8-1 | INTRODUCTION

Preceding chapters have shown how conduction and convection heat transfer may be calculated with the aid of both mathematical analysis and empirical data. We now wish to consider the third mode of heat transfer—thermal radiation. *Thermal radiation* is that electromagnetic radiation emitted by a body as a result of its temperature. In this chapter, we shall first describe the nature of thermal radiation, its characteristics, and the properties that are used to describe materials insofar as the radiation is concerned. Next, the transfer of radiation through space will be considered. Finally, the overall problem of heat transfer by thermal radiation will be analyzed, including the influence of the material properties and the geometric arrangement of the bodies on the total energy that may be exchanged.

8-2 | PHYSICAL MECHANISM

There are many types of electromagnetic radiation; thermal radiation is only one. Regardless of the type of radiation, we say that it is propagated at the speed of light, \(3 \times 10^8\) m/s. This speed is equal to the product of the wavelength and frequency of the radiation,

\[ c = \lambda \nu \]

where

- \(c\) = speed of light
- \(\lambda\) = wavelength
- \(\nu\) = frequency

The unit for \(\lambda\) may be centimeters, angstroms (1 Å = 10^-8 cm), or micrometers (1 \(\mu\)m = 10^-6 m). A portion of the electromagnetic spectrum is shown in Figure 8-1. Thermal radiation lies in the range from about 0.1 to 100 \(\mu\)m, while the visible-light portion of the spectrum is very narrow, extending from about 0.35 to 0.75 \(\mu\)m.

The propagation of thermal radiation takes place in the form of discrete quanta, each quantum having an energy of

\[ E = h\nu \quad [8-1] \]

where \(h\) is Planck’s constant and has the value

\[ h = 6,625 \times 10^{-34} \text{ J} \cdot \text{s} \]
A very rough physical picture of the radiation propagation may be obtained by considering each quantum as a particle having energy, mass, and momentum, just as we considered the molecules of a gas. So, in a sense, the radiation might be thought of as a "photon gas" that may flow from one place to another. Using the relativistic relation between mass and energy, expressions for the mass and momentum of the "particles" could thus be derived; namely,

\[ E = m c^2 = h \nu \]

\[ m = \frac{h \nu}{c^2} \]

Momentum \( = \frac{h \nu}{c} \)

By considering the radiation as such a gas, the principles of quantum-statistical thermodynamics can be applied to derive an expression for the energy density of radiation per unit volume and per unit wavelength as

\[ n_\lambda = \frac{8 \pi h \nu \lambda^{-5}}{e^{h \nu/kT} - 1} \] [8-2]

where \( \lambda \) is Boltzmann’s constant, \( 1.38066 \times 10^{-23} \text{ J/molecule} \cdot \text{K} \). When the energy density is integrated over all wavelengths, the total energy emitted is proportional to absolute temperature to the fourth power:

\[ E_v = \sigma T^4 \] [8-3]

Equation (8-3) is called the Stefan-Boltzmann law, \( E_v \) is the energy radiated per unit time and per unit area by the ideal radiator, and \( \sigma \) is the Stefan-Boltzmann constant, which has the value

\[ \sigma = 5.669 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4 \quad [0.1714 \times 10^{-8} \text{ Btu/h} \cdot \text{ft}^2 \cdot \circ \text{R}^4] \]

where \( E_v \) is in watts per square meter and \( T \) is in degrees Kelvin. In the thermodynamic analysis the energy density is related to the energy radiated from a surface per unit time and per unit area. Thus the heated interior surface of an enclosure produces a certain energy density of thermal radiation in the enclosure. We are interested in radiant exchange with surfaces—hence the reason for the expression of radiation from a surface in terms of its temperature. The subscript \( \sigma \) in Equation (8-3) denotes that this is the radiation from a blackbody. We call this blackbody radiation because materials that obey this law appear black to the eye; they appear black because they do not reflect any radiation. Thus a blackbody is

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also considered as one that absorbs all radiation incident upon it. \( E_s \) is called the *emissive power* of a blackbody.

It is important to note at this point that the “blackness” of a surface to thermal radiation can be quite deceiving insofar as visual observations are concerned. A surface coated with lampblack appears black to the eye and turns out to be black for the thermal-radiation spectrum. On the other hand, snow and ice appear quite bright to the eye but are essentially “black” for long-wavelength thermal radiation. Many white paints are also essentially black for long-wavelength radiation. This point will be discussed further in later sections.

### 8-3 RADIATION PROPERTIES

When radiant energy strikes a material surface, part of the radiation is reflected, part is absorbed, and part is transmitted, as shown in Figure 8-2. We define the reflectivity \( \rho \) as the fraction reflected, the absorptivity \( \alpha \) as the fraction absorbed, and the transmissivity \( \tau \) as the fraction transmitted. Thus

\[
\rho + \alpha + \tau = 1 \quad [8-4]
\]

Most solid bodies do not transmit thermal radiation, so that for many applied problems the transmissivity may be taken as zero. Then

\[
\rho + \alpha = 1
\]

Two types of reflection phenomena may be observed when radiation strikes a surface. If the angle of incidence is equal to the angle of reflection, the reflection is called *specular*. On the other hand, when an incident beam is distributed uniformly in all directions after reflection, the reflection is called *diffuse*. These two types of reflection are depicted

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**Figure 8-2** | Sketch showing effects of incident radiation.

**Figure 8-3** | (a) Specular \((\phi_1 = \phi_2)\) and (b) diffuse reflection.
in Figure 8-3. Note that a specular reflection presents a mirror image of the source to the observer. No real surface is either specular or diffuse. An ordinary mirror is quite specular for visible light, but would not necessarily be specular over the entire wavelength range of thermal radiation. Ordinarily, a rough surface exhibits diffuse behavior better than a highly polished surface. Similarly, a polished surface is more specular than a rough surface. The influence of surface roughness on thermal-radiation properties of materials is a matter of serious concern and remains a subject for continuing research.

The emissive power of a body \( E \) is defined as the energy emitted by the body per unit area and per unit time. One may perform a thought experiment to establish a relation between the emissive power of a body and the material properties defined above. Assume that a perfectly black enclosure is available, i.e., one that absorbs all the incident radiation falling upon it, as shown schematically in Figure 8-4. This enclosure will also emit radiation according to the \( T^4 \) law. Let the radiant flux arriving at some area in the enclosure be \( q_1 \). Now suppose that a body is placed inside the enclosure and allowed to come into temperature equilibrium with it. At equilibrium the energy absorbed by the body must be equal to the energy emitted; otherwise there would be an energy flow into or out of the body that would raise or lower its temperature. At equilibrium we may write

\[
E_A = q_1 A \alpha \tag{8-5}
\]

If we now replace the body in the enclosure with a blackbody of the same size and shape and allow it to come to equilibrium with the enclosure at the same temperature, \( E_b A = q_1 A(1) \) since the absorptivity of a blackbody is unity. If Equation (8-5) is divided by Equation (8-6),

\[
\frac{E}{E_b} = \alpha
\]

and we find that the ratio of the emissive power of a body to the emissive power of a blackbody at the same temperature is equal to the absorptivity of the body. This ratio is defined as the emissivity \( \epsilon \) of the body,

\[
\epsilon = \frac{E}{E_b} \tag{8-7}
\]

so that

\[
\epsilon = \alpha \tag{8-8}
\]

Equation (8-8) is called Kirchhoff's identity. At this point we note that the emissivities and absorptivities that have been discussed are the total properties of the particular material;
that is, they represent the integrated behavior of the material over all wavelengths. Real substances emit less radiation than ideal black surfaces as measured by the emissivity of the material. In reality, the emissivity of a material varies with temperature and the wavelength of the radiation.

The Gray Body

A **gray body** is defined such that the monochromatic emissivity $\epsilon_\lambda$ of the body is independent of wavelength. The *monochromatic emissivity* is defined as the ratio of the monochromatic emissive power of the body to the monochromatic emissive power of a blackbody at the same wavelength and temperature. Thus

$$\epsilon_\lambda = \frac{E_\lambda}{E_{b\lambda}}$$

The total emissivity of the body may be related to the monochromatic emissivity by noting that

$$E = \int_0^\infty \epsilon_\lambda E_{b\lambda} \, d\lambda \quad \text{and} \quad E_b = \int_0^\infty E_{b\lambda} \, d\lambda = \sigma T^4$$

so that

$$\epsilon = \frac{E}{E_b} = \frac{\int_0^\infty \epsilon_\lambda E_{b\lambda} \, d\lambda}{\sigma T^4} \quad [8-9]$$

where $E_{b\lambda}$ is the emissive power of a blackbody per unit wavelength. If the gray-body condition is imposed, that is, $\epsilon_\lambda = \text{constant}$, Equation (8-9) reduces to

$$\epsilon = \epsilon_\lambda \quad [8-10]$$

The emissivities of various substances vary widely with wavelength, temperature, and surface condition. Some typical values of the total emissivity of various surfaces are given in Appendix A. We may note that the tabulated values are subject to considerable experimental uncertainty. A rather complete survey of radiation properties is given in Reference 14.

The functional relation for $E_{b\lambda}$ was derived by Planck by introducing the quantum concept for electromagnetic energy. The derivation is now usually performed by methods of statistical thermodynamics, and $E_{b\lambda}$ is shown to be related to the energy density of Equation (8-2) by

$$E_{b\lambda} = \frac{\hbar c \lambda}{4} \quad [8-11]$$

or

$$E_{b\lambda} = \frac{C_1 \lambda^{-5}}{e^{C_2/\lambda T} - 1} \quad [8-12]$$

where

- $\lambda$ = wavelength, $\mu$m
- $T$ = temperature, K
- $C_1 = 3.743 \times 10^8$ W·$\mu$m$^4$/m$^2$ [1.187 × 10$^8$ Btu·$\mu$m$^4$/h·ft$^2$]
- $C_2 = 1.4387 \times 10^4$ $\mu$m·K [2.5896 × 10$^4$ $\mu$m·°R]

A plot of $E_{b\lambda}$ as a function of temperature and wavelength is given in Figure 8-5a. Notice that the peak of the curve is shifted to the shorter wavelengths for the higher temperatures. These maximum points in the radiation curves are related by Wien’s displacement law,

$$\lambda_{max} T = 2897.6 \mu m \cdot K \quad [5215.6 \mu m \cdot °R] \quad [8-13]$$
Figure 8-5 1 (a) Blackbody emissive power as a function of wavelength and temperature; (b) comparison of emissive power of ideal blackbodies and gray bodies with that of a real surface.

Figure 8-5b indicates the relative radiation spectra from a blackbody at 3000°F and a corresponding ideal gray body with emissivity equal to 0.6. Also shown is a curve indicating an approximate behavior for a real surface, which may differ considerably from that of either an ideal blackbody or an ideal gray body. For analysis purposes surfaces are usually considered as gray bodies, with emissivities taken as the integrated average value.
The shift in the maximum point of the radiation curve explains the change in color of a body as it is heated. Since the band of wavelengths visible to the eye lies between about 0.3 and 0.7 μm, only a very small portion of the radiant-energy spectrum at low temperatures is detected by the eye. As the body is heated, the maximum intensity is shifted to the shorter wavelengths, and the first visible sign of the increase in temperature of the body is a dark-red color. With further increase in temperature, the color appears as a bright red, then bright yellow, and finally white. The material also appears much brighter at higher temperatures because a larger portion of the total radiation falls within the visible range.

We are frequently interested in the amount of energy radiated from a blackbody in a certain specified wavelength range. The fraction of the total energy radiated between 0 and λ is given by

$$\frac{E_{\lambda_1}}{E_{\lambda_0}} = \frac{\int_{\lambda_0}^{\lambda_1} E_{\lambda} \, d\lambda}{\int_{\lambda_0}^{\infty} E_{\lambda} \, d\lambda} \quad [8-14]$$

Equation (8-12) may be rewritten by dividing both sides by $T^5$, so that

$$\frac{E_{\lambda_0}}{T^5} = \frac{C_1}{(\lambda T)^5 e^{C_2/\lambda T} - 1} \quad [8-15]$$

Now, for any specified temperature, the integrals in Equation (8-14) may be expressed in terms of the single variable $\lambda T$. The ratio in Equation (8-14) is plotted in Figure 8-6 and tabulated in Table 8-1, along with the ratio in Equation (8-15). If the radiant energy emitted

**Figure 8-6** Fraction of blackbody radiation in wavelength interval.
## Table 8-11 Radiation functions.

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<td>( \frac{W}{m^2 \cdot K \cdot \mu m} \times 10^{11} )</td>
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<td>0.98474</td>
<td></td>
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<td></td>
</tr>
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</table>

between wavelengths \( \lambda_1 \) and \( \lambda_2 \) is desired, then

\[
E_{b\lambda_2-\lambda_1} = E_{b\lambda_2} \left( \frac{E_{b\lambda_2}}{E_{b\lambda_2}} - \frac{E_{b\lambda_2}}{E_{b\lambda_2}} \right) \tag{8-16}
\]

where \( E_{b\lambda_2} \) is the total radiation emitted over all wavelengths,

\[
E_{b\lambda_2} = \sigma T^4 \tag{8-17}
\]

and is obtained by integrating the Planck distribution formula of Equation (8-12) over all wavelengths.

Solar radiation has a spectrum approximating that of a blackbody at 5800 K. Ordinary window glass transmits radiation up to about 2.5 \( \mu m \). Consulting Table 8-1 for \( \lambda T = (2.5)(5800) = 14,500 \mu m \cdot K \), we find the fraction of the solar spectrum below 2.5 \( \mu m \) to be about 0.97. Thus the glass transmits most of the solar radiation incident upon it. In contrast, room radiation at about 300 K below 2.5 \( \mu m \) has \( \lambda T = (2.5)(300) = 750 \mu m \cdot K \), and only a minute fraction (less than 0.001 percent) of this radiation would be transmitted.
through the glass. The glass, which is essentially transparent for visible light, is almost totally opaque for thermal radiation emitted at ordinary room temperatures.

**Construction of a Blackbody**

The concept of a blackbody is an idealization; that is, a perfect blackbody does not exist—all surfaces reflect radiation to some extent, however slight. A blackbody may be approximated very accurately, however, in the following way. A cavity is constructed, as shown in Figure 8-7, so that it is very large compared with the size of the opening in the side. An incident ray of energy is reflected many times on the inside before finally escaping from the side opening. With each reflection there is a fraction of the energy absorbed corresponding to the absorptivity of the inside of the cavity. After the many absorptions, practically all the incident radiation at the side opening is absorbed. It should be noted that the cavity of Figure 8-7 behaves approximately as a blackbody emitter as well as an absorber.

