Linear Programming

The structure of the problems

1- There are two or more products that are formed out of mixtures of two or more ingredients
2- There are machines or other facilities which are used in manufacturing of the various Products , and whose capacities are limited .
3- Alternatively, The quantities of the ingredients are limited.
4- There is a profit function based on the profit contribution of each unit of each products.
5- It is required to find the production quantities of each product that will maximize the profit function.

In general ,if two products are to be made in quantities X_1 AND X_2 :

	Product 1	Product 2	A mount available
Ingredient 1	a_{11}	a_{12}	b_1
Ingredient 2	a_{21}	a_{22}	b_2
Profit contribution (Per unit)	c_1	c_2	

1- $x_1 \geq 0 , x_2 \geq 0$

2- $a_{11}x_1 + a_{12}x_2 \leq b_1$

$a_{21}x_1 + a_{22}x_2 \leq b_2$

3- Profit function or objective function :

$z = c_1x_1 + c_2x_2$

Note :

$2\ products + 2\ restriction \rightarrow graphical\ and\ enumeration$

$3\ products + 2\ restrictions \rightarrow enumeration$

$3\ products + 3\ restrictions \rightarrow simple\ x$

The graphical method :

Example: shown below are the numbers of pounds of each of two ingredients in one unit of each of two chemical compounds , How money unit $x_1 and x_2$ of the two compounds should be produced in order to maximize the profit ?

	Compound 1	Compound 2	Amount available (lb)
Ingredient A	8	4	160
Ingredient B	2	6	60
Profit contribution ($ID/Unit$)	3	4	

Solution :

For A $\rightarrow 8x_1 + 4x_2 \leq 160$

$$2x_1 + x_2 \leq 40 \rightarrow 1$$

For B $\rightarrow 2x_1 + 6x_2 \leq 60$

$$x_1 + 3x_2 \leq 30 \rightarrow 2$$

$$x_1 \geq 0 , x_2 \geq 0$$

Objective function :

$$Z = 3X_1 + 4X_2$$

graphical :

$$Z = C_1X_1 + C_2X_2$$

$$X_2 = \frac{1}{C_2}(Z - C_1X_1)$$

$$\frac{dx_2}{dx_1} = \frac{-C_1}{C_2}$$

$$= -\frac{3}{4}$$

Enumeration :

0	0	0
20	0	60
0	10	40
18	4	70

So maximum Z at $x_1 = 18 \quad and \;\; x_2 = 4$

$$Z_{max.} = 3 \times 18 + 4 \times 4 = 70$$

Example : same as the previous example, but a third material C Such that

$$6x_1 + 5x_2 \le 150$$

Solution :

1- Draw the line PQ which satisfies the new construction

2- The line PQ falls out side the area of feasible solution

3- This will not affect the solution but indicates that some of material C will not be used

$$\mathbf{6 \times 18 + 5 \times 4 = 128} < 150$$

Problem : the composition (in units / Ib) of two fertilizers A and B are shown below .

A - What quantities of A and B should be bout in order to minimize the cost?

B - what is the minimum cost ?

C - what component is available in quantities greater than minimum and by how much ?

Note : Use the graphical method.

	A	B	Minimum need(units)
Nitrate (N)	10	5	300
Phosphorus ([illegible]	6	10	250
Potash ([illegible]	4	5	100
Price [illegible]	5	8	

A -

$X_1, X_2 > 0$

$10X_1 + 5X_2 \geq 300 \rightarrow X_1(0,30), X_2(60,0)$

$6X_1 + 10X_2 \geq 250 \rightarrow X_1\left(0, \frac{250}{6}\right), X_2(25,0)$

$4X_1 + 5X_2 \geq 100 \rightarrow X_1(0,25), X_2(20,0)$

$C = 5X_1 + 8X_2 \quad \rightarrow \frac{dx_2}{dx_1} = -\frac{5}{8}$

So draw line of $x_1 = 80 \; and \; x_2 = 50 \; or \; x_1 = 40 \; and \; x_2 = 25$

