بسم الله الرحمن الرحيم

Textbooks.

The recommended refrences text for the course are:

1- J. R. Holton: An Introduction to dynamic Meteorology 3rd Edition (1992) by Academic press. Note that is a 4th addition available, dated 2004.

2- A. E. Gill: Atmosphere - Ocean Dynamics (1982) by Academic press.

3- WMO Report: compendium of meteorology Vol. 1 part 1 Dynamic Meteorology No: 364.

4- J. T. Houghton: The Physics of Atmospheres 2nd Edition (1986) by Cambridge Univ. press.

5- Haltiner & Martin: Dynamical & physical Meteorology (1957).

Review of Vectors.

Some physical quantities such as Temperature, time and mass, can be described by a single number. Because these quantities are describable by giving only a magnitude, they are called "scalars".

on the other hand physical quantities such as wind velocity, displacement, force and acceleration require both a magnitude and a direction to completely describe them. such quantities are called "Vectors".

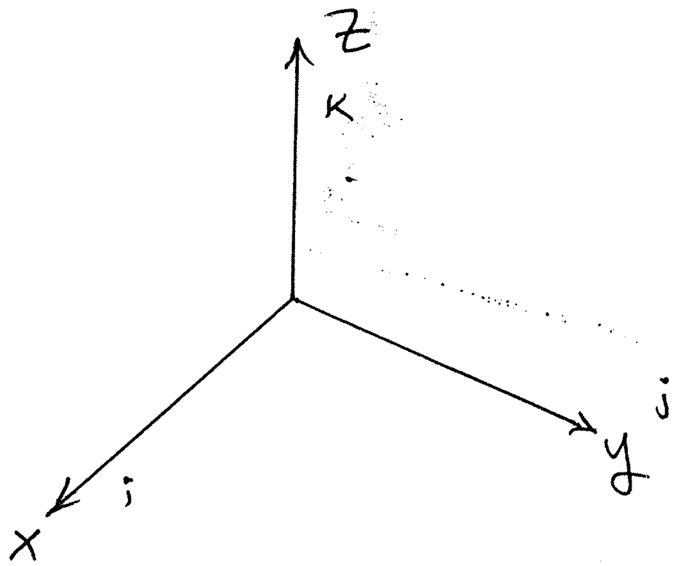
* Vectors have both a magnitude and a direction.
* Vectors are denoted by placing an arrow over the top ($\vec{A}$).
* The magnitude of a vector is denoted either by A or $|\vec{A}|$.

* An vector can be written in terms of components along the coordinate system axes as:

$$\vec{A} = A_x i + A_y j + A_z k .$$

i, j, k are unit vectors along the x, y and z respectively, the magnitude of unit vectors is one.

In meteorology we use the following coordinate system:



The x - coordinate increases eastward.
The y - coordinate increases northward.
The z - coordinate increases upward.

the Velocity along each coordinate direction are defined as:

$u = \frac{dx}{dt}$; u is the speed in eastward direction (zonal wind).

$v = \frac{dy}{dt}$; v is the speed in northward direction (meridional wind).

$w = \frac{dz}{dt}$; w is the speed in upward direction (Vertical wind).

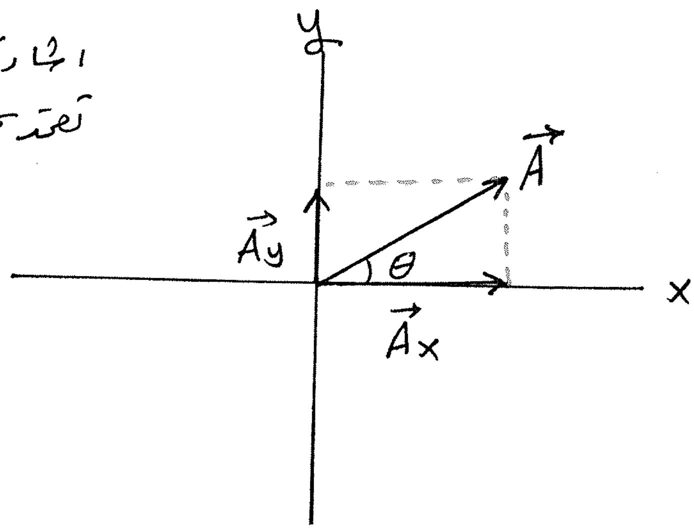
Vector Decomposition:

We can decompose a vector into component vectors along each coordinate axis.

A vector $\vec{A}$ can be decomposed into the vectors.

اشارة المركبتين A_x و A_y
تعتمد على الزاوية θ

A_x − A_y +	A_x + A_y +
A_x − A_y −	A_x + A_y −



$$\cos\theta = \frac{A_x}{A} \Longrightarrow A_x = A\cos\theta.$$

$$\sin\theta = \frac{A_y}{A} \Longrightarrow A_y = A\sin\theta.$$

* The magnitude of $\vec{A}$ is:

$$A = \sqrt{(A_x)^2 + (A_y)^2}$$

* The direction of vector $\vec{A}$ is:

$$\theta = \tan^{-1}\frac{A_y}{A_x}$$

Atmospheric Dynamic: the branch of meteorology dealing with study of the causes and nature of motion of the atmosphere on all scales.

The motion in the atmosphere is governed by a set of equations, Known as the Navier-stokes equations. These equations, Solved numerically by Computers, are used to produce our weather forecasts.

The scales of atmospheric motion are:

	Scale	Time	Distance	Example
1-	planetary Scale	Weeks, or longer	1000 to 40000 Km	trade wind.
2-	Synoptic Scale	Days to weeks	100 to 5000 Km	cyclones
3-	Meso scale	Minutes to hours	1 to 100 Km	Tornado
4-	Microscale	second to minutes	< 1 Km	Turbulence

in order to describe the dynamical behavior of the atmosphere, we treat it as a fluid. The circulation of planet's atmosphere is governed by four basic principles:

1- Newton's law of motion. "second law).
2- conservation of energy. "first law of thermodynamics"
3- conservation of mass. "equation of continuity"
4- the equation of state.

The basics of the equation of motion for the atmosphere is Newton's second law.

$$ma = \sum_{i=1}^{n} F_i$$

Newton's second law is applicable in a so-called inertial system, is a system which is remains fixed relative to the stars.
velocity and acceleration measured in the inertial system are called the absolute velocity V_o and absolute acceleration a_o

We must modify Newton's second law in such away that it applies to a system which is fixed relative to the rotating earth.

$$a = \sum_{i=1}^{n} \frac{F_i}{m}$$

$$\left(\frac{d\vec{V_a}}{dt} \right)_{\substack{fix. \\ sys.}} = \sum_{i=1}^{n} \vec{f_i}$$

where $\vec{f_i}$ is the force per unit mass.

The Total Derivative

metarorological variables such as $P, T, \vec{V}$, can Vary both in space and time, function (x, y, z, t).

Using Taylor series expansion for Temprerature.

$$\Rightarrow dT = \frac{\partial T}{\partial t} dt + \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz.$$

Dividing by dt

$$\frac{dT}{dt} = \frac{\partial T}{\partial t}\frac{dt}{dt} + \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$$

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

where $\frac{dx}{dt} = u$, $\frac{dy}{dt} = v$ and $\frac{dz}{dt} = w$.

which can also be written as

$$\boxed{\frac{dT}{dt} = \frac{\partial T}{\partial t} + \vec{V} \cdot \vec{\nabla} T}$$

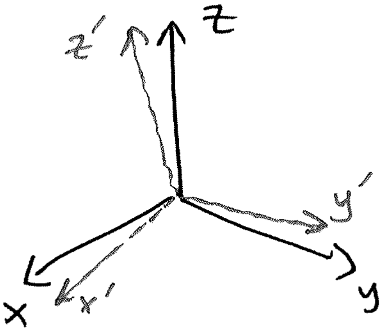
This shows that, the partial derivative is not equal to the total derivative.

Relation between fixed and Rotating system.