### Example 8-1

**Transmission and Absorption in a Glass Plate**

A glass plate 30 cm square is used to view radiation from a furnace. The transmissivity of the glass is 0.5 from 0.2 to 3.5 μm. The emissivity may be assumed to be 0.3 up to 3.5 μm and 0.9 above that. The transmissivity of the glass is zero, except in the range from 0.2 to 3.5 μm. Assuming that the furnace is a blackbody at 2000°C, calculate the energy absorbed in the glass and the energy transmitted.

**Solution**

\[
T' = 2000°C = 2273 \text{ K}
\]

\[
\lambda_1 T' = (0.2)(2273) = 454.6 \mu\text{m} \cdot \text{K}
\]

\[
\lambda_2 T' = (3.5)(2273) = 7955.5 \mu\text{m} \cdot \text{K}
\]

\[
A = (0.3)^2 = 0.09 \text{ m}^2
\]

From Table 8-1

\[
\frac{E_{b\lambda 1}}{\sigma T'^4} = 0 \quad \frac{E_{b\lambda 2}}{\sigma T'^4} = 0.85443
\]

\[
\sigma T'^4 = (5.669 \times 10^{-8})(2273)^4 = 1513.3 \text{ kW/m}^2
\]

Total incident radiation is

\[
0.2 \mu\text{m} < \lambda < 3.5 \mu\text{m} = (1.5133 \times 10^6)(0.85443 - 0)(0.3)^2
\]

\[
= 116.4 \text{ kW} \quad [397 \times 10^3 \text{ Btu/h}]
\]

Total radiation transmitted = (0.5)(116.4) = 58.2 kW

Radiation absorbed

\[
(0.3)(116.4) = 34.92 \text{ kW} \quad \text{for} \quad 0 < \lambda < 3.5 \mu\text{m}
\]

\[
(0.9)(1 - 0.85443)(1513.3)(0.09) = 17.84 \text{ kW} \quad \text{for} \quad 3.5 \mu\text{m} < \lambda < \infty
\]

Total radiation absorbed = 34.92 + 17.84 = 52.76 kW \[180,000 \text{ Btu/h}]

### 8-4 Radiation Shape Factor

Consider two black surfaces \(A_1\) and \(A_2\), as shown in Figure 8-8. We wish to obtain a general expression for the energy exchange between these surfaces when they are maintained at different temperatures. The problem becomes essentially one of determining the amount of
Figure 8-8 Sketch showing area elements used in deriving radiation shape factor.

\[
dq_{1,2\text{net}} = \cos \phi_1 \cos \phi_2 \frac{dA_1 dA_2}{\pi r^2} (E_1 - E_2)
\]

Energy that leaves one surface and reaches the other. To solve this problem the *radiation shape factors* are defined as:

\[
\begin{align*}
F_{1 \rightarrow 2} &= \text{fraction of energy leaving surface 1 that reaches surface 2} \\
F_{2 \rightarrow 1} &= \text{fraction of energy leaving surface 2 that reaches surface 1} \\
F_{i \rightarrow j} &= \text{fraction of energy leaving surface } i \text{ that reaches surface } j
\end{align*}
\]

Other names for the radiation shape factor are *view factor*, *angle factor*, and *configuration factor*. The energy leaving surface 1 and arriving at surface 2 is

\[
E_{f1} A_1 F_{12}
\]

and the energy leaving surface 2 and arriving at surface 1 is

\[
E_{f2} A_2 F_{21}
\]

Since the surfaces are black, all the incident radiation will be absorbed, and the net energy exchange is

\[
E_{f1} A_1 F_{12} - E_{f2} A_2 F_{21} = Q_{1 \rightarrow 2}
\]

If both surfaces are at the same temperature, there can be no heat exchange, that is, \(Q_{1 \rightarrow 2} = 0\). Also, for \(F_1 = F_2\)

\[
E_{f1} = E_{f2}
\]

so that

\[
A_1 F_{12} = A_2 F_{21} \quad \text{[8-18]}
\]

The net heat exchange is therefore

\[
Q_{1 \rightarrow 2} = A_1 F_{12} (E_{f1} - E_{f2}) - A_2 F_{21} (E_{f1} - E_{f2}) \quad \text{[8-19]}
\]

Equation (8-18) is known as a reciprocity relation, and it applies in a general way for any two surfaces \(i\) and \(j\):

\[
A_i F_{ij} = A_j F_{ji} \quad \text{[8-18a]}
\]

Although the relation is derived for black surfaces, it holds for other surfaces also as long as diffuse radiation is involved.
Figure 8-9 | Elevation view of area shown in Figure 8-8.

We now wish to determine a general relation for \( I_{12} \) (or \( I_{21} \)). To do this, we consider the elements of area \( dA_1 \) and \( dA_2 \) in Figure 8-8. The angles \( \phi_1 \) and \( \phi_2 \) are measured between a normal to the surface and the line drawn between the area elements \( r \). The projection of \( dA_1 \) on the line between centers is

\[
dA_1 \cos \phi_1
\]

This may be seen more clearly in the elevation drawing shown in Figure 8-9. We assume that the surfaces are diffuse, that is, that the intensity of the radiation is the same in all directions. The intensity is the radiation emitted per unit area and per unit of solid angle in a certain specified direction. So, in order to obtain the energy emitted by the element of area \( dA_1 \) in a certain direction, we must multiply the intensity by the projection of \( dA_1 \) in the specified direction. Thus the energy leaving \( dA_1 \) in the direction given by the angle \( \phi_1 \) is

\[
I_b \, dA_1 \cos \phi_1 \tag{a}
\]

where \( I_b \) is the blackbody intensity. The radiation arriving at some area element \( dA_2 \) at a distance \( r \) from \( A_1 \) would be

\[
I_b \, dA_1 \cos \phi_1 \, \frac{dA_2}{r^2} \tag{b}
\]

where \( dA_2 \) is constructed normal to the radius vector. The quantity \( dA_2 / r^2 \) represents the solid angle subtended by the area \( dA_2 \). The intensity may be obtained in terms of the emissive power by integrating expression \( (b) \) over a hemisphere enclosing the element of area \( dA_1 \). In a spherical coordinate system like that in Figure 8-10,

\[
dA_2 = r^2 \sin \phi \, d\phi \, d\theta \, d\phi
\]

Figure 8-10 | Spherical coordinate system used in derivation of radiation shape factor.
Then
\[
E_b \, dA_1 = I_b \, dA_1 \int_0^{2\pi} \int_0^{\pi/2} \sin \phi \cos \phi \, d\phi \, d\psi
\]
so that
\[
E_b = \pi I_b
\]  
[8-20]

We may now return to the energy-exchange problem indicated in Figure 8-8. The area element \(dA_\theta\) is given by
\[
dA_\theta = \cos \phi_2 \, dA_2
\]
so that the energy leaving \(dA_1\) that arrives at \(dA_2\) is
\[
\dot{q}_{1\rightarrow 2} = E_{\theta 1} \cos \phi_1 \cos \phi_2 \frac{dA_1 \, dA_2}{\pi r^2}
\]
That energy leaving \(dA_2\) and arriving at \(dA_1\) is
\[
\dot{q}_{2\rightarrow 1} = E_{\theta 2} \cos \phi_2 \cos \phi_1 \frac{dA_2 \, dA_1}{\pi r^2}
\]
and the net energy exchange is
\[
\dot{q}_{\text{net}_{1\rightarrow 2}} = (E_{\theta 1} - E_{\theta 2}) \int_{A_2} \int_{A_1} \cos \phi_1 \cos \phi_2 \frac{dA_1 \, dA_2}{\pi r^2}
\]  
[8-21]

The integral is either \(A_1 \, F_{12}\) or \(A_2 \, F_{21}\) according to Equation (8-19). To evaluate the integral, the specific geometry of the surfaces \(A_1\) and \(A_2\) must be known. We shall work out an elementary problem and then present the results for more complicated geometries in graphical and equation form.

Consider the radiation from the small area \(dA_1\) to the flat disk \(A_2\), as shown in Figure 8-11. The element of area \(dA_2\) is chosen as the circular ring of radius \(x\). Thus
\[
dA_2 = 2\pi x \, dx
\]

**Figure 8-11** | Radiation from a small-area element to a disk.
8-4  Radiation Shape Factor

We note that \( \phi_1 = \phi_2 \) and apply Equation (8-21), integrating over the area \( A_2 \):

\[
dA_1 F_{dA_1-A_2} = dA_1 \int_{a_2} \cos^2 \phi_1 \frac{2\pi \, dx}{\pi \ell^2}
\]

Making the substitutions

\[
r = (R^2 + \chi^2)^{1/2} \quad \text{and} \quad \cos \phi_1 = \frac{R}{(R^2 + \chi^2)^{1/2}}
\]

we have

\[
dA_1 F_{dA_1-A_2} = dA_1 \int_{0}^{D/2} \frac{2R^2 \, dx}{(R^2 + \chi^2)^{3/2}}
\]

Performing the integration gives

\[
dA_1 F_{dA_1-A_2} = -dA_1 \left( \frac{R^2}{R^2 + \chi^2} \right)^{D/2} = dA_1 \frac{D^2}{4R^2 + D^2}
\]

so that

\[
F_{dA_1-A_2} = \frac{D^2}{4R^2 + D^2} \quad \text{[8-22]}
\]

The calculation of shape factors may be extended to more complex geometries, as described in References 3, 5, 24, and 32; 32 gives a very complete catalog of analytical relations and graphs for shape factors. For our purposes we give only the results of a few geometries as shown in Figures 8-12 through 8-16. The analytical relations for these geometries are given in Table 8-2.

**Figure 8-12** | Radiation shape factor for radiation between parallel rectangles.
Figure 8-13 1 Radiation shape factor for radiation between parallel equal coaxial disks.

Real-Surface Behavior

Real surfaces exhibit interesting deviations from the ideal surfaces described in the preceding paragraphs. Real surfaces, for example, are not perfectly diffuse, and hence the intensity of emitted radiation is not constant over all directions. The directional-emittance characteristics of several types of surfaces are shown in Figure 8-17. These curves illustrate the characteristically different behavior of electric conductors and nonconductors. Conductors emit more energy in a direction having a large azimuth angle. This behavior may be satisfactorily explained with basic electromagnetic wave theory, and is discussed in Reference 24. As a result of this basic behavior of conductors and nonconductors, we may anticipate the appearance of a sphere which is heated to incandescent temperatures, as shown in Figure 8-18. An electric conducting sphere will appear bright around the rim since more energy is emitted at large angles $\phi$. A sphere constructed of a nonconducting material will have the opposite behavior and will appear bright in the center and dark around the edge.
Figure 8-14 1 Radiation shape factor for radiation between perpendicular rectangles with a common edge.

Reflectance and absorptance of thermal radiation from real surfaces are a function not only of the surface itself but also of the surroundings. These properties are dependent on the direction and wavelength of the incident radiation. But the distribution of the intensity of incident radiation with wavelength may be a very complicated function of the temperatures and surface characteristics of all the surfaces that incorporate the surroundings. Let us denote the total incident radiation on a surface per unit time, per unit area, and per unit wavelength as \( G_x \). Then the total absorptivity will be given as the ratio of the total energy absorbed to the total energy incident on the surface, or

\[
\alpha = \frac{\int_0^\infty \alpha_x G_x \, d\lambda}{\int_0^\infty G_x \, d\lambda}. \tag{8-23}
\]

If we are fortunate enough to have a gray body such that \( \epsilon_x = \epsilon = \text{constant} \), this relation simplifies considerably. It may be shown that Kirchoff’s law [Equation (8-8)] may be written for monochromatic radiation as

\[
\epsilon_x = \alpha_x. \tag{8-24}
\]

Therefore, for a gray body, \( \alpha_x = \text{constant} \), and Equation (8-23) expresses the result that the total absorptivity is also constant and independent of the wavelength distribution of incident radiation. Furthermore, since the emissivity and absorptivity are constant over all wavelengths for a gray body, they must be independent of temperature as well. Unhappily, real surfaces are not always “gray” in nature, and significant errors may ensue by assuming gray-body behavior. On the other hand, analysis of radiation exchange using real-surface behavior is so complicated that the ease and simplification of the gray-body assumption is justified by the practical utility it affords. References 10, 11, and 24 present comparisons of heat-transfer calculations based on both gray and nongray analyses.
Figure 8-15 | Radiation shape factors for two concentric cylinders of finite length. (a) Outer cylinder to itself; (b) outer cylinder to inner cylinder.
Figure 8-16 | Radiation shape factor for radiation between two parallel coaxial disks.

![Diagram](image)

Table 8-2 | Radiation shape factor relations.

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Shape factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Parallel, equal rectangles</td>
<td>$F_1 = \pi (2/\pi x)[\ln[(1 + x^2)(1 + y^2)/(1 + x^2 + y^2)]^{1/2} + x(1 + y^2)/(1 + y^2)]^{1/2}$</td>
</tr>
<tr>
<td>(Fig. 8-12)</td>
<td>$\times [\tan^{-1}[x/(1 + y^2)]^{1/2} + \tan^{-1}[y/(1 + x^2)]^{1/2} - x\tan^{-1} x - y\tan^{-1} y]$</td>
</tr>
<tr>
<td>$x = X/D, y = Y/D$</td>
<td></td>
</tr>
<tr>
<td>2. Parallel, equal, coaxial disks</td>
<td>$F_1 = [X - (X^2 - 4)^{1/2}]^2$</td>
</tr>
<tr>
<td>(Fig. 8-13)</td>
<td></td>
</tr>
<tr>
<td>$R = d/2, X = (2R^2 + 1)/2R$</td>
<td></td>
</tr>
<tr>
<td>3. Perpendicular rectangles with a common edge</td>
<td>$F_1 = \pi W \ln[(1 + x^2)(1 + y^2)/(1 + x^2 + y^2)]^{1/2} + x(1 + y^2)/(1 + y^2)]^{1/2}$</td>
</tr>
<tr>
<td>(Fig. 8-14)</td>
<td>$\times [\tan^{-1}[x/(1 + y^2)]^{1/2} + \tan^{-1}[y/(1 + x^2)]^{1/2} - x\tan^{-1} x - y\tan^{-1} y]$</td>
</tr>
<tr>
<td>$H = Z/X, W = Y/X$</td>
<td></td>
</tr>
<tr>
<td>4. Finite, coaxial cylinders</td>
<td>$F_2 = \pi (1/X) - \pi X\cos^{-1}(B/A)/2 - \pi X(4A - 4X^2 + 4)/2 \cos^{-1}(B/X)$</td>
</tr>
<tr>
<td>(Fig. 8-15)</td>
<td>$+ B \sin^{-1}(1/X - \pi A/2))$</td>
</tr>
<tr>
<td>$X = r_2/r_1, Y = L/r_1$</td>
<td></td>
</tr>
<tr>
<td>$A = X^2 + Y^2 - 1$</td>
<td></td>
</tr>
<tr>
<td>$B = Y^2 - X^2 + 1$</td>
<td></td>
</tr>
<tr>
<td>5. Parallel, coaxial disks</td>
<td>$F_1 = \pi [X - (X^2 - 4(R_2/R_1)^2)]^{1/2}]^2$</td>
</tr>
<tr>
<td>(Fig. 8-16)</td>
<td></td>
</tr>
<tr>
<td>$R_1 = r_1/L$</td>
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</tr>
<tr>
<td>$R_2 = r_2/L$</td>
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</tr>
<tr>
<td>$X = 1 + (1 + R_2^2)/R_1^2$</td>
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</table>
**Figure 8-17** Typical directional behavior of emissivity for conductors and nonconductors. $\varepsilon_\theta$ is emissivity at angle $\theta$ measured from normal to surface. Nonconductor curves are for (a) wet ice, (b) wood, (c) glass, (d) paper, (e) clay, (f) copper oxide, and (g) aluminum oxide.