For minimum cost from graph $x_1 \; of \; A = 25$, $x_2 \; of \; b = 10$

B - $C = 5 \times 25 + 8 \times 10 = 205 \; I.D$

C - Check $x_1 = 25$, $x_2 = 10$ in each equation

$10x_1 + 5x_2 = 250 + 50 = 300 \quad O.K$

$6x_1 + 10x_2 = 150 + 100 = 250 \quad O.K$

$4x_1 + 5x_2 = 100 + 50 = 150 > 100$

More than minimum by 50 unit

Allocation problems

The assignment problem : the general problem is to take (n) resources (e.g. workers)

And assign resources to(n) recipients (e.g. machines or jobs) , knowing what the cost (or time) of each cost is :

$3 \times 3 \rightarrow 3! = 3 \times 2 = 6 \ ways$

$4 \times 4 \rightarrow 4! = 4 \times 3 \times 2 = 24 \ ways$

$5 \times 5 \rightarrow 5! = 5 \times 4 \times 3 \times 2 = 120 \ ways$

- The task cannot conveniently be done
- The Hungarian method , demonstrated in the follow example , quickly solves the problem.

Example : four workers are to be assigned to four jobs, it being known that the time taken by each on each jobs would be as shown below . find an assignment which would minimize the sum of the times taken

	Jobs				
		A	B	C	D
	a	15	18	21	24
	b	19	23	22	18
	c	26	17	16	19
	d	19	21	23	17

Solution :

1- نطرح اقل قيمة من كل عامود للحصول على (صفر) واحد **على الأقل في كل عامود** :

	A	B	C	D
a	0	1	5	7
b	4	6	6	1
c	11	0	0	2
d	4	4	7	0

2- نطرح اقل قيمة من كل سطر لم يكن يحتوي على صفر (هنا فقط السطر الثاني) ثم نؤشر الاصفار بأقل مايمكن من الخطوط (هنا فقط ثلاث خطوط)
إذن نحتاج إلى خط رابع

	A	B	C	D
a	0	1	5	7
b	3	5	5	0
c	11	0	0	2
d	4	4	7	0

3- نطرح اقل قيمة غير معطاة من كل القيم غير المعطاة (هنا bA=3) ثم نضيفها إلى نقاط التقاطع (هنا aD=7 ، cD=2) فينتج :

	A	B	C	D
a	0	1	5	10
b	0	2	2	0
c	11	0	0	5
d	1	1	4	0

4- نؤشر الاصفار بأقل مايمكن من الخطوط (هنا فقط ثلاث خطوط ونحتاج إلى خط رابع)
5- نعيد الخطوة رقم ثلاثة أعلاه مرة أخرى فينتج :

	A	B	C	D
a	0	0	4	10
b	0	1	1	0
c	12	0	0	6
d	1	0	3	0

ثم نغطي الاصفار بأقل مايمكن من خطوط (هنا أربعة خطوط وهو المطلوب) المصفوفة الناتجة تمثل الحل الأمثل .
6- نستخرج (Assignment) المطلوب ونبدأ بالصفر الوحيد وهو cC :

cC, aA, bD ,dB $\rightarrow Time = 16 + 15 + 18 + 21 = 70$

or

cC ,aB , bA, dD $\rightarrow Time = 16 + 18 + 19 + 17 = 70$

The transportation problem : it is similar the assignment problem except that one or more items may be assignment to any given cell

- Supply = demand or S=D

Example : three different plants producing materials required to be transported to three ware houses . find the optimum transportation cost .

Transportation cost matrix

	Ware houses			
		A	B	C
	a	4	16	8
	b	8	24	16
	c	8	16	24

Solution :

	Ware house				
		A	B	C	Plant output
	a	4 50	16 20	8	70 20
	b	8	24 40	16 60	100 40
	c	8	16 30	24	30
	Ware house requirements (D)	50	90 70 30	60	200 200

So $aA + aB + bB + bC + cB = Transportation\ coscost$

Number of basic variables :

$*m + n - 1 = 3 + 3 - 1 = 5$

Transportation cost= $4 \times 50 + 16 \times 20 + 24 \times 40 + 16 \times 60 + 16 \times 30$

=2920