$\vec{A}$ is a vector in a fixed system

$$\vec{A} = A_x i + A_y j + A_z K \quad \text{(fixed system)}.$$

$$\vec{A}^{*} = A_x' i' + A_y' j' + A_z' K' \quad \text{(Rotated system)}.$$



$$\vec{A} = A_x i + A_y j + A_z K = A_x' i' + A_y' j' + A_z' K'.$$

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt} i + \frac{dA_y}{dt} j + \frac{dA_z}{dt} K = \frac{dA_x'}{dt} i' + \frac{dA_y'}{dt} j' + \frac{dA_z'}{dt} K + A_x' \frac{di'}{dt} + A_y' \frac{dj'}{dt} + A_z' \frac{dK'}{dt}.$$

for any vector

$$\frac{d}{dt}(\text{any vector}) = \vec{\Omega} \times (\text{the vector})$$

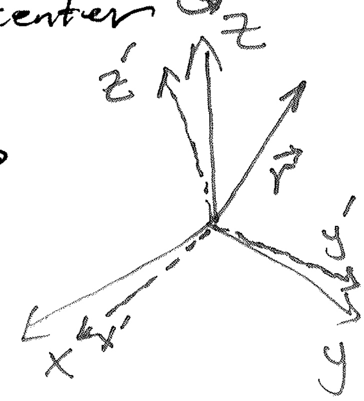
$$\frac{di'}{dt} = \vec{\Omega} \times i', \quad \frac{dj'}{dt} = \vec{\Omega} \times j', \quad \frac{dK'}{dt} = \vec{\Omega} \times K'$$

Thus
$$\frac{dA_x}{dt} i + \frac{dA_y}{dt} j + \frac{dA_z}{dt} K = \frac{dA_x'}{dt} i' + \frac{dA_y'}{dt} j' + \frac{dA_z'}{dt} K + A_x'(\vec{\Omega} \times i') + A_y'(\vec{\Omega} \times j') + A_z'(\vec{\Omega} \times K')$$

$$\left(\frac{d\vec{A}}{dt}\right)_{\substack{\text{fix.}\\ \text{sys}}} = \left(\frac{d\vec{A}}{dt}\right)_{\substack{\text{rotg}\\ \text{sys}}} + \vec{\Omega} \times \vec{A}$$

Equation of motion in the Absolute and Rotating Coordinates.

let $\vec{r}$ is position vector from origin (center of earth)



Recall that : $\left(\frac{d\vec{A}}{dt}\right)_{\substack{fix\\sys}} = \left(\frac{d\vec{A}}{dt}\right)_{\substack{rota.\\syst}} + \vec{\Omega} \times \vec{A}$

$$\left(\frac{d\vec{r}}{dt}\right)_{\substack{fix.\\sys.}} = \left(\frac{d\vec{r}}{dt}\right)_{\substack{rota.\\sys.}} + \vec{\Omega} \times \vec{r}$$

$$\boxed{\vec{V}_a = \vec{V}_r + \vec{\Omega} \times \vec{r}}$$

$$\left(\frac{d\vec{V}_a}{dt}\right)_{\substack{fix\\sys}} = \left(\frac{dV_a}{dt}\right)_{\substack{rota\\sys}} + \vec{\Omega} \times \vec{V}_a$$

$$\left(\frac{d\vec{V}_a}{dt}\right)_{\substack{fix\\sys}} = \frac{d}{dt}\left[\vec{V}_r + \vec{\Omega} \times \vec{r}\right]_{\substack{rota\\sys}} + \vec{\Omega} \times \left[\vec{V}_r + \vec{\Omega} \times \vec{r}\right]$$

$$\left(\frac{d\vec{V}_a}{dt}\right)_{\substack{fix\\sys}} = \left(\frac{d\vec{V}_r}{dt}\right)_{\substack{rota\\sys}} + \frac{d}{dt}\left(\vec{\Omega} \times \vec{r}\right)_{\substack{rota\\sys}} + \vec{\Omega} \times \left[\vec{V}_r + \vec{\Omega} \times \vec{r}\right]$$

$$\left(\frac{dV_a}{dt}\right)_{\substack{fix\\sys}} = \left(\frac{d\vec{V}_r}{dt}\right)_{\substack{rota\\sys}} + \frac{d}{dt}\underline{\left(\vec{\Omega} \times \vec{r}\right)} + \vec{\Omega} \times \vec{V}_r + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)$$

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$$\therefore \underbrace{\left(\frac{d\vec{V}_a}{dt}\right)}_{\text{fix sys}} = \underbrace{\left(\frac{d\vec{V}_r}{dt}\right)}_{\text{rota sys}} + 2(\vec{\Omega} \times \vec{V}_r) - \vec{\Omega}^2 \vec{r}$$

Apply Newton's second law.

$$\left(\frac{d\vec{V}_a}{dt}\right) = \left(\frac{d\vec{V}_r}{dt}\right) + 2(\vec{\Omega} \times \vec{V}_r) - \vec{\Omega}^2 \vec{r} = \sum f_i$$

$$\boxed{\therefore \underbrace{\frac{d\vec{V}_a}{dt}}_{\text{Abs. Accel}} = \underbrace{\frac{d\vec{V}}{dt}}_{\text{Rel. Accel}} + \underbrace{2(\vec{\Omega} \times \vec{V})}_{\text{Coriolis Accel}} - \underbrace{\vec{\Omega}^2 \vec{R}}_{\text{Centripetal Accel}} = \underbrace{\sum f_i}_{\text{Net real forces}}}$$

The forces on the right hand side of equation are:

1- Gravity $\vec{g}$ 2- PGF $-\frac{1}{\rho}\vec{\nabla} P$ 3- Friction $f_{friction}$.

$$\frac{d\vec{V}}{dt} + 2(\vec{\Omega} \times \vec{V}) - \vec{\Omega}^2 \vec{R} = \vec{g} - \frac{1}{\rho}\vec{\nabla} P + \vec{f}$$

$$\boxed{\underbrace{\frac{d\vec{V}}{dt}}_{\text{Rela. accel.}} = \underbrace{-\frac{1}{\rho}\vec{\nabla} P + \vec{g} + \vec{f}}_{\text{Realforce}} \underbrace{- 2(\vec{\Omega} \times \vec{V}) + \vec{\Omega}^2 \vec{R}}_{\text{apparent forces due to earth rotation}}}$$

This is THE EQUATION OF MOTION used in meteorology - it is the back bone of atmosphere science.

$$\left(\frac{d\vec{V_a}}{dt}\right)_{\substack{fix\\sys}} = \left(\frac{d\vec{V_r}}{dt}\right)_{\substack{rota\\sys}} + \left(\frac{d\Omega}{dt} \times r\right)_{\substack{rota\\sys}} + \left(\vec{\Omega} \times \frac{d\vec{r}}{dt}\right)_{\substack{rota\\sys}} + \vec{\Omega} \times \vec{V_r} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

$$\left(\frac{d\vec{\Omega}}{dt} \times \vec{r}\right)_{\substack{rota\\sys}} = \text{Zero}$$ --- show that.

Thus :

$$\left(\frac{d\vec{V_a}}{dt}\right)_{\substack{fix\\sys}} = \left(\frac{d\vec{V_r}}{dt}\right)_{\substack{rota\\sys}} + 0 + \left(\vec{\Omega} \times \frac{d\vec{r}}{dt}\right) + \vec{\Omega} \times \vec{V_r} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

Recall that $\vec{V_r} = \dfrac{d\vec{r}}{dt}$

$$\left(\frac{d\vec{V_a}}{dt}\right)_{\substack{fix\\sys}} = \left(\frac{d\vec{V_r}}{dt}\right)_{\substack{rota\\sys}} + 2\left(\vec{\Omega} \times \vec{V_r}\right) + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

Recall that $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

Thus $\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = (\vec{\Omega} \cdot \vec{r})\vec{\Omega} - (\vec{\Omega} \cdot \vec{\Omega})\vec{r}$

where $\vec{\Omega} \perp \vec{r} \Longrightarrow \boxed{\vec{\Omega} \cdot \vec{r} = 0}$ show that

$$\therefore \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = 0 - (\vec{\Omega} \cdot \vec{\Omega})\vec{r}$$

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = -\Omega^2 \vec{r}$$

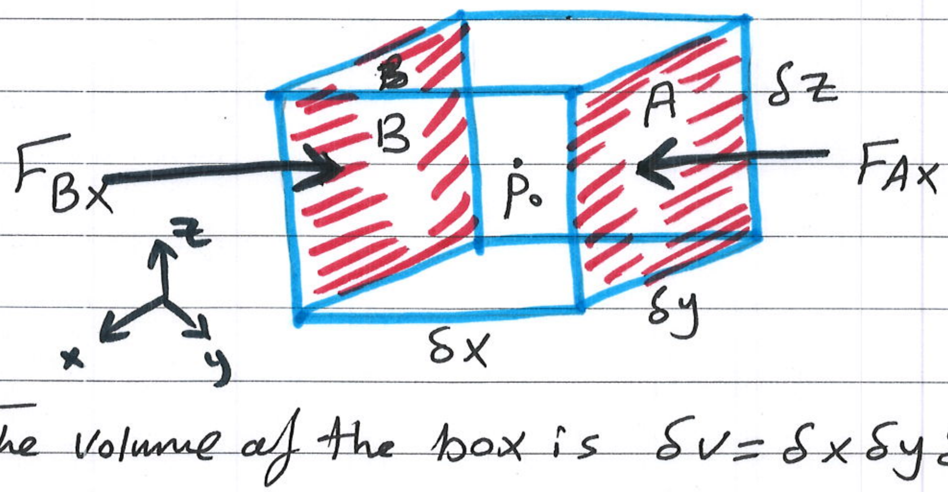
$$\frac{d\vec{V}}{dt} = PGF + \underset{\text{force}}{\text{Gravitional}} + \underset{\text{"friction"}}{\text{viscous}} - 2\vec{\Omega}\times\vec{V} + \vec{\Omega}^2\vec{R}$$

The Forces in the Equation of motion:

1/ The pressure gradient force.