**Figure 8-18** Effect of directional emittance on appearance of an incandescent sphere: (a) electrical conductor; (b) electrical nonconductor.

---

**Example 8-2**

Two parallel black plates 0.5 by 1.0 m are spaced 0.5 m apart. One plate is maintained at 1000°C and the other at 500°C. What is the net radiant heat exchange between the two plates?

**Solution**

The ratios for use with Figure 8-12 are

\[
\frac{Y}{D} = \frac{0.5}{0.5} = 1.0, \quad \frac{X}{D} = \frac{1.0}{0.5} = 2.0
\]

so that $F_{12} = 0.285$. The heat transfer is calculated from

\[
q = A_1 F_{12}(E_M - E_{12}) = \sigma A_1 F_{12}(T_1^4 - T_2^4)
\]

\[
= (5.669 \times 10^{-8})(0.5)(0.285)(1273^4 - 773^4)
\]

\[
= 18.33 \text{ kW} \quad [62,540 \text{ Btu/h}]
\]
8-5 | RELATIONS BETWEEN SHAPE FACTORS

Some useful relations between shape factors may be obtained by considering the system shown in Figure 8-19. Suppose that the shape factor for radiation from $A_3$ to the combined area $A_{1,2}$ is desired. This shape factor must be given very simply as

$$F_{3-1,2} = F_{3-1} + F_{3-2} \quad [8-25]$$

that is, the total shape factor is the sum of its parts. We could also write Equation (8-25) as

$$A_3 F_{3-1,2} = A_3 F_{3-1} + A_3 F_{3-2} \quad [8-26]$$

and making use of the reciprocity relations

$$A_3 F_{3-1,2} = A_{1,2} F_{1,2-3}$$
$$A_3 F_{3-1} = A_1 F_{1-3}$$
$$A_3 F_{3-2} = A_2 F_{2-3}$$

the expression could be rewritten

$$A_{1,2} F_{1,2-3} = A_1 F_{1-3} + A_2 F_{2-3} \quad [8-27]$$

which simply states that the total radiation arriving at surface 3 is the sum of the radiations from surfaces 1 and 2. Suppose we wish to determine the shape factor $F_{1-3}$ for the surfaces in Figure 8-20 in terms of known shape factors for perpendicular rectangles with a common edge. We may write

$$F_{1-2,3} = F_{1-2} + F_{1-3}$$

in accordance with Equation (8-25). Both $F_{1-2,3}$ and $F_{1-2}$ may be determined from Figure 8-14, so that $F_{1-3}$ is easily calculated when the dimensions are known. Now consider the somewhat more complicated situation shown in Figure 8-21. An expression for the shape factor $F_{1-4}$ is desired in terms of known shape factors for perpendicular rectangles.

Figure 8-19 | Sketch showing some relations between shape factors.

$$F_{3-1,2} = F_{3-1} + F_{3-2}$$
$$A_3 F_{3-1,2} = A_1 F_{3-1} + A_2 F_{3-2}$$
$$A_{1,2} F_{1,2-3} = A_1 F_{1-3} + A_2 F_{2-3}$$

$A_3$

$A_1$

$A_2$

$A_{1,2}$
with a common edge. We write

\[ A_{1,2}F_{1,2-3,4} = A_1F_{1-3,4} + A_2F_{2-3,4} \]  \[ a \]

in accordance with Equation (8-25). Both \( F_{1,2-3,4} \) and \( F_{2-3,4} \) can be obtained from Figure 8-14, and \( F_{1-3,4} \) may be expressed

\[ A_1F_{1-3,4} = A_1F_{1-3} + A_1F_{1-4} \]  \[ b \]

Also

\[ A_{1,2}F_{1,2-3} = A_1F_{1-3} + A_2F_{2-3} \]  \[ c \]

Solving for \( A_1F_{1-3} \) from \( c \), inserting this in \( b \), and then inserting the resultant expression for \( A_1F_{1-3,4} \) in \( a \) gives

\[ A_{1,2}F_{1,2-3,4} = A_{1,2}F_{1,2-3} - A_2F_{2-3} + A_1F_{1-4} + A_2F_{2-3,4} \]  \[ d \]

Notice that all shape factors except \( F_{1-4} \) may be determined from Figure 8-14. Thus

\[ F_{1-4} = \frac{1}{A_1} \left( A_{1,2}F_{1,2-3,4} - A_2F_{2-3} + A_1F_{1-4} - A_2F_{2-3,4} \right) \]  \[ 8-28 \]

In the foregoing discussion the tacit assumption has been made that the various bodies do not see themselves, that is,

\[ F_{11} = F_{22} = F_{33} = 0 \cdots \]

To be perfectly general, we must include the possibility of concave curved surfaces, which may then see themselves. The general relation is therefore

\[ \sum_{j=1}^{n} F_{ij} = 1.0 \]  \[ 8-29 \]

where \( F_{ij} \) is the fraction of the total energy leaving surface \( i \) that arrives at surface \( j \). Thus for a three-surface enclosure we would write

\[ F_{11} + F_{12} + F_{13} = 1.0 \]

and \( F_{11} \) represents the fraction of energy leaving surface 1 that strikes surface 1. A certain amount of care is required in analyzing radiation exchange between curved surfaces.
Hamilton and Morgan [5] have presented generalized relations for parallel and perpendicular rectangles in terms of shape factors which may be obtained from Figures 8-12 and 8-14. The two situations of interest are shown in Figures 8-22 and 8-23. For the perpendicular rectangles of Figure 8-22 it can be shown that the following reciprocity relations apply [5]:

$$A_1 F_{13'1} = A_3 F_{31'} = A_3 F_{31} = A_1 F_{13} \quad [8-30]$$

By making use of these reciprocity relations, the radiation shape factor $F_{13'}$ may be expressed by

$$A_1 F_{13'} = \frac{1}{2} \left[ K_{1,2,3,4,5,6}^2 - K_{2,3,4,5,6}^2 - K_{1,2,5,6}^2 + K_{3,4,5,6}^2 - K_{4,5,6}^2 - (1 \times 2 \times 3 \times 4 \times 5 \times 6) \right.$$ \hspace{2cm} [8-31]

$$- K_{1,2,3,4,5,6}^2 (4 \times 5 \times 6) + K_{1,2,5,6}^2 \times (5 \times 6) + K_{3,4,5}^2 (4 \times 5) + K_{3,4,5}^2 (4 \times 5) + K_{3,5,6}^2 (1 \times 2 \times 3 \times 5 \times 6) + K_{4,5}^2 (2 \times 3 \times 4 \times 5) + K_{2,5}^2 (2 \times 5) - K_2 (5 \times 6) - K_5 (2 \times 5) + K_{51}$$
where the $K$ terms are defined by

\[
K_{m-n} = A_{m} F_{m-n} \quad \text{[8-32]}
\]

\[
K_{(m)^2} = A_{m} F_{m-m} \quad \text{[8-33]}
\]

The generalized parallel-rectangle arrangement is depicted in Figure 8-23. The reciprocity relations that apply to this situation are given in Reference 5 as

\[
A_1 F_{19} = A_3 F_{37} = A_6 F_{61} = A_7 F_{73} \quad \text{[8-34]}
\]

Making use of these relations, it is possible to derive the shape factor $F_{19}$ as

\[
A_1 F_{19} = \frac{1}{4} \left[ K_{1,2,3,4,5,6,7,8,9} - K_{1,2,5,6,7,8} - K_{2,3,4,5,8,9} + K_{4,5,7,8,9} - K_{4,5,6,7,8} - K_{4,5,6,7,8} - K_{5,6,7,8} - K_{5,6,7,8} + K_{5,6,7,8} + K_{5,6,7,8} + K_{5,6,7,8} \right] \quad \text{[8-35]}
\]

The nomenclature for the $K$ terms is the same as given in Equations (8-32) and (8-33).

### Shape-Factor Algebra for Open Ends of Cylinders

**EXAMPLE 8-3**

Two concentric cylinders having diameters of 10 and 20 cm have a length of 20 cm. Calculate the shape factor between the open ends of the cylinders.

#### Solution

We use the nomenclature of Figure 8-15 for this problem and designate the open ends as surfaces 3 and 4. We have $L/r_2 = 20/10 = 2.0$ and $r_1/r_2 = 0.5$; so from Figure 8-15 or Table 8-2 we obtain

\[
F_{21} = 0.4126 \quad F_{22} = 0.3286
\]

Using the reciprocity relation [Equation (8-18)] we have

\[
A_1 F_{12} = A_2 F_{21} \quad \text{and} \quad F_{12} = (d_2/d_1) F_{21} = (20/10)(0.4126) = 0.8253
\]

For surface 2 we have

\[
F_{21} + F_{22} + F_{23} + F_{24} = 1.0
\]

From symmetry $F_{23} = F_{24}$ so that

\[
F_{23} = F_{24} = \left( \frac{1}{2} \right) (1 - 0.4126 - 0.3286) = 0.1294
\]

Using reciprocity again,

\[
A_2 F_{23} = A_3 F_{32}
\]

and

\[
F_{32} = \frac{\pi (20)(20)}{\pi (20^2 - 10^2)^2} 0.1294 = 0.6901
\]

We observe that $F_{11} = F_{33} = F_{44} = 0$ and for surface 3

\[
F_{31} + F_{32} + F_{34} = 1.0 \quad \text{[a]}
\]

So, if $F_{31}$ can be determined, we can calculate the desired quantity $F_{34}$. For surface 1

\[
F_{12} + F_{13} + F_{14} = 1.0
\]
and from symmetry \( F_{13} = F_{14} \) so that
\[
F_{13} = \left( \frac{1}{2} \right) \left( 1 - 0.8253 \right) = 0.0874
\]

Using reciprocity gives
\[
A_1 F_{13} = A_3 F_{31} \\
F_{31} = \frac{\pi (20)^2}{2 \pi^2 (10)^2 / 2} = 0.233
\]

Then, from Equation (a)
\[
F_{34} = 1 - 0.233 - 0.6901 = 0.0769
\]

---

### EXAMPLE 8-4  
Shape-Factor Algebra for Truncated Cone

A truncated cone has top and bottom diameters of 10 and 20 cm and a height of 10 cm. Calculate the shape factor between the top surface and the side and also the shape factor between the side and itself.

**Solution**

We employ Figure 8-16 for solution of this problem and take the nomenclature as shown, designating the top as surface 2, the bottom as surface 1, and the side as surface 3. Thus, the desired quantities are \( F_{23} \) and \( F_{33} \). We have \( L / r_1 = 10 / 10 = 1.0 \) and \( r_2 / L = 5 / 10 = 0.5 \). Thus, from Figure 8-16
\[
F_{12} = 0.12
\]

From reciprocity [Equation (8-18)]
\[
A_1 F_{12} = A_2 F_{21} \\
F_{21} = (20 / 10)^2 (0,12) = 0.48
\]

and
\[
F_{22} = 0
\]

so that
\[
F_{21} + F_{23} = 1.0
\]

and
\[
F_{23} = 1 - 0.48 = 0.52
\]

For surface 3,
\[
F_{31} + F_{32} + F_{33} = 1.0 \quad [a]
\]

so we must find \( F_{31} \) and \( F_{32} \) in order to evaluate \( F_{33} \). Since \( F_{13} = 0 \), we have
\[
F_{12} + F_{13} = 1.0 \quad \text{and} \quad F_{13} = 1 - 0.12 = 0.88
\]

and from reciprocity
\[
A_1 F_{13} = A_3 F_{31} \quad [b]
\]

The surface area of the side is
\[
A_3 = \pi (r_1 + r_2) \left[ (r_1 - r_2)^2 + L^2 \right]^{1/2} \\
= \pi (5 + 10) (5^2 + 10^2)^{1/2} = 526.9 \text{ cm}^2
\]
So, from Equation (b)

\[ F_{31} = \frac{\pi (10^2)}{526.9} = 0.88 = 0.525 \]

A similar procedure applies with surface 2 so that

\[ F_{32} = \frac{\pi (5)^2}{526.9} = 0.52 = 0.0775 \]

Finally, from Equation (a)

\[ F_{33} = 1 - 0.525 - 0.0775 = 0.397 \]

Shape-Factor Algebra for Cylindrical Reflector

**EXAMPLE 8-5**

The long circular half-cylinder shown in Figure Example 8-5 has a diameter of 60 cm and a square rod 20 by 20 cm placed along the geometric centerline. Both are surrounded by a large enclosure. Find \( F_{12} \), \( F_{13} \), and \( F_{11} \) in accordance with the nomenclature in the figure.

**Figure Example 8-5**

Half-cylinder, \( d = 60 \) cm

1

2

3

Large room

20 cm

**Solution**

From symmetry we have

\[ F_{21} = F_{23} = 0.5 \] \[ [a] \]

In general, \( F_{11} + F_{12} + F_{13} = 1 \). To aid in the analysis we create the fictitious surface 4 shown as the dashed line. For this surface, \( F_{41} = 1 \). Now, all radiation leaving surface 1 will arrive either at 2 or at 3. Likewise, this radiation will arrive at the imaginary surface 4, so that

\[ F_{14} = F_{12} + F_{13} \] \[ [b] \]

From reciprocity,

\[ A_1 F_{14} = A_4 F_{41} \]

From the areas, for unit length,

\[ A_1 = \pi d/2 = \pi (0.6)/2 = 0.942 \]
\[ A_4 = 0.2 + (2)(0.1)^2 + (0.2)^2)^{1/2} = 0.647 \]
\[ A_2 = 4(0.2) = 0.8 \]

so that

\[ F_{14} = \frac{A_4}{A_1} F_{41} = \frac{(0.647)(1.0)}{0.942} = 0.686 \] \[ [c] \]
We also have, from reciprocity,
\[ \lambda_2 F_{21} = \lambda_1 F_{12} \]
so
\[ F_{12} = \frac{\lambda_2}{\lambda_1} F_{21} = \frac{(0.8)(0.5)}{0.942} = 0.425 \quad [d] \]
Combining (b), (c), and (d) gives
\[ F_{13} = 0.686 - 0.425 = 0.261 \]
Finally,
\[ F_{11} = 1 - F_{12} - F_{13} = 1 - 0.425 - 0.261 = 0.314 \]
This example illustrates how one may make use of clever geometric considerations to calculate the radiation shape factors.

