (2,2,0) = (-2,2,3)$

$U \times V = (-10, 5, 10), \| U \times V \| = \sqrt{(-10)^2 + 5^2 + 10^2} = \sqrt{225} = 15$

Area $= \frac{1}{2} \| U \times V \| = \frac{1}{2} 15 = 7.5$

NOTE: 1- $U \cdot V$ is constant

2- $U \times V$ is a vector

3-the vector $U \times V$ is perpendicularly to both U & V

4-The vector $i=(1,0,0), j=(0,1,0), k=(0,0,1)$

EXAMPLE:

Find the volume of the parallelogram that generated by $A=(2,1,3), B=(-1,3,2), C=(1,1,-3)$

Sol. $V = \| A \cdot (B \times C) \|$

$$B \times C = \begin{vmatrix} i & j & k \\ -1 & 3 & 2 \\ 1 & 1 & -3 \end{vmatrix} = -11i - j - 4k$$

$$A \cdot (-11i - j - 4k) = -22 - 2 - 12 = -36$$

$$\| A \cdot E \| = \| -36 \| = 36$$

EXAMPLE

Find the volume of parallelepiped, where

$$A=(2,1,3), B=(1,2,0), C=(3,0,1)$$