The pressure force is due to the variation of the atmospheric pressure from point to point. The pressure at a given point and a given time is independent of direction.

Consider the air parcel like the box in the fig. the pressure at the center of box is p_0.



The volume of the box is $\delta V = \delta x \delta y \delta z$

The pressure at the surface A is

$$P_{Ax} = p_0 + \frac{\partial P}{\partial x}\left(\frac{\delta x}{2}\right).$$

The Force at the surface A is:

$$\vec{F}_{Ax} = -\left[P_o + \frac{\partial P}{\partial x}\left(\frac{\delta x}{2}\right)\right]\delta y \delta z\, i$$

The pressure at the surface B is:

$$P_{Bx} = P_o - \frac{\partial P}{\partial x}\left(\frac{\delta x}{2}\right)$$

The force at the surface B is

$$\vec{F}_{Bx} = \left[P_o - \frac{\partial P}{\partial x}\left(\frac{\delta x}{2}\right)\right]\delta y \delta z\, i$$

the net pressure force in the x-direction is

$$\vec{F}_x = \vec{F}_{Bx} - \vec{F}_{Ax}$$

$$\vec{F}_x = \left[P_o - \frac{\partial P}{\partial x}\left(\frac{\delta x}{2}\right)\right]\delta y \delta z\, i - \left[P_o + \frac{\partial P}{\partial x}\left(\frac{\delta x}{2}\right)\right]\delta y \delta z\, i$$

$$\vec{F}_x = -2\frac{\partial P}{\partial x}\left(\frac{\delta x}{2}\right)\delta y \delta z\, i$$

$$\vec{F}_x = -\frac{\partial P}{\partial x}\,\delta V\, i$$

$$\delta V = \delta x \delta y \delta z$$

since $\delta V = \frac{m}{\rho}$ where m is mass and ρ is density.

$$\vec{F}_x = -\frac{\partial p}{\partial x}\,\delta V i = -\frac{m}{\rho}\frac{\partial p}{\partial x}\,i \qquad \div m$$

$$\vec{f}_x = \frac{\vec{F}_x}{m} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\,i \qquad x\text{-direction.}$$

Simillarly for y and z directions

$$\vec{f}_y = -\frac{1}{\rho}\frac{\partial p}{\partial y}\,j. \qquad y\text{-direction}$$

$$\vec{f}_z = -\frac{1}{\rho}\frac{\partial p}{\partial z}\,k. \qquad z\text{-direction.}$$

The net total of force per unit mass:

$$\vec{f} = \vec{f}_x + \vec{f}_y + \vec{f}_z$$

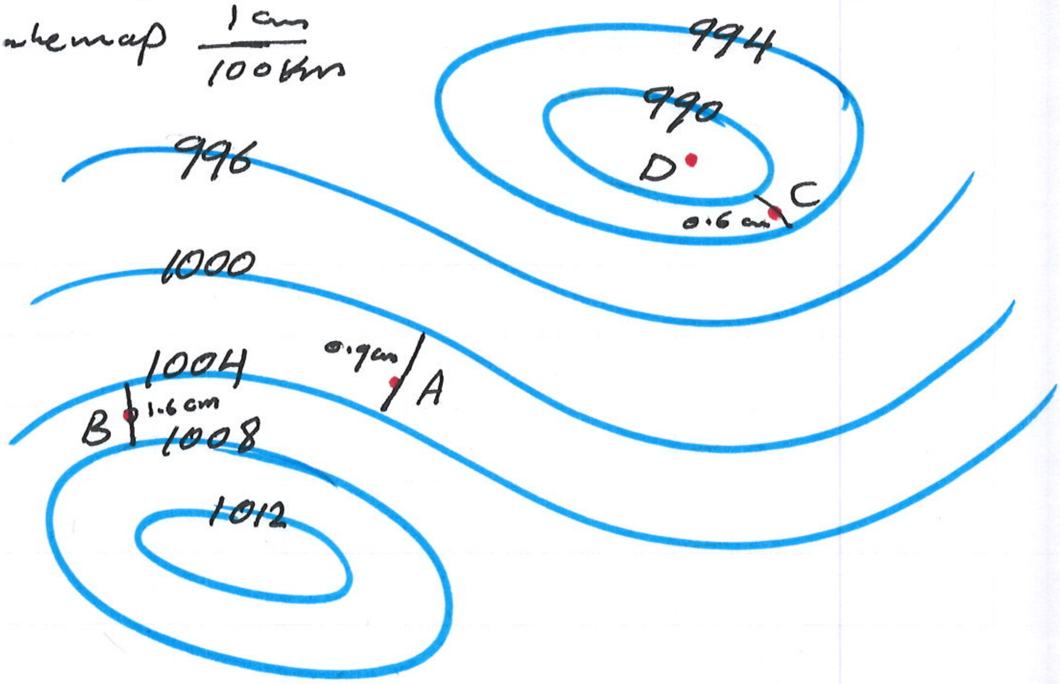
$$\vec{f} = -\frac{1}{\rho}\left[\frac{\partial p}{\partial x}i + \frac{\partial p}{\partial y}j + \frac{\partial p}{\partial z}k\right].$$

$$\boxed{\vec{f} = -\frac{1}{\rho}\vec{\nabla}p = -\frac{1}{\rho}\cdot \text{grad}\,p.}$$

The direction of the pressure gradient force $\vec{f}$ is in the opposite direction of the pressure gradient $\vec{\nabla}p$, and $\nabla p \approx \frac{\Delta p}{\Delta n}$ where Δp is the contour interval for the isobars and Δn is the horizontal distance between the isobars.

At the four points shown in the picture below, estimate the magnitude of the acceleration due to the pressure gradient force. Assume a density of 1.23 Kg/m^3. The isobars are labeled in mb.

Scale map $\frac{1\,cm}{100\,km}$



$$\vec{f} = -\frac{1}{\rho}\vec{\nabla}P$$

$$|\vec{f}| = \frac{1}{\rho}\nabla P = \frac{1}{\rho}\frac{\Delta P}{\Delta n}$$

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__point B:__

$$\Delta P = 1008 - 1004 = 4\,mb = 4\,hPa$$
$$= 4\times10^2\,Pa = 4\times10^2\,N/m^2$$
$$\Delta n = 1.6\,cm \times \frac{100\,Km}{1\,cm} = 160\,Km$$
$$\Delta n = 160\times10^3\,m$$

$$|\vec{f}| = \frac{1}{1.23}\,\frac{4\times10^2}{160\times10^3} = 0.002\ m/sec^2$$

point A :

$$|\vec{f}| = \frac{1}{\rho}\frac{\Delta p}{\Delta n}$$

$$\Delta p = 1004 - 1000 = 4 mb = 4 \times 10^{2} N/m^{2}$$

$$\Delta n = 0.9 cm \times \frac{100 Km}{1 cm} = 90 Km$$

$$|\vec{f}| = \frac{1}{1.23} \times \frac{4 \times 10^{2}}{90 \times 10^{3}} = 0.0036 m/sec^{2}$$

point C :

$$\Delta p = 994 - 990 = 4 mb = 4 \times 10^{2} N/m^{2}$$

$$\Delta n = 0.6 cm \times \frac{100 Km}{1} = 60 Km$$

$$|\vec{f}| = \frac{1}{1.23} \times \frac{4 \times 10^{2}}{60 \times 10^{3}} = 0.0054 m/se^{2}$$

point D

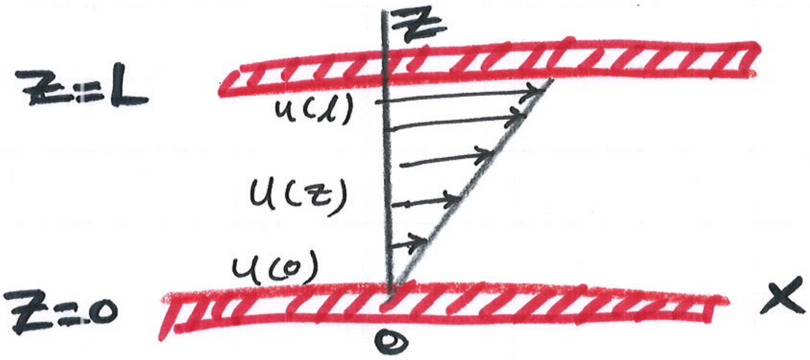
$$|\vec{f}| = \text{Zero}.$$

because $\Delta p = \text{Zero}$.