8-6 | HEAT EXCHANGE BETWEEN NONBLACKBODIES

The calculation of the radiation heat transfer between black surfaces is relatively easy because all the radiant energy that strikes a surface is absorbed. The main problem is one of determining the geometric shape factor, but once this is accomplished, the calculation of the heat exchange is very simple. When nonblackbodies are involved, the situation is much more complex, for all the energy striking a surface will not be absorbed; part will be reflected back to another heat-transfer surface, and part may be reflected out of the system entirely. The problem can become complicated because the radiant energy can be reflected back and forth between the heat-transfer surfaces several times. The analysis of the problem must take into consideration these multiple reflections if correct conclusions are to be drawn.

We shall assume that all surfaces considered in our analysis are diffuse, gray, and uniform in temperature and that the reflective and emissive properties are constant over all the surface. Two new terms may be defined:

- \( G \) = irradiation
  
  = total radiation incident upon a surface per unit time and per unit area

- \( J \) = radiosity
  
  = total radiation that leaves a surface per unit time and per unit area

In addition to the assumptions stated above, we shall also assume that the radiosity and irradiation are uniform over each surface. This assumption is not strictly correct, even for ideal gray diffuse surfaces, but the problems become exceedingly complex when this analytical restriction is not imposed. Sparrow and Cess [10] give a discussion of such problems. As shown in Figure 8-24, the radiosity is the sum of the energy emitted and the energy reflected when no energy is transmitted, or

\[ J = \epsilon E_b + \rho G \quad [8-36] \]

where \( \epsilon \) is the emissivity and \( E_b \) is the blackbody emissive power. Since the transmissivity is assumed to be zero, the reflectivity may be expressed as

\[ \rho = 1 - \alpha = 1 - \epsilon \]

so that

\[ J = \epsilon E_b + (1 - \epsilon)G \quad [8-37] \]
The net energy leaving the surface is the difference between the radiosity and the irradiation:

\[ \frac{q}{A} = J - G = \epsilon E_b + (1 - \epsilon)G - G \]

Solving for \( G \) in terms of \( J \) from Equation (8-37),

\[ q = \frac{\epsilon A}{1 - \epsilon}(E_b - J) \]

or

\[ q = \frac{E_b - J}{(1 - \epsilon)/\epsilon A} \tag{8-38} \]

At this point we introduce a very useful interpretation for Equation (8-38). If the denominator of the right side is considered as the surface resistance to radiation heat transfer, the numerator as a potential difference, and the heat flow as the “current,” then a network element could be drawn as in Figure 8-24(b) to represent the physical situation. This is the first step in the network method of analysis originated by Oppenheim [20].

Now consider the exchange of radiant energy by two surfaces, \( A_1 \) and \( A_2 \), shown in Figure 8-25. Of that total radiation leaving surface 1, the amount that reaches surface 2 is

\[ J_1 A_1 F_{12} \]

and of that total energy leaving surface 2, the amount that reaches surface 1 is

\[ J_2 A_2 F_{21} \]
The net interchange between the two surfaces is

\[ q_{1-2} = J_1 A_1 F_{12} - J_2 A_2 F_{21} \]

But

\[ A_1 F_{12} = A_2 F_{21} \]

so that

\[ q_{1-2} = (J_1 - J_2) A_1 F_{12} = (J_1 - J_2) A_2 F_{21} \]

or

\[ q_{1-2} = \frac{J_1 - J_2}{1/A_1 F_{12}} \] \[ 8.39 \]

We may thus construct a network element that represents Equation (8.39), as shown in Figure 8-25. The two network elements shown in Figures 8-24 and 8-25 represent the essentials of the radiation-network method. To construct a network for a particular radiation heat-transfer problem we need only connect a “surface resistance” \( (1 - \varepsilon) / \varepsilon A \) to each surface and a “space resistance” \( 1 / A_1 F_i \) between the radiosity potentials. For example, two surfaces that exchange heat with each other and nothing else would be represented by the network shown in Figure 8-26. In this case the net heat transfer would be the overall potential difference divided by the sum of the resistances:

\[ q_{\text{net}} = \frac{E_{A1} - E_{A2}}{(1 - \varepsilon_1) / \varepsilon_1 A_1 + 1/A_1 F_{12} + (1 - \varepsilon_2) / \varepsilon_2 A_2} \]

\[ \sigma(T_1^4 - T_2^4) \]

\[ = \frac{(1 - \varepsilon_1) / \varepsilon_1 A_1 + 1/A_1 F_{12} + (1 - \varepsilon_2) / \varepsilon_2 A_2} \]

\[ 8.40 \]

A network for a three-body problem is shown in Figure 8-27. In this case each of the bodies exchanges heat with the other two. The heat exchange between body 1 and body 2 would be

\[ q_{1-2} = \frac{J_1 - J_2}{1/A_1 F_{12}} \]

and that between body 1 and body 3,

\[ q_{1-3} = \frac{J_1 - J_3}{1/A_1 F_{13}} \]

To determine the heat flows in a problem of this type, the values of the radiosities must be calculated. This may be accomplished by performing standard methods of analysis used in dc circuit theory. The most convenient method is an application of Kirchhoff’s current law to the circuit, which states that the sum of the currents entering a node is zero. Example 8-6 illustrates the use of the method for the three-body problem.

**Figure 8-26** | Radiation network for two surfaces that see each other and nothing else.
Figure 8-27  | Radiation network for three surfaces that see each other and nothing else.

Insulated Surfaces and Surfaces with Large Areas

As we have seen, \((E_b - J)\) represents the potential difference for heat flow through the surface resistance \((1 - \epsilon)/\epsilon A\). If a surface is perfectly insulated, or re-radiates all the energy incident upon it, it has zero heat flow and the potential difference across the surface resistance is zero, resulting in \(J = E_b\). But, the insulated surface does not have zero surface resistance. In effect, the \(J\) node in the network is floating, that is, it does not draw any current. On the other hand, a surface with a very large area \((A \rightarrow \infty)\) has a surface resistance approaching zero, which makes it behave like a blackbody with \(\epsilon = 1.0\). It, too, will have \(J = E_b\) because of the zero surface resistance. Thus, these two cases—insulated surface and surface with a large area—both have \(J = E_b\), but for entirely different reasons. We will make use of these special cases in several examples.

A problem that may be easily solved with the network method is that of two flat surfaces exchanging heat with one another but connected by a third surface that does not exchange heat, i.e., one that is perfectly insulated. This third surface nevertheless influences the heat-transfer process because it absorbs and re-radiates energy to the other two surfaces that exchange heat. The network for this system is shown in Figure 8-28. Notice that node \(J_3\) is not connected to a radiation surface resistance because surface 3 does not exchange energy. A surface resistance \((1 - \epsilon)/\epsilon A\) exists, but because there is no heat current flow there is no

Figure 8-28  | Radiation network for two plane or convex surfaces enclosed by a third surface that is nonconducting but re-radiating (insulated).
potential difference, and \( J_3 = I_{13} \). Notice also that the values for the space resistances have been written

\[
I_{13} = 1 - F_{12} \\
F_{23} = 1 - F_{21}
\]

since surface 3 completely surrounds the other two surfaces. For the special case where surfaces 1 and 2 are convex, that is, they do not see themselves and \( F_{11} = F_{22} = 0 \), Figure 8-28 is a simple series-parallel network that may be solved for the heat flow as

\[
q_{\text{net}} = \frac{\sigma A_1 (T_1^4 - T_2^4)}{\frac{A_1 + A_2 + 2 A_1 F_{12}}{A_2 - A_1 (F_{12})^2} + \left( \frac{1}{\epsilon_1} - 1 \right) + \frac{A_1}{A_2} \left( \frac{1}{\epsilon_2} - 1 \right)} \tag{8-41}
\]

where the reciprocity relation

\[ A_1 F_{12} = A_2 F_{21} \]

has been used to simplify the expression. It is to be noted again that Equation (8-41) applies only to surfaces that do not see themselves; that is, \( F_{11} = F_{22} = 0 \). If these conditions do not apply, one must determine the respective shape factors and solve the network accordingly.

Example 8-7 gives an appropriate illustration of a problem involving an insulated surface. This network, and others that follow, assume that the only heat exchange is by radiation. Conduction and convection are neglected for now.

**EXAMPLE 8-6**

Hot Plates Enclosed by a Room

Two parallel plates 0.5 by 1.0 m are spaced 0.5 m apart, as shown in Figure Example 8-6. One plate is maintained at 1000°C and the other at 500°C. The emissivities of the plates are 0.2 and 0.5, respectively. The plates are located in a very large room, the walls of which are maintained at 27°C. The plates exchange heat with each other and with the room, but only the plate surfaces facing each other are to be considered in the analysis. Find the net transfer to each plate and to the room.

**Figure Example 8-6** 1 (a) Schematic. (b) Network.

(a) Room at 27°C

(b)  
\[
E_{b_1} = \sigma T_1^4 \\
E_{b_2} = \sigma T_3^4 \\
J_1 = 8.0 \\
J_2 = 7.02 \\
J_3 = 2.0 \\
E_{b_1} = \sigma T_1^4 = J_1 \\
E_{b_2} = \sigma T_3^4 = J_3 \\
E_{b_1} = \sigma T_1^4 = J_1 \\
E_{b_2} = \sigma T_3^4 = J_3 \\
\]

**Solution**

This is a three-body problem, the two plates and the room, so the radiation network is shown in Figure 8-27. From the data of the problem

\[
T_1 = 1000^\circ \text{C} = 1273 \text{ K} \quad A_1 = A_2 = 0.5 \text{ m}^2 \\
T_2 = 500^\circ \text{C} = 773 \text{ K} \quad \epsilon_1 = 0.2 \quad \epsilon_2 = 0.5 \\
T_3 = 27^\circ \text{C} = 300 \text{ K} 
\]
Because the area of the room \( A_3 \) is very large, the resistance \( (1 - \epsilon_3) / \epsilon_3 A_3 \) may be taken as zero and we obtain \( E_{i3} = J_3 \). The shape factor \( F_{12} \) was given in Example 8-2:

\[
\begin{align*}
F_{12} &= 0.285 = I_{21} \\
F_{13} &= 1 - F_{12} = 0.715 \\
F_{23} &= 1 - F_{21} = 0.715
\end{align*}
\]

The resistances in the network are calculated as

\[
\begin{align*}
\frac{1 - \epsilon_1}{\epsilon_1 A_1} &= \frac{1 - 0.2}{0.2(0.5)} = 8.0 \\
\frac{1 - \epsilon_2}{\epsilon_2 A_2} &= \frac{1 - 0.5}{0.5(0.5)} = 2.0 \\
\frac{1}{A_1 F_{12}} &= \frac{1}{0.5(0.285)} = 7.018 \\
\frac{1}{A_2 F_{13}} &= \frac{1}{0.5(0.715)} = 2.797
\end{align*}
\]

Taking the resistance \( (1 - \epsilon_3) / \epsilon_3 A_3 \) as zero, we have the network as shown. To calculate the heat flows at each surface we must determine the nodal junctions \( J_1 \) and \( J_2 \). The network is solved by setting the sum of the nodal heat currents entering nodes \( J_1 \) and \( J_2 \) to zero:

node \( J_1 \):

\[
\frac{E_{i1} - J_1}{8.0} + \frac{J_2 - J_1}{7.018} + \frac{E_{i3} - J_1}{2.797} = 0 \quad \text{[a]}
\]

node \( J_2 \):

\[
\frac{J_1 - J_2}{7.018} + \frac{E_{i2} - J_2}{2.797} + \frac{E_{i3} - J_2}{2.0} = 0 \quad \text{[b]}
\]

Now

\[
\begin{align*}
E_{i1} &= \sigma T_1^4 = 148.87 \text{ kW/m}^2 \quad \text{[47,190 Btu/h · ft}^2] \\
E_{i2} &= \sigma T_2^4 = 20.241 \text{ kW/m}^2 \quad \text{[6416 Btu/h · ft}^2] \\
E_{i3} &= \sigma T_3^4 = 0.4592 \text{ kW/m}^2 \quad \text{[145.6 Btu/h · ft}^2]
\end{align*}
\]

Inserting the values of \( E_{i1}, E_{i2} \) and \( E_{i3} \) into Equations (a) and (b), we have two equations and two unknowns \( J_1 \) and \( J_2 \) that may be solved simultaneously to give

\[
J_1 = 33,469 \text{ kW/m}^2 \quad J_2 = 15,054 \text{ kW/m}^2
\]

The total heat lost by plate 1 is

\[
q_1 = \frac{E_{i3} - J_1}{(1 - \epsilon_1)/\epsilon_1 A_1} = \frac{148.87 - 33,469}{8.0} = 14,425 \text{ kW}
\]

and the total heat lost by plate 2 is

\[
q_2 = \frac{E_{i2} - J_2}{(1 - \epsilon_2)/\epsilon_2 A_2} = \frac{20.241 - 15,054}{2.0} = 2,594 \text{ kW}
\]

The total heat received by the room is

\[
q_3 = \frac{J_1 - J_3}{1/A_1 F_{13}} + \frac{J_2 - J_3}{1/A_2 F_{23}} = \frac{33,469 - 0.4592}{2.797} + \frac{15,054 - 0.4592}{2.797} = 17,020 \text{ kW} \quad \text{[58,070 Btu/h]}
\]

From an overall-balance standpoint we must have

\[
q_3 = q_1 + q_2
\]

because the net energy lost by both plates must be absorbed by the room.
EXAMPLE 8-7  

Surface in Radiant Balance

Two rectangles 50 by 50 cm are placed perpendicularly with a common edge. One surface has 
\( T_1 = 1000 \) K, \( \varepsilon_1 = 0.6 \), while the other surface is insulated and in radiant balance with a large 
surrounding room at 300 K. Determine the temperature of the insulated surface and the heat lost 
by the surface at 1000 K.

Figure Example 8-7  
(a) Schematic. (b) Network.