2- The viscous Force.

viscous force is due to friction caused by interactions of molecules of a fluid.

IF we consider a layer of incompressible fluid between two horizontal plates separated by a distance (L) as shown in the fig.



The layer in contact with the upper and lower plates will moved as the following:

at $z=L$ the fluid moves at speed $u(L)=u_0$ and at $z=0$ the fluid is motionless $u(0)=0$.

$$F \propto \frac{A u_0}{L}$$

where u_0 is peak speed, A is area and L is depth.

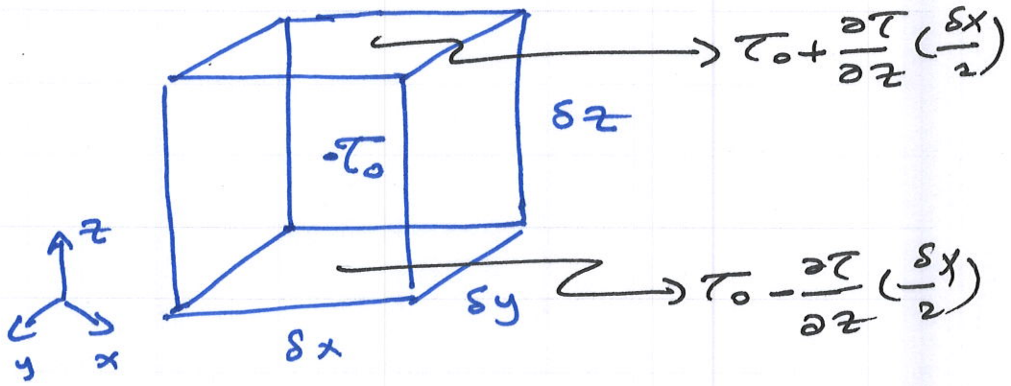
$$\therefore F = \mu \frac{A u_0}{L}$$

where μ dynamical viscosity coefficient.

The shearing stress $\tau = \frac{F}{A}$

Thus $\tau_x = \frac{F_x}{A} = \mu \frac{u_0}{L} = \mu \frac{\partial u}{\partial z}$

From the fig. along the top of the cube the force in x-direction is: $\tau_0 + \frac{\partial \tau}{\partial z} \left(\frac{\delta x}{2} \right)$



on the bottom face of the cube, the force in the x-direction is: $\tau_0 - \frac{\partial \tau}{\partial z} \left(\frac{\delta x}{2} \right)$

The net force is: $\vec{F}_x = \left[\sum \vec{\tau}_x \right] A$

$$\vec{F}_x = \left\{ \left[\tau_0 - \frac{\partial \tau}{\partial z} \left(\frac{\delta x}{2} \right) \right] - \left[\tau_0 + \frac{\partial \tau}{\partial z} \left(\frac{\delta x}{2} \right) \right] \right\} \delta y \, \delta z \, i$$

$$\vec{F}_x = - \frac{\partial \tau}{\partial z} \delta x \, \delta y \, \delta z$$

$$\vec{F}_x = -\frac{\partial \tau}{\partial z}\,\delta v\, i$$

$$\delta v = \frac{m}{\rho} \Rightarrow \vec{F}_x = -\frac{\partial \tau}{\partial z}\frac{m}{\rho}\, i$$

$$\therefore \frac{\vec{F}_x}{m} = \vec{f}_x = -\frac{1}{\rho}\frac{\partial \tau}{\partial z}\, i$$

$$\vec{f}_x = -\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) i \qquad \text{x-direction}$$

H.W $$\vec{f}_y = -\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) j \qquad \text{y-direction.}$$

H.W $$\vec{f}_z = -\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right) k \qquad \text{z-direction}$$

The total force per unit mass:

$$\vec{f} = \vec{f}_x + \vec{f}_y + \vec{f}_z$$

$$\vec{f} = -\frac{1}{\rho}\left[\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)i + \frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)j + \frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right)k\right]$$

$$\vec{f} = -\frac{\mu}{\rho}\left[\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)i + \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}\right)j + \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial z}\right)k\right].$$

$$\vec{f} = -\frac{\mu}{\rho}\frac{\partial}{\partial z}\left[\left(\frac{\partial u}{\partial z}\right)i + \left(\frac{\partial v}{\partial z}\right)j + \frac{\partial w}{\partial z}k\right].$$

$$\vec{f} = -\frac{\mu}{\rho}\frac{\partial}{\partial z}\left[\frac{\partial \vec{V}}{\partial z}\right]. \qquad \text{where } \vec{V} = ui + vj + wk.$$

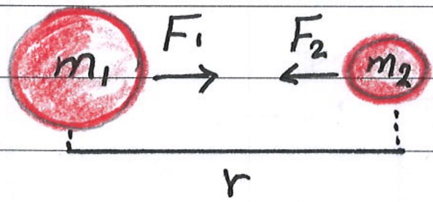
The acceleration due to the viscous force can be written as:

$$\boxed{\vec{f} = -\frac{1}{\rho}\frac{\partial \tau}{\partial z}}$$

The viscosity of the atmosphere is small, so we can ignore it.

3) The Gravitional force.

Newton's Law of universal gravitation states that "any two element of mass in the universe attracts each other with a force proportional to their masses and inversely proportional to the square of the distance between them".



Newton's Law can be written as a vector equation:

$$\vec{g} = -G \frac{m_1 m_2}{r^3} \vec{r}$$

where $\vec{g}$ attraction of m_1 on m_2 (force of gravition)

$\vec{r}$ is vector from m_1 to m_2

G is universal gravitional constant $= 6.66 \times 10^{-11}\ N m^2 kg^{-2}$

IF we assume $m_2 = 1$ kg

$$\therefore \vec{g} = -G \frac{m_1}{r^3} \vec{r}$$

IF $m_1 = M$ ' total mass of earth is equal to 5.988×10^{24} Kg.

The acceleration due to the gravitational force at the surface of earth ($r = a = 6378$ Km) is

$$\vec{g}_* = -G \frac{M}{a^2} \vec{r}$$

At some altitude Z above the surface of the earth, the acceleration due to the gravitational force is:

$$\vec{g}_* = -G \frac{M}{(a+z)^2} \vec{r}$$

* $\vec{r}$ is the vector from the center of earth to partical in the atmosphere.

* $\vec{g}_*$ is directed toward the center of earth.

4/ Centrifugal force.

The earth is no inertial system, but a rotating system; therfore Centrifugal and Coriolis force occur.

The term $\Omega^2 \vec{R}$ represents the centrifugal acceleration due to the earth's rotation, the centrifugal force is always directed away from the axis of rotation. The centrifugal force is combined with the gravitational force to define a new force called GRAVITY.

The acceleration due to gravity force is defined as:

$$\vec{g} = \vec{g}_* + \Omega^2 \vec{R}$$

when you see gravity in an equation of motion, Keep in mind that it is a combination of the gravitational acceleration plus centrifugal accele.

The gravity in an equation of motion has a plus sign, but Keep in mind that if written in component from it lies solely in the negative K-direc.

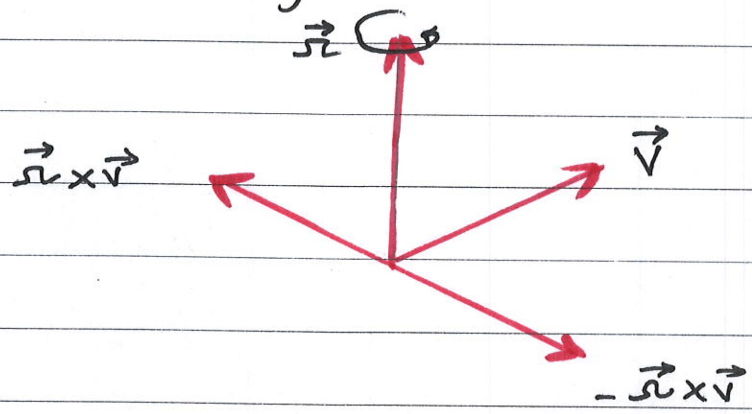
$$\vec{g} = -gK.$$

5/ The Coriolis force.

Coriolis force an air parcel with velocity in a Coordinate system with angular velocity Ω is:

$$\text{Coriolis force} = -2\vec{\Omega} \times \vec{V}$$

The minus sign is important, just like in advection. The coriolis force acts perpendicular to the direction of velocity vector:

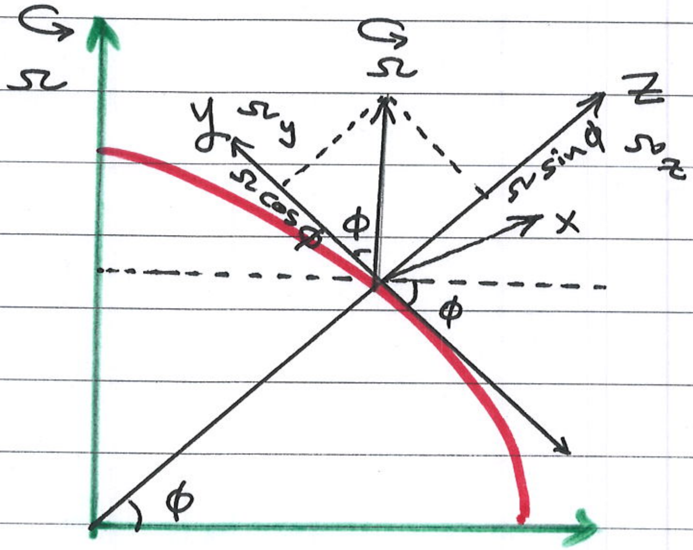


in the northern hemisphere, where Ω is counterclockwise, the Coriolis force acts to the right of the velocity vector, it tends to deflect air parcel to the right of their direction of motion.

The coriolis force changes only the direction of particle's motion, not it's speed, because the force is always perpendicular to the direction of motion.

The components of coriolis force.

From the Fig.



$\Omega_x = 0$

$\Omega_y = \Omega\cos\phi$

$\Omega_z = \Omega\sin\phi$, So..

$$\text{Coriolis force} = -2\vec{\Omega}\times\vec{V} = \begin{vmatrix} i & j & k \\ 0 & \Omega_y & \Omega_z \\ u & v & w \end{vmatrix}$$

$$-2\vec{\Omega}\times\vec{V} = -2(\Omega_y w - \Omega_z v)i + (0 - \Omega_z u)j - (0 - \Omega_y u)k$$

$$-2\vec{\Omega}\times\vec{V} = (2\Omega_z v - 2\Omega_y w)i - \Omega_z uj + \Omega_y uk.$$

$$-2\vec{\Omega}\times\vec{V} = (2\Omega\sin\phi\, v - 2\Omega\cos\phi\, w)i - \Omega\sin\phi\, uj + \Omega\cos\phi\, uk$$

let $f = 2\Omega\sin\phi$ and $e = 2\Omega\cos\phi$

Thus $\boxed{-2\vec{\Omega}\times\vec{V} = (fv - ew)i - fuj + euk.}$

The components of the equation of motion:

The Equation of motion is:

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\vec{\nabla}p + \vec{g}_* + \vec{f} - 2\vec{\Omega}\times\vec{V} + \vec{\Omega}^2\vec{R}$$

we not that $\vec{g}_*$ and $\Omega^2\vec{R}$ are the only forces which depend ~~on~~ solely on position.

$$\vec{g} = \vec{g}_* + \Omega^2\vec{R}$$

where $\vec{g}$ is gravity.

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\vec{\nabla}p + \vec{g} + \vec{f} - 2\vec{\Omega}\times\vec{V} \quad \ldots\ldots (1)$$

Now we find the x, y and z component of each term in equation (1).

$$\frac{d\vec{V}}{dt} = \frac{du}{dt}i + \frac{dv}{dt}j + \frac{dw}{dt}k \quad \ldots\ldots (A)$$

$$-\frac{1}{\rho}\vec{\nabla}p = -\frac{1}{\rho}\frac{\partial p}{\partial x}i - \frac{1}{\rho}\frac{\partial p}{\partial y}j - \frac{1}{\rho}\frac{\partial p}{\partial z}k \quad \ldots\ldots (B)$$

$$\vec{g} = -gk \quad \ldots\ldots (C)$$

$$\vec{f} = -\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)i - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)j - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right)k \quad \ldots (D)$$

$$-2\vec{\Omega}\times\vec{V} = (fv - ew)i - fuj + euk \quad \ldots\ldots (E)$$

put equations A, B, C, D and E in Equation (1).

$$\frac{du}{dt}i + \frac{dv}{dt}j + \frac{dw}{dt}k = -\frac{1}{\rho}\frac{\partial p}{\partial x}i - \frac{1}{\rho}\frac{\partial p}{\partial y}j - \frac{1}{\rho}\frac{\partial p}{\partial z}k$$

$$-gk + \left(-\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)i - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)j - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right)k\right.$$

$$+ (fv - ew)i - fuj + euk.$$

The components of this equation are:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) + fv - ew \qquad \text{x-Dir.}$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) - fu \qquad \text{y-Dir.}$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right) - g + eu \qquad \text{z-Dir.}$$

To assess which terms can be neglected. we assign an "order of magnitude" to all the variables and parameters in the equations.

For synoptic scales the following order of magnitude are:-

name	symbol	order of magnitude
Horizontal velocity	U	10 m/sec.
vertical velocity	W	0.01 m/sec.
Horizontal distance	L	1000 Km $= 10^6$ m
vertical distance	H	10 Km $= 10^4$ m
Horizontal pressure	Δp	10 mb
density	ρ	1 kg/m^3
Time	$\frac{L}{U}$	1 day (10^5 sec)
viscosity	μ	$1.46 \times 10^{-5}\ \frac{m^2}{sec}$
omega	Ω	$7.3 \times 10^{-5}\ sec^{-1}$
latitude	ϕ	45°
radius of earth	R	6.378×10^6 m.

Using these scales and parameters, in the x-momentum equation we have the following order of magnitude.

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) + fv - ew.$$

$$\frac{\partial u}{\partial t} = \frac{U^2}{L} = \frac{10^2}{10^6} = 10^{-4} \; m/sec^2$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\Delta P}{\rho L} = \frac{10^3}{1 * 10^6} = 10^{-3} \; m/sec^2$$

$$\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) = \mu\frac{\partial^2 u}{\partial z^2} = \frac{\nu U}{H^2} = \frac{1.5\times 10^{-5}\times 10}{10^8}$$

$$\therefore \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) = 10^{-12} \; m/sec^2$$

$$fv = 2\Omega v\sin\phi = 2\Omega U\sin\phi = 2 * 7.3\times 10^{-5} * 10 \times \sin 45^{\circ}$$

$$fv = 10^{-3} \; m/sec^2$$

$$ew = 2\Omega W\cos\phi = 2 * 7.3\times 10^{-5} * 10^{-2} * 0.7$$

$$ew = 10^{-6} \; m/sec^2.$$

A similar analysis for y and Z equations.

Many of terms are very small compared to others, and can therfore be ignored without much loss of accuracy. We can therefore ignore the viscous terms and coriolis term that involves the vertical velocity.

Thus the components of equation of motion are:

$$\boxed{\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv}$$

$$\boxed{\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu.}$$

$$\boxed{\frac{\partial p}{\partial z} = -\rho g} \longrightarrow \text{Hydrostatic Equation.}$$

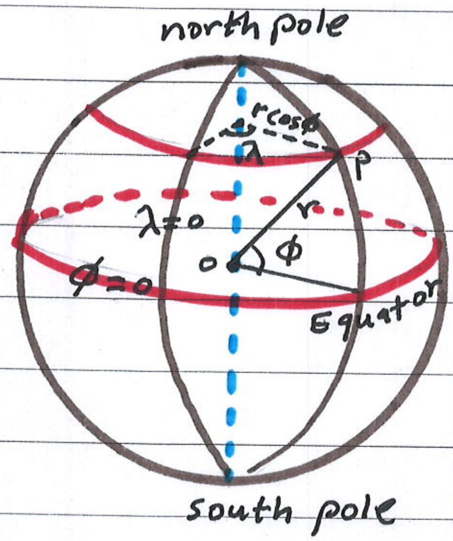
Hydrostatic equation is expressing a balance between gravity acting downwards and the vertical component of the pressure force acting upwards

This system of equation is quite often used in analysis of dynamical problems.