(a) 
Room at 300 K

\( T_1 = 1000 \) K

Insulated

(b) 

![Network diagram]

---

Solution

Although this problem involves two surfaces that exchange heat and one that is insulated or 
non-radiating, Equation (8-41) may not be used for the calculation because one of the 
heat-exchanging surfaces (the room) is not convex. The radiation network is shown in Figure Example 8-7 
where surface 3 is the room and surface 2 is the insulated surface. Note that \( J_3 = E_{b3} \) because the room is 
large and \((1 - \varepsilon_3)/\varepsilon_3 A_3 \) approaches zero. Because surface 2 is insulated it has zero heat transfer 
and \( J_2 = E_{b2} \). \( J_2 \) “floats” in the network and is determined from the overall radiant balance. From 
Figure 8-14 the shape factors are

\[ F_{12} = 0.2 = F_{21} \]

Because \( F_{11} = 0 \) and \( F_{22} = 0 \) we have

\[ F_{12} + F_{13} = 1.0 \quad \text{and} \quad F_{13} = 1 - 0.2 = 0.8 = F_{23} \]

\[ A_1 = A_2 = (0.5m)^2 = 0.25 \text{ m}^2 \]

The resistances are

\[ \frac{1 - \varepsilon_1}{\varepsilon_1 A_1} = \frac{0.4}{(0.6)(0.25)} = 2.667 \]

\[ \frac{1}{A_1 F_{13}} = \frac{1}{A_2 F_{23}} = \frac{1}{(0.25)(0.8)} = 5.0 \]

\[ \frac{1}{A_1 F_{12}} = \frac{1}{(0.25)(0.2)} = 20.0 \]

We also have

\[ E_{b3} = (5.669 \times 10^{-8})(1000)^4 = 5.669 \times 10^4 \text{ W/m}^2 \]

\[ J_3 = E_{b3} = (5.669 \times 10^{-8})(300)^4 = 459.2 \text{ W/m}^2 \]

The overall circuit is a series-parallel arrangement and the heat transfer is

\[ q = \frac{E_{b1} - E_{b2}}{R_{\text{equiv}}} \]

We have

\[ R_{\text{equiv}} = 2.667 + \frac{1}{\frac{1}{5} + 1/(20 + 5)} = 6.833 \]
and
\[ q = \frac{56.690 - 459.2}{6.833} = 8.229 \text{ kW} \quad [28.086 \text{ Btu/h}] \]

This heat transfer can also be written
\[ q = \frac{E_{11} - J_1}{(1 - e_1)e_1 A_1} \]

Inserting the values we obtain
\[ J_1 = 34.745 \text{ W/m}^2 \]

The value of \( J_2 \) is determined from proportioning the resistances between \( J_1 \) and \( J_3 \), so that
\[ \frac{J_1 - J_2}{20} = \frac{J_1 - J_3}{20 + 5} \]

and
\[ J_2 = 7316 = \frac{E_{12}}{\sigma} \]

Finally, we obtain the temperature of the insulated surface as
\[ T_2 = \left( \frac{7316}{5.669 \times 10^{-8}} \right)^{1/4} = 599.4 \text{ K} \quad [619^\circ \text{F}] \]

**Comment**

Note, once again, that we have made use of the \( J = E_1 \) relation in two instances in this example, but for two different reasons. \( J_2 = E_{12} \), because surface 2 is insulated and there is zero current flow through the surface resistance, while \( J_3 = E_{13} \), because the surface resistance for surface 3 approaches zero as \( A_3 \rightarrow \infty \).

### 8-7 INFINITE PARALLEL SURFACES

When two infinite parallel planes are considered, \( A_1 \) and \( A_2 \) are equal; and the radiation shape factor is unity since all the radiation leaving one plane reaches the other. The network is the same as in Figure 8-26, and the heat flow per unit area may be obtained from Equation (8-40) by letting \( A_1 = A_2 \) and \( F_{12} = 1.0 \). Thus
\[ q = \frac{\sigma(T_1^4 - T_2^4)}{1/e_1 + 1/e_2 - 1} \quad [8-42] \]

When two long concentric cylinders as shown in Figure 8-29 exchange heat we may again apply Equation (8-40). Rewriting the equation and noting that \( F_{12} = 1.0 \),
\[ q = \frac{\sigma A_1(T_1^4 - T_2^4)}{1/e_1 + (A_1/A_2)(1/e_2 - 1)} \quad [8-43] \]

The area ratio \( A_1/A_2 \) may be replaced by the diameter ratio \( d_1/d_2 \) when cylindrical bodies are concerned.

**Convex Object in Large Enclosure**

Equation (8-43) is particularly important when applied to the limiting case of a convex object completely enclosed by a very large concave surface. In this instance \( A_1/A_2 \rightarrow 0 \)
**Figure 8-29** Radiation exchange between two cylindrical surfaces.

\[ q = \frac{\sigma A_1 (T_1^4 - T_2^4)}{1 - \frac{A_1}{A_2} \left( \frac{1}{e_1} - 1 \right)} \]

**Figure 8-30** Radiation heat transfer between simple two-body diffuse, gray surfaces. In all cases \( f_{12} = 1.0 \).

- Small convex object in large enclosure
  \[ q = A_1 \varepsilon_1 (T_1^4 - T_2^4) \]
  for \( A_1 / A_2 \to 0 \)

- Infinite parallel planes
  \[ \langle q / A \rangle = \frac{\sigma (T_1^4 - T_2^4)}{1 - e_1 / e_2 - 1} \]
  with \( A_1 = A_2 \)

- Infinite concentric cylinders
  \[ q = \frac{\sigma A_1 (T_1^4 - T_2^4)}{1 / e_1 + 1 / e_2 - 1 / e_1 / e_2} \]
  with \( A_1 / A_2 = r_1 / r_2 \), \( r_1 / l \to 0 \)

- Concentric spheres
  \[ q = \frac{\sigma A_1 (T_1^4 - T_2^4)}{1 / e_1 + 1 / e_2 - 1 / e_1 / e_2} \]
  for \( A_1 / A_2 = (r_1 / r_2)^2 \)

and the following simple relation results:

\[ q = \sigma A_1 \varepsilon_1 (T_1^4 - T_2^4) \] \[ \textbf{[8-43a]} \]

This equation is readily applied to calculate the radiation-energy loss from a hot object in a large room.

Some of the radiation heat-transfer cases for simple two-body problems are summarized in Figure 8-30. In this figure, both surfaces are assumed to be gray and diffuse.
Open Hemisphere in Large Room

EXAMPLE 8-8

The 30-cm-diameter hemisphere in Figure Example 8-8 is maintained at a constant temperature of 500°C and insulated on its back side. The surface emissivity is 0.4. The opening exchanges radiant energy with a large enclosure at 30°C. Calculate the net radiant exchange.

![Figure Example 8-8](image)

**Solution**

This is an object completely surrounded by a large enclosure but the inside surface of the sphere is not convex; that is, it sees itself, and therefore we are not permitted to use Equation (8-43a). In the figure we take the inside of the sphere as surface 1 and the enclosure as surface 2. We also create an imaginary surface 3 covering the opening. We actually have a two-surface problem (surfaces 1 and 2) and therefore may use Equation (8-40) to calculate the heat transfer. Thus,

\[
E_{bh1} = \sigma T_1^4 = \sigma(773)^4 = 20,241 \text{ W/m}^2
\]

\[
E_{bh2} = \sigma T_2^4 = \sigma(303)^4 = 478 \text{ W/m}^2
\]

\[A_1 = 2\pi r^2 = (2)\pi(0.15)^2 = 0.1414 \text{ m}^2\]

\[
\frac{1 - \epsilon_1}{\epsilon_1} = \frac{0.6}{0.4(0.1414)} = 10.61
\]

\[
A_2 \rightarrow \infty
\]

so that

\[
\frac{1 - \epsilon_2}{\epsilon_2 A_2} \rightarrow 0
\]

Now, at this point we recognize that all of the radiation leaving surface 1 that will eventually arrive at enclosure 2 will also hit the imaginary surface 3 (i.e., \( F_{12} = F_{13} \)). We also recognize that

\[A_1 F_{13} = A_3 F_{13}\]

But, \( F_{31} = 1.0 \) so that

\[F_{13} = F_{12} = \frac{A_3}{A_1} = \frac{2\pi r^2}{2\pi r^2} = 0.5\]

Then \( 1/A_1 F_{12} = 1/(0.1414)(0.5) = 14.14 \) and we can calculate the heat transfer by inserting the quantities in Equation (8-40):

\[
q = \frac{20,241 - 478}{10.61 + 14.14 + 0} = 799 \text{ W}
\]

**Apparent Emissivity of a Cavity**

Consider the cavity shown in Figure 8-31 having an internal concave surface area \( A_i \) and emissivity \( \epsilon_i \) radiating out through the opening with area \( A_o \). The cavity exchanges radiant energy with a surrounding at \( T_s \) having an area that is large compared to the area of the opening. We want to determine a relationship for an apparent emissivity of the opening in terms of the above variables. If one considers the imaginary surface \( A_o \) covering the

![Figure 8-31](image)
opening and exchanging heat with $A_i$ we have

$$F_{oi} = 1.0$$

and, from reciprocity,

$$A_i F_{oi} = A_i F_{oj}$$

But, $F_{oi} = F_{ij}$ so that

$$A_i F_{ij} = A_0$$  \[8-44\]

The net radiant exchange of surface $A_i$ with the large enclosure $A_o$ is given by

$$q_{i-o} = \left( E_{bi} - E_{bo} \right)/A_i (1 - \epsilon_i)/\epsilon_i A_i + 1/A_i F_{ij}$$  \[8-45\]

and the net radiant energy exchange of an imaginary surface $A_o$ having an apparent emissivity $\epsilon_o$ with the large surroundings is given by Equation (8-43a) as

$$q_{i-o} = \epsilon_i A_i \left( E_{bi} - E_{bo} \right)$$  \[8-46\]

for $A_o$ at the same temperature at the cavity surface $A_i$. Substituting (8-44) in (8-45) and equating (8-45) and (8-46) gives, after algebraic manipulation,

$$\epsilon_o = \epsilon_i A_i \left[ A_o + \epsilon_i (A_i - A_o) \right]$$  \[8-47\]

We can observe the following behavior for $\epsilon_o$ in limiting cases:

$$\epsilon_o = \epsilon_i \quad \text{for} \quad A_o = A_i \quad \text{or no cavity at all}$$

**Figure 8-32** Apparent emissivity of cavity.
and

\[ \epsilon_{ud} \rightarrow 1.0 \quad \text{for} \quad A_i \gg A_0 \]

or a very large cavity. A plot of Equation (8-47) is given in Figure 8-32.

The apparent emissivity concept may also be used to analyze transient problems that admit to the lumped capacity approximation. Such an example is discussed in Appendix D, section D-6. In addition, an example is given of multiple lumped capacity formulation applied to the heating of a box of electronic components exchanging energy by convection and radiation with an enclosure.

**Effective Emissivity of Finned Surface**

_A repeating finned surface having the relative dimensions shown in Figure Example 8-9 is utilized to produce a higher effective emissivity than that for a flat surface alone. Calculate the effective emissivity of the combination of fin tip and open cavity for surface emissivities of 0.2, 0.5, and 0.8._

**Figure Example 8-9**

![Diagram of finned surface](image)

**Solution**

For unit depth in the z-dimension we have

\[ A_1 = 10, \quad A_2 = 5, \quad A_3 = (2)(25) + 10 = 60 \]

The apparent emissivity of the open cavity area \( A_1 \) is given by Equation (8-47) as

\[ \epsilon_{ud} = \epsilon A_3 / [A_1 + \epsilon (A_3 - A_1)] = 60\epsilon / (10 + 50\epsilon) \]  \[a\]

For constant _surface emissivity_ the emitted energy from the total area \( A_1 + A_2 \) is

\[ (\epsilon_{ud} A_1 + \epsilon A_2) E_b \]  \[b\]

and the energy emitted per unit area for that total area is

\[ [(\epsilon_{ud} A_1 + \epsilon A_2) / (A_1 + A_2)] E_b \]  \[c\]
The coefficient of \( E_R \) is the effective emissivity, \( \varepsilon_{\text{eff}} \) of the combination of the flat surface and open cavity. Inserting Equation (a) in (c) gives the following numerical values:

For \( \varepsilon = 0.2 \), \( \varepsilon_{\text{eff}} = 0.4667 \)
For \( \varepsilon = 0.5 \), \( \varepsilon_{\text{eff}} = 0.738 \)
For \( \varepsilon = 0.8 \), \( \varepsilon_{\text{eff}} = 0.907 \)

One could employ these effective values to calculate the radiation performance of such a finned surface in conjunction with applicable radiation properties of surrounding surfaces.

8-8 | RADIATION SHIELDS

One way of reducing radiant heat transfer between two particular surfaces is to use materials that are highly reflective. An alternative method is to use radiation shields between the heat-exchange surfaces. These shields do not deliver or remove any heat from the overall system; they only place another resistance in the heat-flow path so that the overall heat transfer is retarded. Consider the two parallel infinite planes shown in Figure 8-33a. We have shown that the heat exchange between these surfaces may be calculated with Equation (8-42). Now consider the same two planes, but with a radiation shield placed between them, as in Figure 8-33b. The heat transfer will be calculated for this latter case and compared with the heat transfer without the shield.

Since the shield does not deliver or remove heat from the system, the heat transfer between plate 1 and the shield must be precisely the same as that between the shield and plate 2, and this is the overall heat transfer. Thus

\[
\left( \frac{q_1}{A} \right)_{1-3} = \left( \frac{q_1}{A} \right)_{3-2} = \frac{q}{A}
\]

\[
\frac{q}{A} = \frac{\sigma(T_3^4 - T_2^4)}{1/\varepsilon_1 + 1/\varepsilon_3 - 1} = \frac{\sigma(T_1^4 - T_2^4)}{1/\varepsilon_3 + 1/\varepsilon_2 - 1}
\]

[8-48]

The only unknown in Equation (8-48) is the temperature of the shield \( T_3 \). Once this temperature is obtained, the heat transfer is easily calculated. If the emissivities of all three surfaces are equal, that is, \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 \), we obtain the simple relation

\[
T_3^4 = \frac{1}{2}(T_1^4 + T_2^4)
\]

[8-49]

**Figure 8-33** | Radiation between parallel infinite planes with and without a radiation shield.

\[ q/A \]

(a) 1 2

(b) 1 3 2

Shield
and the heat transfer is
\[
\frac{q}{A} = \frac{\frac{1}{\epsilon_1} \beta (T_1^4 - T_2^4)}{1/\epsilon_1 + 1/\epsilon_3 - 1}
\]

But since \( \epsilon_3 = \epsilon_2 \), we observe that this heat flow is just one-half of that which would be experienced if there were no shield present. The radiation network corresponding to the situation in Figure 8-33b is given in Figure 8-34.

By inspecting the network in Figure 8-34, we see that the radiation heat transfer is impeded by the insertion of three resistances more than would be present with just two surfaces facing each other: an extra space resistance and two extra surface resistances for the shield. The higher the reflectivity of the shield (i.e., the smaller its emissivity), the greater will be the surface resistances inserted. Even for a black shield, with \( \epsilon = 1 \) and zero surface resistance, there will still be an extra space resistance inserted in the network. As a result, insertion of any surface that intercepts the radiation path will always cause some reduction in the heat-transfer rate, regardless of its surface emissive properties.

Multiple-radiation-shield problems may be treated in the same manner as that outlined above. When the emissivities of all surfaces are different, the overall heat transfer may be calculated most easily by using a series radiation network with the appropriate number of elements, similar to the one in Figure 8-34. If the emissivities of all surfaces are equal, a rather simple relation may be derived for the heat transfer when the surfaces may be considered as infinite parallel planes. Let the number of shields be \( n \). Considering the radiation network for the system, all the “surface resistances” would be the same since the emissivities are equal. There would be two of these resistances for each shield and one for each heat-transfer surface. There would be \( n + 1 \) “space resistances,” and these would all be unity since the radiation shape factors are unity for the infinite parallel planes. The total resistance in the network would thus be
\[
R(\text{n shields}) = (2n + 2) \frac{1 - \epsilon}{\epsilon} + (n + 1)(1) = (n + 1) \left( \frac{2}{\epsilon} - 1 \right)
\]

The resistance when no shield is present is
\[
R(\text{no shield}) = \frac{1}{\epsilon} + \frac{1}{\epsilon} - 1 = \frac{2}{\epsilon} - 1
\]

We note that the resistance with the shields in place is \( n + 1 \) times as large as when the shields are absent. Thus
\[
\left( \frac{q}{A} \right) \text{ with shields} = \frac{1}{n + 1} \left( \frac{q}{A} \right) \text{ without shields}\]

if the temperatures of the heat-transfer surfaces are maintained the same in both cases. The radiation-network method may also be applied to shield problems involving cylindrical systems. In these cases the proper area relations must be used in formulating the resistance elements.
Notice that the analyses above, dealing with infinite parallel planes, have been carried out on a per-unit-area basis because all areas are the same.

**Heat-Transfer Reduction with Parallel-Plate Shield**

Two very large parallel planes with emissivities 0.3 and 0.8 exchange heat. Find the percentage reduction in heat transfer when a polished-aluminum radiation shield ($e = 0.04$) is placed between them.

**Solution**
The heat transfer without the shield is given by

\[ \frac{q}{A} = \frac{\sigma(T_1^4 - T_2^4)}{1/\epsilon_1 + 1/\epsilon_2 - 1} = 0.279\sigma(T_1^4 - T_2^4) \]

The radiation network for the problem with the shield in place is shown in Figure 8-34. The resistances are

\[ \frac{1 - \epsilon_1}{\epsilon_1} = \frac{1 - 0.3}{0.3} = 2.333 \]
\[ \frac{1 - \epsilon_3}{\epsilon_3} = \frac{1 - 0.04}{0.04} = 24.0 \]
\[ \frac{1 - \epsilon_2}{\epsilon_2} = \frac{1 - 0.8}{0.8} = 0.25 \]

The total resistance with the shield is

\[ 2.333 + (2)(24.0) + (2)(1) + 0.25 = 52.583 \]

and the heat transfer is

\[ \frac{q}{A} = \frac{\sigma(T_1^4 - T_2^4)}{52.583} = 0.01902\sigma(T_1^4 - T_2^4) \]

so that the heat transfer is reduced by 93.2 percent.

**EXAMPLE 8-11**

Open Cylindrical Shield in Large Room

The two concentric cylinders of Example 8-3 have $T_1 = 1000$ K, $\epsilon_1 = 0.8$, $\epsilon_2 = 0.2$ and are located in a large room at 300 K. The outer cylinder is in radiant balance. Calculate the temperature of the outer cylinder and the total heat lost by the inner cylinder.
Solution

The network for this problem is shown in Figure Example 8-11. The room is designated as surface 3 and $J_3 = E_{b_3}$, because the room is very large (i.e., its surface resistance is very small). In this problem we must consider the inside and outside of surface 2 and thus have subscripts $i$ and $o$ to designate the respective quantities. The shape factors can be obtained from Example 8-3 as

$$F_{12} = 0.8253 \quad F_{13} = 0.1747$$
$$F_{23i} = (2)(0.1294) = 0.2588 \quad F_{23o} = 1.0$$

Also,

$$A_1 = \pi (0.1)(0.2) = 0.06283 \, \text{m}^2$$
$$A_2 = \pi (0.2)(0.2) = 0.12566 \, \text{m}^2$$
$$E_{b_1} = (5.669 \times 10^{-8})(1000)^4 = 5.669 \times 10^4 \, \text{W/m}^2$$
$$E_{b_3} = (5.669 \times 10^{-8})(300)^4 = 459.2 \, \text{W/m}^2$$

and the resistances may be calculated as

$$\frac{1 - \epsilon_1}{\epsilon_1 A_1} = 3.979 \quad \frac{1 - \epsilon_2}{\epsilon_2 A_2} = 31.83$$
$$\frac{1}{A_1 F_{12}} = 19.28 \quad \frac{1}{A_2 F_{23i}} = 30.75$$
$$\frac{1}{A_2 F_{23o}} = 7.958 \quad \frac{1}{A_1 F_{13}} = 91.1$$

The network could be solved as a series-parallel circuit to obtain the heat transfer, but we will need the radiosities anyway, so we set up three nodal equations to solve for $J_1$, $J_{2i}$, and $J_{2o}$. We sum the currents into each node and set them equal to zero:

**node $J_1$:**

$$\frac{E_{b_1} - J_1}{3.979} + \frac{E_{b_3} - J_1}{91.1} + \frac{J_{2i} - J_1}{19.28} = 0$$

**node $J_{2i}$:**

$$\frac{J_1 - J_{2i}}{19.28} + \frac{E_{b_3} - J_{2i}}{30.75} + \frac{J_{2o} - J_{2i}}{(2)(31.83)} = 0$$

**node $J_{2o}$:**

$$\frac{E_{b_3} - J_{2o}}{7.958} + \frac{J_{2i} - J_{2o}}{(2)(31.83)} = 0$$

These equations have the solution

$$J_1 = 49,732 \, \text{W/m}^2$$
$$J_{2i} = 26,444 \, \text{W/m}^2$$
$$J_{2o} = 3346 \, \text{W/m}^2$$

The heat transfer is then calculated from

$$q = \frac{E_{b_3} - J_1}{(1 - \epsilon_1)/\epsilon_1 A_1} = \frac{56,690 - 49,732}{3.979} = 1749 \, \text{W} \quad [5968 \, \text{Btu/h}]$$

From the network we see that

$$E_{b_2} = \frac{J_{2i} + J_{2o}}{2} = \frac{26,444 + 3346}{2} = 14,895 \, \text{W/m}^2$$

and

$$T_2 = \left( \frac{14,895}{5,669 \times 10^{-8}} \right)^{1/4} = 716 \, \text{K} \quad [829^\circ \text{F}]$$
If the outer cylinder had not been in place acting as a “shield” the heat loss from cylinder 1 could have been calculated from Equation (8-43a) as

\[ q = \varepsilon \varepsilon' A \left( T_1^\gamma - T_3^\gamma \right) \]

\[ = (0.8)(0.06283)(56,690 - 459.2) = 2826 \text{ W} \] [9644 Btu/h]

**8-9 | GAS RADIATION**

Radiation exchange between a gas and a heat-transfer surface is considerably more complex than the situations described in the preceding sections. Unlike most solid bodies, gases are in many cases transparent to radiation. When they absorb and emit radiation, they usually do so only in certain narrow wavelength bands. Some gases, such as N₂, O₂, and others of nonpolar symmetrical molecular structure, are essentially transparent at low temperatures, while CO₂, H₂O, and various hydrocarbon gases radiate to an appreciable extent.

The absorption of radiation in gas layers may be described analytically in the following way, considering the system shown in Figure 8-35. A monochromatic beam of radiation having an intensity \( I_\lambda \) impinges on the gas layer of thickness \( dx \). The decrease in intensity resulting from absorption in the layers is assumed to be proportional to the thickness of the layer and the intensity of radiation at that point. Thus

\[ dI_\lambda = -\alpha_\lambda dx \] [8-51]

where the proportionality constant \( \alpha_\lambda \) is called the *monochromatic absorption coefficient*. Integrating this equation gives

\[ \int_{I_\lambda}^{I_{\lambda'}} \frac{dI_\lambda}{I_\lambda} = \int_0^x -\alpha_\lambda dx \]

or

\[ \frac{I_{\lambda'}}{I_{\lambda}} = e^{-\alpha_\lambda x} \] [8-52]

Equation (8-52) is called Beer’s law and represents the familiar exponential-decay formula experienced in many types of radiation analyses dealing with absorption. In accordance with our definitions in Section 8-3, the monochromatic transmissivity will be given as

\[ \tau_\lambda = e^{-\alpha_\lambda x} \] [8-53]

If the gas is nonreflecting, then

\[ \tau_\lambda + \alpha_\lambda = 1 \]

and

\[ \alpha_\lambda = 1 - e^{-\alpha_\lambda x} \] [8-54]

As we have mentioned, gases frequently absorb only in narrow wavelength bands. For example, water vapor has an absorptivity of about 0.7 between 1.4 and 1.5 \( \mu \)m, about 0.8 between 1.6 and 1.8 \( \mu \)m, about 1.0 between 2.6 and 2.8 \( \mu \)m, and about 1.0 between 5.5 and 7.0 \( \mu \)m. As we have seen in Equation (8-54), the absorptivity will also be a function of the thickness of the gas layer, and there is a temperature dependence as well.

The calculation of gas-radiation properties is quite complicated, and References 23 to 25 should be consulted for detailed information.
8-10 | RADIATION NETWORK FOR AN ABSORBING AND TRANSMITTING MEDIUM

The foregoing discussions have shown the methods that may be used to calculate radiation heat transfer between surfaces separated by a completely transparent medium. The radiation-network method is used to great advantage in these types of problems.

Many practical problems involve radiation heat transfer through a medium that is both absorbing and transmitting. The various glass substances are one example of this type of medium; gases are another. Some approximate transmissivities or glass substances over the wavelength range of $0.5 \mu m < \lambda < 2.5 \mu m$ are given in Table 8-3.

Keeping in mind the complications involved with the band absorption characteristics of gases, we shall now examine a simplified radiation network method for analyzing transmitting absorbing systems.

To begin, let us consider a simple case, that of two nontransmitting surfaces that see each other and nothing else. In addition, we let the space between these surfaces be occupied by a transmitting and absorbing medium. The practical problem might be that of two large planes separated by either an absorbing gas or a transparent sheet of glass or plastic. The situation is shown schematically in Figure 8-36. The transparent medium is designated by the subscript $m$. We make the assumption that the medium is nonreflecting and that Kirchhoff’s identity applies, so that

$$\alpha_m + \tau_m = 1 = \epsilon_m + \tau_m \quad [8-55]$$

The assumption that the medium is nonreflecting is a valid one when gases are considered. For glass or plastic plates this is not necessarily true, and reflectivities of the order

<table>
<thead>
<tr>
<th>Glass</th>
<th>$\tau (0.5 \mu m &lt; \lambda &lt; 2.5 \mu m)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soda lime glass</td>
<td></td>
</tr>
<tr>
<td>Thickness</td>
<td></td>
</tr>
<tr>
<td>1.6 mm</td>
<td>0.9</td>
</tr>
<tr>
<td>6.4 mm</td>
<td>0.75</td>
</tr>
<tr>
<td>9.5 mm</td>
<td>0.7</td>
</tr>
<tr>
<td>12.7 mm</td>
<td>0.65</td>
</tr>
<tr>
<td>Aluminum silicate, thickness = 12.7 mm</td>
<td>0.85</td>
</tr>
<tr>
<td>Borosilicate</td>
<td>12.7 mm</td>
</tr>
<tr>
<td>Fused silica</td>
<td>12.7 mm</td>
</tr>
<tr>
<td>Pyrex</td>
<td>12.7 mm</td>
</tr>
</tbody>
</table>

Figure 8-36 | Radiation system consisting of a transmitting medium between two planes.
of 0.1 are common for many glass substances. In addition, the transmissive properties of
glasses are usually limited to a narrow wavelength band between about 0.2 and 4 \mu m. Thus
the analysis that follows is highly idealized and serves mainly to furnish a starting point
for the solution of problems in which transmission of radiation must be considered. Other
complications with gases are mentioned later in the discussion. When both reflection and
transmission must be taken into account, the analysis techniques discussed in Section 8-12
must be employed.

Returning to the analysis, we note that the medium can emit and transmit radiation
from one surface to the other. Our task is to determine the network elements to use in
describing these two types of exchange processes. The transmitted energy may be analyzed
as follows. The energy leaving surface 1 that is transmitted through the medium and arrives
at surface 2 is

\[ J_1 A_1 F_{12 \tau_m} \]

and the energy leaving surface 2 and arrives at surface 1 is

\[ J_2 A_2 F_{21 \tau_m} \]

The net exchange in the transmission process is therefore

\[ q_{1-2 \text{transmitted}} = A_1 F_{12 \tau_m} (J_1 - J_2) \]
\[ q_{1-2 \text{transmitted}} = \frac{J_1 - J_2}{1/A_1 F_{12 \tau_m} (1 - \epsilon_m)} \quad [8-56] \]

and the network element that may be used to describe this process is shown in Figure 8-37.

Now consider the exchange process between surface 1 and the transmitting medium.
Since we have assumed that this medium is nonreflecting, the energy leaving the medium
(other than the transmitted energy, which we have already considered) is precisely the
energy emitted by the medium

\[ J_n = \epsilon_m E_{in} \]

And of the energy leaving the medium, the amount which reaches surface 1 is

\[ A_n F_{n1} J_n = A_n F_{n1} \epsilon_m E_{in} \]

Of that energy leaving surface 1, the quantity that reaches the transparent medium is

\[ J_1 A_1 F_{1n} \epsilon_n \]

At this point we note that absorption in the medium means that the incident radiation has “reached” the medium. Consistent with the above relations, the net energy exchange between the medium and surface 1 is the difference between the amount emitted by the medium toward surface 1 and that absorbed which emanated from surface 1. Thus

\[ q_{n-1 \text{net}} = A_n F_{n1} \epsilon_m E_{in} - J_1 A_1 F_{1n} \epsilon_n \]

Using the reciprocity relation

\[ A_1 F_{1n} = A_n F_{n1} \]

we have

\[ q_{n-1 \text{net}} = \frac{E_{in1} - J_1}{1/A_1 F_{1n} \epsilon_n} \quad [8-57] \]

This heat-exchange process is represented by the network element shown in Figure 8-38. The
total network for the physical situation of Figure 8-36 is shown in Figure 8-39.
If the transport medium is maintained at some fixed temperature, then the potential \( E_{in} \) is fixed according to
\[ E_{in} = \alpha_i \frac{A_i}{\alpha_i} \]

On the other hand, if no net energy is delivered to the medium, then \( E_{in} \) becomes a floating node, and its potential is determined by the other network elements.