Coordinate system (spherical system).

It is partical to use aspherical coordinate system with origin at the center of the earth to describe dynamical aspects.

The coordinate system is rotating together with the planet.



λ longitude in easterly direction.

ϕ latitude in northerly direction.

Z geometric altitude.

R radius of planet $r = R + Z$.

$u = r\cos\phi \dfrac{d\lambda}{dt}$ Zonal component of wind $\vec{V}$ measured towards the east.

$v = r \dfrac{d\phi}{dt}$ meridional component of wind $\vec{V}$ measured towards the north.

$w = \dfrac{dz}{dt}$ Vertical component of $\vec{V}$.

Equation of motion in spherical coordinates.

The whole set in spherical coordinates without friction.

$$\frac{du}{dt} = \frac{uv}{r}\tan\phi - \frac{uw}{r} + 2\Omega\sin\phi\, v - 2\Omega\cos\phi\, w - \frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial \lambda}$$

$$\frac{dv}{dt} = -\frac{u^2}{r}\tan\phi - \frac{uw}{r} - 2\Omega\sin\phi\, u - \frac{1}{\rho r}\frac{\partial p}{\partial \phi}$$

$$\frac{dw}{dt} = \frac{u^2+v^2}{r} + 2\Omega\cos\phi\, u - g - \frac{1}{\rho}\frac{\partial p}{\partial r}.$$

In the Lagrangian frame we have to take care of the total derivative.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial}{\partial \lambda} + \frac{v}{r}\frac{\partial}{\partial \phi} + w\frac{\partial}{\partial z}.$$

additional equations.

The momentum equation "equations of motion" in component form comprise a system of three equations with 4 unknown quantities (u, v, p and ρ).

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu$$

$$\frac{\partial p}{\partial z} = -\rho g.$$

They are not a closed set, because there are four dependent variables and only three equations. We need to come up with some more equations in order to close the set.

The three additional equations are:

1- Gas Equation.
2- Thermodynamic Equation.
3- Continuity Equation.

1- Gas Equation.

A perfect gas (ideal gas) obeys the physical laws of Boyle and Charles. the general equation of state is:

$$P = \rho R T$$

where:

P: pressure.
ρ: density.
R: gas constant (287 $J Kg^{-1} K^{-1}$
T: Temperature.

Boyle's Law : $P_1 V_1 = P_2 V_2$ where T is constant.

Charles Law : $\frac{P_1}{T_1} = \frac{P_2}{T_2}$ where V is constant.

Combing Boyles and charle's Law we get:

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2} = C$$

where C depends on the mass of gas it is equal to 287 $J Kg^{-1} K^{-1}$ (specific gas constant).

$\therefore \frac{P\alpha}{T} = R$ where $\alpha = \frac{1}{\rho}$

$$\boxed{\therefore P = \rho R T}$$

2- Thermodynamic Equation.

The thermodynamic energy equation comes from 1st law of Thermodynmics (conservation of energy).

The first law of thermodynamic can be written by:

$$dH = du + dw.$$

where:

dH: is the amount of heat added per unit mass.

du: is the change in energy per unit mass, $du = C_v\, dT$

dw: is the work done by unit mass on system, $dw = p\,d\alpha$.

$$\boxed{\therefore dH = C_v\, dT + p\,d\alpha} \quad \text{——— } ①$$

Derived equation of state $p\alpha = RT$ we get:

$$p\,d\alpha + \alpha\, dp = R\,dT$$

$$p\,d\alpha = R\,dT - \alpha\, dp \quad \text{——— } ②$$

put ② in ① we get:

$$dH = C_v dT + R dT - \alpha dp$$

$$dH = (C_v + R) dT - \alpha dp.$$

Recall that : $R = C_p - C_v$

$\therefore C_p = R + C_v$

where $C_p = 1004 \frac{J}{Kg\,K}$

$C_v = 717 \frac{J}{Kg\,K}$

C_p and C_v are specific heat at constant p and v.

$$\boxed{\therefore dH = C_p dT - \alpha dp}$$

$$\cancel{\frac{dH}{dt} = C_p \frac{dT}{dt} - \alpha \frac{dp}{dt}}$$

dH: is the Heat added per unit mass.

So the equation of first law of thermodyamic are:

$$dH = C_v dT + p d\alpha$$

$$dH = C_p dT - \alpha dp$$

Addibatic Assmption.

It is assumed that $dH=0$, this assumption can be made whenever the motion is so fast, so that the heat exchange between the particale and the surrounding is negligible.

For addibatic motion the equations of first law of thermodynamics becomes:

$$C_v dT + p d\alpha = 0 \quad \text{—— (1)}$$

$$C_p dT - \alpha dp = 0 \quad \text{—— (2)}$$

The equation of state is $p = \rho R T = \frac{1}{\alpha} R T$ —— (3)

put (3) in (1)

$$C_v dT + \frac{RT}{\alpha} d\alpha = 0$$

$$C_v dT = -RT \frac{d\alpha}{\alpha}$$

$$C_v \frac{dT}{T} = -R \frac{d\alpha}{\alpha} \quad \div C_v$$

$$\int_{T_1}^{T} \frac{dT}{T} = -\frac{R}{C_v} \int_{\alpha_1}^{\alpha} \frac{d\alpha}{\alpha}$$

$$\ln T \Big|_{T_1}^{T} = -\frac{R}{C_v} \ln \alpha \Big|_{\alpha_1}^{\alpha}$$

$$\ln (T - T_1) = -\frac{R}{C_v}\left(\ln(\alpha - \alpha_1)\right)$$

بأخذ e للطرفين

$$\ln \frac{T}{T_1} = -\frac{R}{C_v} \ln \frac{\alpha}{\alpha}$$

$$\boxed{\therefore \frac{T}{T_1} = \left(\frac{\alpha}{\alpha_1}\right)^{\frac{-R}{C_v}}} \quad \text{——— (4)}$$

if T increases α will decrease and vice versa.

From equation (2)

$$C_p\, dT - \alpha\, dp = 0$$

From State equation $p = \frac{1}{\alpha} R T$ $\quad \boxed{\therefore \alpha = \frac{RT}{p}}$ — (5)

put (5) in (2)

$$C_p\, dT - \frac{RT}{P}\, dp$$

$$\int_{T_1}^{T} \frac{dT}{T} = \frac{R}{C_p} \int_{P_1}^{P} \frac{dP}{P}$$

$$\ln \frac{T}{T_1} = \frac{R}{C_p} \ln \frac{P}{P_1}$$

$$\boxed{\therefore \frac{T}{T_1} = \left(\frac{P}{P_1}\right)^{\frac{R}{C_p}}} \qquad (6)$$

If T increases p will increase and vice versa.

From equation (4) and (6) we get:

$$\left(\frac{\alpha}{\alpha_1}\right)^{\frac{-\cancel{R}}{C_v}} = \left(\frac{P}{P_1}\right)^{\frac{\cancel{R}}{C_p}}$$

$$\boxed{\therefore \frac{\alpha}{\alpha_1} = \left(\frac{P}{P_1}\right)^{\frac{-C_v}{C_p}}}$$

An increase in p correspound to decrease in α and vice versa.

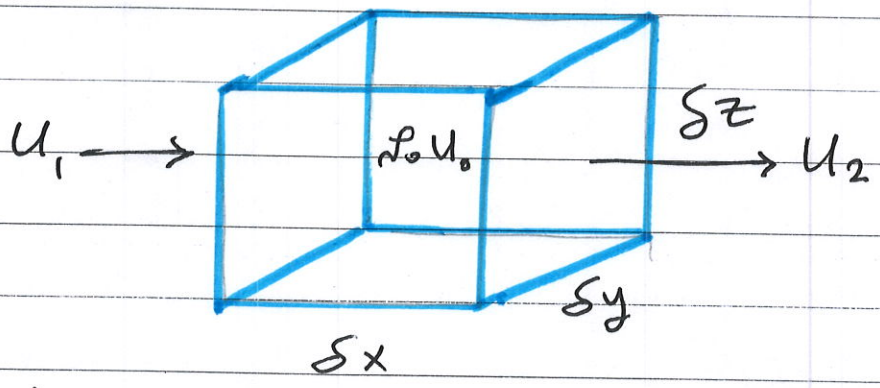
3 - The continuity Equation.