In reality, the radiation shape factors \( F_{1-2}, F_{1-n}, \) and \( F_{2-n} \) are unity for this example, so that the expression for the heat flow could be simplified to some extent; however, these shape factors are included in the network resistances for the sake of generality in the analysis.

When the practical problem of heat exchange between gray surfaces through an absorbing gas is encountered, the major difficulty is that of determining the transmissivity and emissivity of the gas. These properties are functions not only of the temperature of the gas, but also of the thickness of the gas layer; that is, thin gas layers transmit more radiation than thick layers. The usual practical problem almost always involves more than two heat-transfer surfaces, as in the simple example given above. As a result, the transmissivities between the various heat-transfer surfaces can be quite different, depending on their geometric orientation. Since the temperature of the gas will vary, the transmissive and emissive properties will vary with their location in the gas. One way of handling this situation is to divide the gas body into layers and set up a radiation network accordingly, letting the potentials of the various nodes “float,” and thus arriving at the gas-temperature distribution.

Even with this procedure, an iterative method must eventually be employed because the radiation properties of the gas are functions of the unknown “floating potentials.” Naturally, if the temperature of the gas is uniform, the solution is much easier.

We shall not present the solution of a complex gas-radiation problem since the tedious effort required for such a solution is beyond the scope of our present discussion; however, it is worthwhile to analyze a two-layer transmitting system in order to indicate the general scheme of reasoning that might be applied to more complex problems.

Consider the physical situation shown in Figure 8-40. Two radiating and absorbing surfaces are separated by two layers of transmitting and absorbing media. These two layers might represent two sheets of transparent media, such as glass, or they might represent the division of a separating gas into two parts for purposes of analysis. We designate the two transmitting and absorbing layers with the subscripts \( \mu \) and \( \nu \). The energy exchange between surface 1 and \( \mu \) is given by
\[ q_{1-\mu} = A_1 F_{1\mu} \epsilon_\mu j_1 - A_\mu F_{\mu \mu} \epsilon_\mu E_{in} = \frac{J_1 - E_{1\mu}}{1/A_1 F_{1\mu} \epsilon_\mu} \quad [8-58] \]
and that between surface 2 and \( \nu \) is
\[ q_{2-\nu} = A_1 F_{2\nu} \epsilon_\nu j_2 - A_\nu F_{\nu \nu} \epsilon_\nu E_{in} = \frac{J_2 - E_{2\nu}}{1/A_2 F_{2\nu} \epsilon_\nu} \quad [8-59] \]
Figure 8-40 | Radiation system consisting of two transmitting layers between two planes.

\[
q_{1-2} = A_1 F_{12} J_1 \tau_n \varepsilon_n = A_1 F_{12} J_1 (1 - \varepsilon_n) (1 - \varepsilon_n)
\]

and of that energy leaving surface 2, the amount arriving at surface 1 is

\[
q_{2-1} = A_2 F_{21} J_2 \tau_n \varepsilon_n = A_2 F_{21} J_2 (1 - \varepsilon_n) (1 - \varepsilon_n)
\]

so that the net energy exchange by transmission between surfaces 1 and 2 is

\[
q_{1-2}^{\text{transmitted}} = \frac{A_1 F_{12} (1 - \varepsilon_n) (1 - \varepsilon_n) J_1 - J_2}{1/A_1 F_{12} (1 - \varepsilon_n) (1 - \varepsilon_n)} \tag{8.60}
\]

and the network element representing this transmission is shown in Figure 8-41. Of that energy leaving surface 1, the amount that is absorbed in \( n \) is

\[
q_{1-n} = A_1 F_{1n} J_1 \tau_n \varepsilon_n = A_1 F_{1n} J_1 (1 - \varepsilon_n) \varepsilon_n
\]

Also,

\[
q_{n-1} = A_n F_{n1} J_n \tau_n \varepsilon_n = A_n F_{n1} J_n (1 - \varepsilon_n) \varepsilon_n
\]

since

\[
J_n = \varepsilon_n E_{b,n}
\]

The net exchange between surface 1 and \( n \) is therefore

\[
q_{1-n}^{\text{net}} = A_1 F_{1n} (1 - \varepsilon_n) \varepsilon_n (J_1 - E_{b,n}) = \frac{J_1 - E_{b,n}}{1/A_1 F_{1n} (1 - \varepsilon_n) \varepsilon_n} \tag{8.61}
\]

and the network element representing this situation is shown in Figure 8-42. In like manner, the net exchange between surface 2 and \( n \) is

\[
q_{2-n}^{\text{net}} = \frac{J_2 - E_{b,n}}{1/A_2 F_{2n} (1 - \varepsilon_n) \varepsilon_n}
\]

Of that radiation leaving \( n \), the amount absorbed in \( n \) is

\[
q_{n-n} = J_n A_n F_{nn} \varepsilon_n = A_n J_n \varepsilon_n E_{b,n}
\]

and

\[
q_{n-n} = A_n J_n \varepsilon_n E_{b,n}
\]
so that the net energy exchange between \( m \) and \( n \) is

\[
q_{m-n,\text{net}} = A_{m,n} F_{m,n} \varepsilon_r \varepsilon_p (E_{m,r} - E_{n,r}) = \frac{E_{m,r} - E_{n,r}}{1/A_{m,n} F_{m,n} \varepsilon_r \varepsilon_p} \tag{8-62}
\]

and the network element representing this energy transfer is given in Figure 8-43.

The final network for the entire heat-transfer process is shown in Figure 8-44, with the surface resistances added. If the two transmitting layers \( m \) and \( n \) are maintained at given temperatures, the solution to the network is relatively easy to obtain because only two unknown potentials \( J_1 \) and \( J_2 \) need be determined to establish the various heat-flow quantities. In this case the two transmitting layers will either absorb or lose a certain quantity of energy, depending on the temperature at which they are maintained.

When no net energy is delivered to the transmitting layers, nodes \( E_{m,r} \) and \( E_{n,r} \) must be left “floating” in the analysis, and for this particular system four nodal equations would be required for a solution of the problem.

---

**EXAMPLE 8-12**

Network for Gas Radiation Between Parallel Plates

Two large parallel planes are at \( T_1 = 800 \text{ K} \), \( \varepsilon_1 = 0.3 \), \( T_2 = 400 \text{ K} \), \( \varepsilon_2 = 0.7 \) and are separated by a gray gas having \( \varepsilon_g = 0.2 \), \( \tau_g = 0.8 \). Calculate the heat-transfer rate between the two planes and the temperature of the gas using a radiation network. Compare with the heat transfer without presence of the gas.

**Solution**

The network shown in Figure 8-39 applies to this problem. All the shape factors are unity for large planes and the various resistors can be computed on a unit-area basis as

\[
\begin{align*}
\frac{1 - \varepsilon_1}{\varepsilon_1} &= \frac{0.7}{0.3} = 2.333 \\
\frac{1 - \varepsilon_2}{\varepsilon_2} &= \frac{0.3}{0.7} = 0.4286 \\
F_{12} &= \frac{1}{F_{12}} = \frac{1}{1 - 0.2} = \frac{1}{0.2} = 5.0 \\
E_{b1} &= aT_1^4 = 23,220 \text{ W/m}^2 \\
E_{b2} &= aT_2^4 = 1451 \text{ W/m}^2
\end{align*}
\]
The equivalent resistance of the center “triangle” is
\[ R = \frac{1}{1/1.25 + 1/(5.0 + 5.0)} = 1.111 \]
The total heat transfer is then
\[ \frac{q}{A} = \frac{E_{I_3} - E_{I_2}}{\sum R} = \frac{23.200 - 1451}{2.333 + 1.111 + 0.4286} = 5616 \text{ W/m}^2 \]
If there were no gas present the heat transfer would be given by Equation (8-42):
\[ \frac{q}{A} = \frac{23.200 - 1451}{1/0.3 + 1/0.7 - 1} = 5781 \text{ W/m}^2 \]
The radiosities may be computed from
\[ \frac{q}{A} = (E_{I_3} - h_1) \left( \frac{\epsilon_1}{1 - \epsilon_1} \right) = (E_{I_2} - h_2) \left( \frac{\epsilon_2}{1 - \epsilon_2} \right) = 5616 \text{ W/m}^2 \]
which gives \( J_1 = 10,096 \text{ W/m}^2 \) and \( J_2 = 3858 \text{ W/m}^2 \). For the network \( I_{I_{art}} \) is just the mean of these values
\[ I_{I_{art}} = \frac{1}{2} (10,096 + 3858) = 6977 = \sigma T_e^4 \]
so that the temperature of the gas is
\[ T_e = 592.3 \text{ K} \]

8-11 | RADIATION EXCHANGE WITH SPECULAR SURFACES

All the preceding discussions have considered radiation exchange between diffuse surfaces. In fact, the radiation shape factors defined by Equation (8-21) hold only for diffuse radiation because the radiation was assumed to have no preferred direction in the derivation of this relation. In this section we extend the analysis to take into account some simple geometries containing surfaces that may have a specular type of reflection. No real surface is completely diffuse or completely specular. We shall assume, however, that all the surfaces to be considered emit radiation diffusely but that they may reflect radiation partly in a specular manner and partly in a diffuse manner. We therefore take the reflectivity to be the sum of a specular component and a diffuse component:
\[ \rho = \rho_s + \rho_d \]
It is still assumed that Kirchhoff’s identity applies so that
\[ \epsilon = \alpha = 1 - \rho \]
The net heat lost by a surface is the difference between the energy emitted and absorbed:
\[ q = A(\epsilon E_b - \alpha G) \]
We define the diffuse radiosity \( J_D \) as the total diffuse energy leaving the surface per unit area and per unit time, or
\[ J_D = \epsilon E_b + \rho_d G \]
Solving for the irradiation \( Q \) from Equation (8-66) and inserting in Equation (8-65) gives

\[
q = \frac{A}{\rho D} \left[ E_b (\epsilon + \rho D) - J_D \right]
\]

or, written in a different form,

\[
q = \frac{E_b - J_D}{\epsilon A (1 - \rho_s)}
\]

where \( 1 - \rho_s \) has been substituted for \( \epsilon + \rho D \). It is easy to see that Equation (8-67) may be represented with the network element shown in Figure 8-45. A quick inspection will show that this network element reduces to that in Figure 8-24 for the case of a surface that reflects in only a diffuse manner (i.e., for \( \rho_s = 0 \)).

Now let us compute the radiation exchange between two specular-diffuse surfaces. For the moment, we assume that the surfaces are oriented as shown in Figure 8-46. In this arrangement any diffuse radiation leaving surface 1 that is specularly reflected by 2 will not be reflected directly back to 1. This is an important point, for in eliminating such reflections we are considering only the direct diffuse exchange between the two surfaces. In subsequent paragraphs we shall show how the specular reflections must be analyzed. For the surfaces in Figure 8-46 the diffuse exchanges are given by

\[
q_{1 \rightarrow 2} = J_{1D} A_1 F_{12}(1 - \rho_{2s})
\]

\[
q_{2 \rightarrow 1} = J_{2D} A_2 F_{21}(1 - \rho_{1s})
\]

Equation (8-68) expresses the diffuse radiation leaving 1 that arrives at 2 and that may contribute to a diffuse radiosity of surface 2. The factor \( 1 - \rho_s \) represents the fraction absorbed plus the fraction reflected diffusely. The inclusion of this factor is important because we are considering only diffuse direct exchange, and thus must leave out the specular-reflection contribution for now. The net exchange is given by the difference between Equations (8-68) and (8-69), according to Reference 21.

\[
q_{12} = \frac{J_{1D}(1 - \rho_{1s}) - J_{2D}(1 - \rho_{2s})}{A_1 F_{12}(1 - \rho_{1s})(1 - \rho_{2s})}
\]

The network element representing Equation (8-70) is shown in Figure 8-47.

To analyze specular reflections we utilize a technique presented in References 12 and 13. Consider the enclosure with four long surfaces shown in Figure 8-48. Surfaces 1, 2, and 4 reflect diffusely, while surface 3 has both a specular and a diffuse component of reflection. The dashed lines represent mirror images of the surfaces 1, 2, and 4 in surface 3. (A specular reflection produces a mirror image.) The nomenclature 2 (3) designates the mirror image of surface 2 in mirror 3.

Now consider the radiation leaving 2 that arrives at 1. There is a direct diffuse radiation of

\[
\langle q_{2 \rightarrow 1} \rangle_{	ext{diffuse}} = J_2 A_2 F_{21}
\]

Part of the diffuse radiation from 2 is specularly reflected in 3 and strikes 1. This specularly reflected radiation acts like diffuse energy coming from the image surface 2 (3). Thus we may write

\[
\langle q_{2 \rightarrow 1} \rangle_{	ext{specular}} = J_2 A_2 F_{23}(1/\rho_3)
\]

The radiation shape factor \( F_{23} \) is the one between surface 2 (3) and surface 1. The reflectivity \( \rho_3 \) is inserted because only this fraction of the radiation gets to 1. Of course,
\( \lambda_2 = \lambda_{2,3} \). We now have

\[
q_{2 \rightarrow 1} = J_2 \lambda_2 (F_{21} + \rho_{b2} F_{21;1}) \tag{8-73}
\]

Similar reasoning leads to

\[
q_{1 \rightarrow 2} = J_1 \lambda_1 (F_{12} + \rho_{b1} F_{12;1}) \tag{8-74}
\]

Combining Equations (8-73) and (8-74) and making use of the reciprocity relation \( \lambda_1 F_{12} = \lambda_2 F_{21} \) gives

\[
q_{12} = \frac{J_1 - J_2}{1/[\lambda_1 (F_{12} + \rho_{b1} F_{12;1})]} \tag{8-75}
\]

The network element represented by Equation (8-75) is shown in Figure 8-49.

Analogous network elements may be developed for radiation between the other surfaces in Figure 8-48, so that the final complete network becomes as shown in Figure 8-50. It is to be noted that the elements connecting to \( J_{3D} \) are simple modifications of the one shown in Figure 8-47 since \( \rho_{13} = \rho_{23} = \rho_{43} = 0 \). An interesting observation can be made about this network for the case where \( \rho_{3D} = 0 \). In this instance surface 3 is completely specular and

\[
J_{3D} = \epsilon_3 E_{33}
\]

so that we are left with only three unknowns, \( J_1, J_2, \) and \( J_4 \), when surface 3 is completely specular-reflecting.

Now let us complicate the problem a step further by letting the enclosure have two specular-diffuse surfaces, as shown in Figure 8-51. In this case multiple images may be formed as shown. Surface 1 (3, 2) represents the image of 1 after it is viewed first through 3 and then through 2. In other words, it is the image of surface 1 (3) in mirror 2. At the same location is surface 1 (2, 3), which is the image of surface 1 (2) in mirror 3.