The Continuity equation comes from the law of conservation of mass," there is no sources or sinks of mass anywhere in the atmosphere.

Consider the box at a fixed point in space. The net change in mass is found by adding up the mass fluxes entering and leaving through each face of the box.

the volume of box is $\delta V = \delta x \delta y \delta z$

The mass equal $m = \rho \delta V$.



The change of mass per unit time is:

$$\frac{\partial m}{\partial t} = \frac{\partial \rho}{\partial t} \delta V.$$

The net inflow in the x-direction is:

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$$\rho_1 u_1 \delta y \delta z - \rho_2 u_2 \delta y \delta z = 0$$

$$\frac{\partial \rho}{\partial t}\delta V=\left[\rho_0 u_0-\frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\,\delta z-$$

$$\left[\rho_0 u_0+\frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\,\delta z.$$

$$\frac{\partial \rho}{\partial t}\delta V=-\frac{\partial(\rho u)}{\partial x}\delta V \quad \text{—— } x$$

Similarly the net inflow in y and z direction are:

$$\frac{\partial \rho}{\partial t}\delta V=-\frac{\partial(\rho v)}{\partial y}\delta V. \quad \text{—— } y$$

$$\frac{\partial \rho}{\partial t}\delta V=-\frac{\partial(\rho w)}{\partial z}\delta V \quad \text{—— } z$$

the Total change of mass per unit time will be:

$$\frac{\partial \rho}{\partial t}\cancel{\delta V}=-\left[\frac{\partial}{\partial x}\rho u+\frac{\partial}{\partial y}\rho v+\frac{\partial}{\partial z}\rho w\right]\cancel{\delta V}$$

$$\frac{\partial \rho}{\partial t}=-\vec{\nabla}\cdot(\rho\vec{V})$$

$$\boxed{\therefore\ \frac{\partial \rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{V})=0}$$ The first forms.

$$\text{or}\quad \frac{\partial \rho}{\partial t} + \rho\, \vec{\nabla}\cdot\vec{V} + \vec{V}\cdot\vec{\nabla}\rho = 0$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\, \vec{\nabla}\cdot\vec{V} = 0$$

$$\boxed{\therefore\ \frac{d\rho}{dt} + \rho\,(\vec{\nabla}\cdot\vec{V}) = 0}$$ The second forms.

The physical meaning of continuity equation (first form) is that:
The Change in density at a fixed point in space is dependent upon the divergence of the mass flux.

* If there is divergence of the mass flux then
  $\vec{\nabla}\cdot\rho\vec{V} > 0$ and density will decrease.
* If there is convergence of mass flux then
  $\vec{\nabla}\cdot\rho\vec{V} < 0$ and density will be increase.

$\vec{\nabla}\cdot(\rho\vec{V}) > 0$ اذا كان هناك تباعد في فيض الكتلة في عنصر ما فان الكثافة سوف تتناقص.

$\vec{\nabla}\cdot(\rho\vec{V}) < 0$ اذا كان هناك تقارب في فيض الكتلة في عنصر ما فان الكثافة سوف تتزايد.

The Governing Equations. المعادلات الحاكمة

The equations that govern the atmosphere on Synoptic Scale are:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + fv \quad \text{—} (1)$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - fu \quad \text{—} (2)$$

$$\frac{\partial P}{\partial z} = -\rho g \quad \text{—} (3)$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0 \quad \text{—} (4)$$

$$dH = c_v\frac{dT}{dt} + P\frac{d\alpha}{dt} \quad \text{—} (5)$$

$$P = \rho R T \quad \text{—} (6)$$

we have six equations with six unknows (u, v, w, P, T and ρ).

Theoretically, they can be solved to predict or diagnose the future value of the 6 Variables.

STATIC EQUILLBRIUM IN THE ATMOSPHERS.

In theoretical considerations and in order of magnitude estimations it is often advantageous to use rather simple distributions of the atmospheric Variables, in particular the variations with height of pressure, temprecature, density, etc.
A number of simple atmospheric models have been constructed for such purposes. They are characterized by the fact that the Hydrostatic equation can be integrated exactly using simple functions. So we shall consider some example of atmospheric models.

A- The Homogenous atmosphere:

The density is equal to a constant ρ_0 in this atmosphere and independent on space & time.

$$\rho = \rho_0 = \text{constant}$$

From Hydrostatic Equation $\frac{dP}{dz} = -\rho_0 g$

$$\int_{P_0}^{P} dP = -\rho_0 g \int_{z_0}^{z} dz$$

$$P\Big|_{P_0}^{P} = -\rho_0 g \int_0^{z} Z$$

$$P - P_0 = -\rho_0 g Z.$$

$$p = p_0 - \rho_0 g z \quad \text{——— (1)}$$

where p_0 is pressure for $z = 0$.

The homogenous atmosphere has a finite height H

when $p = 0$ $z = H$

$$0 = p_0 - \rho_0 g H$$

$$\therefore H = \frac{p_0}{\rho_0 g} \qquad p_0 = \rho_0 R T_0$$

$$\therefore H = \frac{\cancel{\rho_0} R T_0}{\cancel{\rho_0} g} \qquad T_0 = 283\,^\circ K$$

$R = 287$

$g = 9.8$

$$\boxed{H \approx 8000 \text{ meter.}}$$

we may define a tempreature in the homogenous atmosphere from gas equation.

$$p = \rho_0 R T \Rightarrow \therefore T = \frac{p}{\rho_0 R} \quad \text{——— (2)}$$

put equation (1) in (2)

$$T = \frac{p_0 - \rho_0 g z}{\rho_0 R} \Rightarrow T = \frac{p_0}{\rho_0 R} - \frac{\cancel{\rho_0} g z}{\cancel{\rho_0} R}$$

$$\boxed{T = T_0 - \frac{g}{R} z}$$

this equation shows that T decreases linearly with height in a homogenous atmosphere.

Q/ From the atmospheric model (homogenous atmosphere) Show that the lapse rate γ is:

$$\gamma = \frac{dT}{dz} = -3.4\ ^{o}K/100m.$$

B - The isothermal atmosphere:

in this model we have $T = T_o = const.$
From the Hydrostatic Equation we get:

$$dP = -\rho g\, dz$$

Recall that $\rho = \dfrac{P}{RT_o}$

$$dP = -\frac{P}{RT_o} g\, dz$$

$$\int_{P_o}^{P} \frac{dP}{P} = -\frac{g}{RT_o} \int_0^z dz$$

$$\ln P \Big|_{P_o}^{P} = -\frac{g}{RT_o} z$$

$$\ln \frac{P}{P_o} = -\frac{g}{RT_o} z$$

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$$\left(\frac{P}{P_o}\right) = e^{\left(-\frac{g}{RT_o} z\right)}$$

This equation shows that the isothermal atmosphere is of infinite extent because $P \to 0$ when $z \to \infty$

$$\boxed{P = P_o\, e^{\left(\frac{-g}{RT_o} z\right)}}$$

The scale height for an isothermal atmosphere is often defined as the height at which the pressure has decreased to e^{-1} of the surface pressure.

$$z = H_s \qquad p = p_0\, e^{\left(-\frac{g}{RT_0} H_s\right)}$$

$$p = p_0\, e^{-1}$$

$$\cancel{p_0}\, \cancel{e}^{\left(-\frac{g}{RT_0} H_s\right)} = \cancel{p_0}\, \cancel{e}^{-1}$$

$$-\frac{g}{RT_0} H_s = -1 \quad \therefore H_s = \frac{RT_0}{g} = 8000\,m$$

or, that the scale height is equal to the height of the homogenouse atmosphere having the same surface Tempreature as the isothermal atmosphere.

The density in the isothermal atmosphere can be calculated from gas equation. $p_0 = \rho_0 R T_0$, $p = \rho R T_0$

$$p = p_0\, e^{\left(-\frac{g}{RT_0} z\right)}$$

$$\rho \cancel{R} \cancel{T_0} = \rho_0 \cancel{R} \cancel{T_0}\, e^{\left(\frac{-g}{RT_0} z\right)}$$

$$\boxed{\therefore \rho = \rho_0\, e^{\left(-\frac{g}{RT_0} z\right)}}$$

Q1

If the surface temparature T_o is $283\,^{\circ}K$, and the Surface pressure p_o is 1000hpa, Calculate the pressure and the temparature at height of 5 Km?

To calculate the pressure we used the equation

$$P = P_o \, e^{-\frac{gZ}{RT_o}}$$

$Z = 5\text{ Km} = 5000\text{ m}$

$$P = 1000 \, e^{-\frac{9.8 \times 5000}{287 \times 283}}$$

$P = 547$ hpa.

to calculate the temparature we used the equation

$$T = T_o - \frac{gZ}{R}$$

$$T = 283 - \frac{9.8 \times 5000}{287}$$

$T = 112\,^{\circ}K$.

Q/ integrate the H.E to find an expression for the temperature as a function of height, what is the temperature at the Middle of the atmosphere?

from H.E $\frac{dP}{dz} = -\rho g$ $\rho = \rho_0$

$dP = -\rho_0 g\, dz$ $\int_{P_0}^{P} dP = -\rho_0 g \int_{z_0}^{z} dz$

$P \Big|_{P_0}^{P} = -\rho_0 g z$ $P - P_0 = -\rho_0 g z$

$P = P_0 - \rho_0 g z$ —— ①

from equation of state $P = \rho_0 R T$

$T = \frac{P}{\rho_0 R}$ —— ② put ① in ②

$T = \frac{P_0 - \rho_0 g z}{\rho_0 R} = \overbrace{\frac{P_0}{\rho_0 R}}^{T_0} - \frac{\cancel{\rho_0} g z}{\cancel{\rho_0} R}$

$\boxed{T = T_0 - \frac{gz}{R}}$

at the middle of the atmosphere

$z = \frac{8000}{2} = 4000\,m$

$T = T_0 - \frac{gz}{R} \Rightarrow T = 273 - \frac{9.8 \times 4 \times 10^3}{287} = \boxed{136.4\,°K}$

Q/

intgrate the hydrostatic equation to find an expression of the pressure as a function of height ?

$$T = T_0 = \text{constant}$$

$$\frac{dp}{dz} = -\rho g \qquad \rho = \frac{p}{RT_0}$$

$$\frac{dp}{dz} = -\frac{gp}{RT_0} \Longrightarrow \frac{dp}{p} = -\frac{g}{RT_0}dz$$

$$\int_{p_0}^{p} \frac{dp}{p} = -\frac{g}{RT_0}\int_{z_0}^{z} dz$$

$$\ln\frac{p}{p_0} = -\frac{g}{RT_0}z$$

$$\frac{p}{p_0} = e^{-\frac{gz}{RT_0}}$$

$$\therefore p = p_0 e^{-\frac{gz}{RT_0}}$$

Q/ Show that " The scale height for an isothermal is equal to the height of the homogenous atmosphere".

To prove that $H_s = H$

$$p = p_o e^{-\frac{gz}{RT_o}} \quad \text{——} ① \text{ ---- isothermal}$$

$$p = p_o e^{-1} \quad \text{——} ②$$

$$\cancel{p_o} \cancel{e}^{-\frac{gz}{RT_o}} = \cancel{p_o} \cancel{e}^{-1}$$

$$\frac{gz}{RT_o} = 1 \quad \text{where } z = H_s$$

$$H_s = \frac{RT_o}{g} = \frac{287 \times 273}{9.8} = 8000\,m$$

$$\frac{dp}{dz} = -\rho_o g \Rightarrow \int_{p_o}^{p} dp = -\rho_o g \int_{z_o}^{z} dz \text{ --- homogenous}$$

$$p - p_o = -\rho_o g z \quad \text{when } p = 0 \quad z = H$$

$$0 = p_o - \rho_o g H \Rightarrow H = \frac{p_o}{\rho_o g} \longrightarrow p_o = \rho_o R T_o$$

$$H = \frac{\cancel{\rho_o} R T_o}{\cancel{\rho_o} g} = \frac{RT_o}{g} = \frac{287 \times 273}{9.8} = 8000\,m$$

$$\therefore H_s = H = 8\,km.$$

3 - The Atmosphere with constant laps rate.

The laps rate is $\gamma = -\frac{dT}{dz}$

$$\int_{T_0}^{T} dT = -\gamma \int_{z_0}^{z} dz \Rightarrow T - T_0 = -\gamma z$$

$$\boxed{\therefore T = T_0 - \gamma z} \quad \text{——— (1)}$$

From hydrostat equation $\frac{dp}{dz} = -\rho g$

$P = \rho R T \qquad \rho = \frac{P}{RT}$

$$\boxed{\frac{dp}{dz} = -\frac{gP}{RT}} \quad \text{——— (2)}$$

put (1) in (2)

$$\frac{dp}{dz} = -\frac{gP}{R(T_0 - \gamma z)}$$

$$\frac{dp}{p} = -\frac{g}{R(T_0 - \gamma z)} dz$$

let us $u = T_0 - \gamma z \qquad du = 0 - \gamma dz$

$$dz = -\frac{du}{\gamma} \qquad \cancel{dz} = \frac{\cancel{\ }}{du}$$

$$\int_{p_0}^{p} \frac{dp}{p} = -\frac{g}{R} \int_{0}^{z} \frac{\frac{-du}{\gamma}}{u}$$

$$\ln \frac{p}{p_0} = +\frac{g}{R\gamma} \int_{0}^{z} \frac{du}{u}$$

$$\ln \frac{p}{p_0} = +\frac{g}{R\gamma} \ln (u) \Big|_{0}^{z} \qquad u = T_0 - \gamma z$$

$$\ln \frac{p}{p_0} = \frac{g}{R\gamma} \ln (T_0 - \gamma z) \Big|_{0}^{z}$$

$$\ln \frac{p}{p_0} = \frac{g}{R\gamma} \left[\ln (T_0 - \gamma z) - \ln (T_0) \right]$$

$$\ln \frac{p}{p_0} = \frac{g}{R\gamma} \ln (T_0 - \gamma z) - \frac{g}{R\gamma} \ln (T_0).$$

where $T = T_0 - \gamma z$

$$\ln \frac{p}{p_0} = \frac{g}{R\gamma} \ln \frac{T}{T_0}$$

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$$\frac{p}{p_0} = \left(\frac{T}{T_0} \right)^{\frac{g}{R\gamma}} \Rightarrow p = p_0 \left(\frac{T}{T_0} \right)^{\frac{g}{R\gamma}}$$

$$p = \rho R T \qquad p_0 = \rho_0 R T_0$$

$$\rho_z R T_z = \rho_0 R T_0 \left(\frac{T}{T_0} \right)^{\frac{g}{R\gamma}}$$

$$\rho \cancel{R} T = \rho_0 \cancel{R} T_0 \left(\frac{T}{T_0}\right)^{\frac{g}{R\gamma}}$$

$$\rho = \rho_0 \left(\frac{T}{T_0}\right)^{-1} \left(\frac{T}{T_0}\right)^{\frac{g}{R\gamma}}$$

$$\boxed{\therefore \rho = \rho_0 \left(\frac{T}{T_0}\right)^{\frac{g}{R\gamma}-1}}$$

4- The Adibatic Atmosphere:

$$\theta = \theta_0 = \text{constant.}$$

$$\theta = T\left(\frac{p}{p_1}\right)^{\frac{R}{c_p}}$$ using chane rule.

$$\cancelto{0}{\frac{1}{\theta}\frac{\partial \theta}{\partial z}} = \frac{1}{T}\frac{\partial T}{\partial z} - \frac{R}{c_p}\frac{1}{p}\frac{\partial p}{\partial z} = 0$$

$$\frac{\partial p}{\partial z} = -\rho g \quad , \quad p = \rho R T \Rightarrow \frac{R}{p} = +\frac{1}{\rho T}$$

$$\frac{1}{T}\frac{\partial T}{\partial z} = \frac{R}{c_p}\frac{1}{p}\frac{\partial p}{\partial z}$$

$$\frac{1}{T}\frac{\partial T}{\partial z} = -\frac{g}{c_p T}$$

$$\frac{1}{T}\frac{\partial T}{\partial z} + \frac{g}{c_p T} = 0$$

$$\frac{\partial T}{\partial z} = -\frac{g}{c_p}$$ $g = 9.8$, $c_p = 1004$

$$\therefore \frac{\partial T}{\partial z} = -\frac{9.8}{1004} = 10\,^{\circ}K/Km.$$