This problem is complicated because multiple specular reflections must be considered. Consider the exchange between surfaces 1 and 4. Diffuse energy leaving 1 can arrive at 4 in

---

**Figure 8-49** Network element for Equation (8-75).

**Figure 8-50** Complete radiation network for system in Figure 8-48.
Figure 8-51 | System with two specular-diffuse surfaces.

<table>
<thead>
<tr>
<th>4 (3)</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (3)</td>
<td>3</td>
</tr>
<tr>
<td>2 (3)</td>
<td>2</td>
</tr>
<tr>
<td>1 (2, 3)</td>
<td>1</td>
</tr>
<tr>
<td>4 (2, 3)</td>
<td>4</td>
</tr>
<tr>
<td>4 (3, 2)</td>
<td>4</td>
</tr>
</tbody>
</table>

1, 4 Diffuse reflecting
2, 3 Specular-diffuse reflecting

five possible ways:

direct: \( J_1 A_4 F_{14} \)
reflection in 2 only: \( J_1 A_4 F_{1(2),4/2} \)
reflection in 3 only: \( J_1 A_4 F_{1(3),4/3} \) [8-76]
reflection first in 2 and then in 3: \( J_1 A_4 F_{1(2),3/2} F_{1(2,3),4} \)
reflection first in 3 and then in 2: \( J_1 A_4 F_{1(3),2/3} F_{1(3),2} F_{1(3,2),4} \) [8-77]

The last shape factor, \( F_{1(3,2),4} \), is zero because surface 1 (3, 2) cannot see surface 4 when looking through mirror 2. On the other hand, \( F_{1(2,3),4} \) is not zero because surface 1 (2, 3) can see surface 4 when looking through mirror 3. The sum of the above terms is given as

\[
q_{1\rightarrow4} = J_1 A_4 F_{14} + \rho_2 F_{1(2,3),4} F_{1(2),4} + \rho_3 F_{1(3,2),4} F_{1(3),2} F_{1(3),2} F_{1(3,2),4} \] [8-77]

In a similar manner,

\[
q_{4\rightarrow1} = J_4 A_4 F_{4(3),2} F_{4(3,2),1} \] [8-78]

Subtracting these two equations and applying the usual reciprocity relations gives the network element shown in Figure 8-52.

Now consider the diffuse exchange between surfaces 1 and 3. Of the energy leaving 1, the amount which contributes to the diffuse radiosity of surface 3 is

\[
q_{1\rightarrow3} = J_1 A_1 F_{13} (1 - \rho_3) + J_1 A_1 F_{1(2,3),3} (1 - \rho_3) \] [8-79]

The first term represents the direct exchange, and the second term represents the exchange after one specular reflection in mirror 2. As before, the factor \( 1 - \rho_3 \) is included to leave out of consideration the specular reflection from 3. This reflection, of course, is taken into account in other terms. The diffuse energy going from 3 to 1 is

\[
q_{3\rightarrow1} = J_3 A_3 F_{31} + J_3 A_3 F_{3(2),1} \] [8-80]

The first term is the direct radiation, and the second term is that which is specularly reflected in mirror 2. Combining Equations (8-79) and (8-80) gives the network element shown in Figure 8-53.

Figure 8-52 | Network element representing exchange between surfaces 1 and 4 of Figure 8-51.

\[
q_{14} = \frac{J_1 A_1 (F_{14} + \rho_2 F_{1(2),4} + \rho_3 F_{1(3),4} + \rho_3 \rho_2 F_{1(2,3),4})}{J_4}
\]
The above two elements are typical for the enclosure of Figure 8-51 and the other elements may be constructed by analogy. Thus the final complete network is given in Figure 8-54.

If both surfaces 2 and 3 are pure specular reflectors, that is,
\[
\rho_{23} = \rho_{32} = 0
\]

we have
\[
J_{2D} = c_2 E_{12}, \quad J_{3D} = c_3 E_{13}
\]

and the network involves only two unknowns, \( J_1 \) and \( J_4 \), under these circumstances.

We could complicate the calculation further by installing the specular surfaces opposite each other. In this case there would be an infinite number of images, and a series solution would have to be obtained; however, the series for such problems usually converge rather rapidly. The reader should consult Reference 13 for further information on this aspect of radiation exchange between specular surfaces.

8-12 | RADIATION EXCHANGE WITH TRANSMITTING, REFLECTING, AND ABSORBING MEDIA

We now consider a simple extension of the presentations in Sections 8-10 and 8-11 to analyze a medium where reflection, transmission, and absorption modes are all important.
As in Section 8-10, we shall analyze a system consisting of two parallel diffuse planes with a medium in between that may absorb, transmit, and reflect radiation. For generality we assume that the surface of the transmitting medium may have both a specular and a diffuse component of reflection. The system is shown in Figure 8-55.

For the transmitting medium \( m \) we have

\[
\alpha_m + \rho_m D + \rho_m r + \tau_m = 1 \tag{8-81}
\]

Also

\[
\epsilon_m = \alpha_m
\]

The diffuse radiosity of a particular surface of the medium is defined by

\[
J_{mD} = \epsilon_m E_{m} + \rho_m D G \tag{8-82}
\]

where \( G \) is the irradiation on the particular surface. Note that \( J_{mD} \) no longer represents the total diffuse energy leaving a surface. Now it represents only emission and diffuse reflection. The transmitted energy will be analyzed with additional terms. As before, the heat exchange is written

\[
q = A(cE_m - \sigma G) \tag{8-83}
\]

Solving for \( G \) from Equation (8-82) and making use of Equation (8-81) gives

\[
q = \frac{E_{bn} - J_{mD}}{\rho_m D [1 - \tau_m - \rho_m r]} \tag{8-84}
\]

The network element representing Equation (8-84) is shown in Figure 8-56. This element is quite similar to the one shown in Figure 8-45, except that here we must take the transmissivity into account.

The transmitted heat exchange between surfaces 1 and 2 is the same as in Section 8-10; that is,

\[
q = \frac{J_1 - J_2}{1/A_1 A_2 \tau_{m1}} \tag{8-85}
\]

The heat exchange between surface 1 and \( m \) is computed in the following way. Of that energy leaving surface 1, the amount that arrives at \( m \) and contributes to the diffuse radiosity of \( m \) is

\[
q_{1 \rightarrow m} = J_1 A_1 F_{1m}(1 - \tau_m - \rho_m r) \tag{8-86}
\]

The diffuse energy leaving \( m \) that arrives at 1 is

\[
q_{m \rightarrow 1} = J_{mD} A_m F_{m1} \tag{8-87}
\]

Subtracting (8-87) from (8-86) and using the reciprocity relation

\[
A_1 F_{1m} = A_m F_{m1}
\]

gives

\[
q_{1m} = \frac{J_1 - J_{mD}}{1/A_1 F_{1m}(1 - \tau_m - \rho_m r)} \tag{8-88}
\]

The network element corresponding to Equation (8-89) is quite similar to the one shown in Figure 8-50. An equation similar to Equation (8-89) can be written for the radiation exchange between surface 2 and \( m \). Finally, the complete network may be drawn as in Figure 8-57. It is to be noted that \( J_{mD} \) represents the diffuse radiosity of the left side of \( m \), while \( J_{m' D} \) represents the diffuse radiosity of the right side of \( m \).
If \( \dot{m} \) is maintained at a fixed temperature, then \( J_1 \) and \( J_2 \) must be obtained as a solution to nodal equations for the network. On the other hand, if no net energy is delivered to \( \dot{m} \), then \( E_{\text{in}} \) is a floating node, and the network reduces to a simple series-parallel arrangement. In this latter case the temperature of \( \dot{m} \) must be obtained by solving the network for \( E_{\text{in}} \).

We may extend the analysis a few steps further by distinguishing between specular and diffuse transmission. A specular transmission is one where the incident radiation goes "straight through" the material, while a diffuse transmission is encountered when the incident radiation is scattered in passing through the material, so that it emerges from the other side with a random spatial orientation. As with reflected energy, the assumption is made that the transmissivity may be represented with a specular and a diffuse component:

\[
\tau = \tau_s + \tau_d
\]  

[8-89]

The diffuse radiosity is still defined as in Equation (8-82), and the net energy exchange with a transmitting surface is given by Equation (8-84). The analysis of transmitted energy exchange with other surfaces must be handled somewhat differently, however.

Consider, for example, the arrangement in Figure 8-58. The two diffuse opaque surfaces are separated by a specular-diffuse transmitting and reflecting plane. For this example all planes are assumed to be infinite in extent. The specular-transmitted exchange between surfaces 1 and 3 may be calculated immediately with

\[
(q_{13})_{\text{specular\-transmitted}} = \frac{J_1 - J_3}{1/A_1 F_{13} \tau_d}
\]  

[8-90]

The diffuse-transmitted exchange between 1 and 3 is a bit more complicated. The energy leaving 1 that is transmitted diffusely through 2 is

\[
J_1 A_1 F_{12} \tau_d
\]

Of this amount transmitted through 2, the amount that arrives at 3 is

\[
(q_{13})_{\text{diffuse\-transmitted}} = J_1 A_1 F_{12} \tau_d F_{23}
\]  

[8-91]

Similarly, the amount leaving 3 that is diffusely transmitted to 1 is

\[
(q_{31})_{\text{diffuse\-transmitted}} = J_3 A_3 F_{32} \tau_d F_{21}
\]  

[8-92]
Now, by making use of the reciprocity relations, $A_1 F_{12} = A_2 F_{21}$ and $A_3 F_{32} = A_2 F_{23}$, subtraction of Equation (8-92) from Equation (8-91) gives

$$\langle q_{12} \rangle_{\text{net diffuse-transmitted}} = \frac{J_1 - J_3}{1/A_1 F_{12} F_{23} t_2 t_3} \quad \text{[8-93]}$$

**Apparent Emissivity of Cavity with Transparent Cover**

Using similar reasoning to that which enabled us to arrive at a relation for the apparent emissivity of a cavity in Equation (8-47), we may consider the effect a transparent covering may have on $\varepsilon_2$. The covered cavity is indicated in Figure 8-59 with the characteristics of the cover described by

$$\varepsilon_2 + \rho_2 + t_2 = 1.0$$

The corresponding radiation network for this cavity exchanging heat with a large surrounding at $T_s$ is shown in Figure 8-60. As in Equation (8-47), we define the apparent emissivity
Figure 8-60  
Radiation network for cavity with partially transparent cover.

\[
q = \epsilon_\alpha A_2 (E_{1\alpha} - E_{1\beta}) \quad [8-94]
\]

The shape factors in the radiation network are determined as

\[
F_{21} = 1, \quad A_1 F_{12} = A_2 F_{21} = A_2, \quad F_{23} = 1, \quad A_2 F_{23} = A_2
\]

But, \( F_{12} = F_{13} \) so that \( A_1 F_{13} = A_2 \). The heat exchange is determined from the network as

\[
q = (E_{1\alpha} - E_{1\beta}) / \Sigma R \quad [8-95]
\]

where \( \Sigma R \) is the equivalent resistance for the series parallel network. Performing the necessary algebraic manipulation to evaluate \( \Sigma R \), and equating the heat transfers in (8-95) and (8-94) gives the relation for the apparent emissivity as

\[
\epsilon_\alpha/(\tau_2 + \epsilon_2/2) = K/[A_2/A_1](1 - \epsilon_1 + K) \quad [8-96]
\]

where

\[
K = \epsilon_1/(\tau_2 + \epsilon_2/2) \quad [8-96a]
\]

We may note the following behavior for three limiting conditions.

1. For \( \tau_2 \to 1 \), we have an open cavity and the behavior approaches that described by Equation (8-47).
2. For \( \tau_2 \to 1 \) and \( A_2 = A_1 \), we have neither cavity nor cover and \( \epsilon_\alpha \to \epsilon_1 \).
3. For \( A_1 \gg A_2 \) we have a very large cavity with \( \epsilon_\alpha \to \tau_2 + \epsilon_2/2 \).

The behavior of \( \epsilon_\alpha \) is displayed graphically in Figure 8-61.

**EXAMPLE 8-13  Cavity with Transparent Cover**

The rectangular cavity between the fins of Example 8-9 has \( \epsilon_1 = 0.5 \) along with a cover placed over the opening with the properties

\[
\tau_2 = 0.5, \quad \rho_2 = 0.1, \quad \epsilon_2 = 0.4
\]

Calculate the apparent emissivity of the covered opening.
Figure 8-61 | Apparent emissivity of cavity with partially transparent cover.

Solution

Per unit depth in the z direction we have \( A_1 = 225 + 25 + 10 = 60 \) and \( A_2 = 10 \). We may evaluate \( K \) from Equation (8-96a)

\[
K = 0.5/(0.5 + 0.4/2) = 5/7
\]

The value of \( \varepsilon_a \) is then computed from Equation (8-96) as

\[
\varepsilon_a = (0.5 + 0.4/2)(5/7)/[(10/60)(1 - 0.5) + 5/7] = 0.6269
\]

If there were no cover present, the value of \( \varepsilon_a \) would be given by Equation (8-47) as

\[
\varepsilon_a = (0.5)(60)/(1 + 0.5)(60 - 10) = 0.8571
\]

Obviously, the presence of the cover reduces the heat transfer for values of \( \tau_2 < 1.0 \).

Transmitting and Reflecting System

for Furnace Opening

A furnace at 1000°C has a small opening in the side that is covered with a quartz window having the following properties:

\[
\begin{align*}
0 < \lambda < 4 \text{ \mu m} & \quad \tau = 0.9 & \quad \varepsilon = 0.1 & \quad \rho = 0 \\
4 < \lambda < \infty & \quad \tau = 0 & \quad \varepsilon = 0.8 & \quad \rho = 0.2
\end{align*}
\]
The interior of the furnace may be treated as a blackbody. Calculate the radiation lost through the quartz window to a room at 30°C. Diffuse surface behavior is assumed.

**Figure Example 8-14**

![Diagram of a network with nodes and edges labeled with variables and constants.]

**Solution**

The diagram for this problem is shown in Figure Example 8-14. Because the room is large it may be treated as a blackbody also. We shall analyze the problem by calculating the heat transfer for each wavelength band and then adding them together to obtain the total. The network for each band is a modification of Figure 8-57, as shown here for black furnace and room. We shall make the calculation for unit area; then

\[ A_1 = A_2 = A_3 = 1.0 \]
\[ F_{12} = 1.0 \quad F_{13} = 1.0 \quad F_{32} = 1.0 \]

The total emissive powers are

\[ E_{b_1} = (5.669 \times 10^{-8})(1273)^4 \approx 1.4887 \times 10^5 \text{ W/m}^2 \]
\[ E_{b_3} = (5.669 \times 10^{-8})(303)^4 = 477.8 \text{ W/m}^2 \]

To determine the fraction of radiation in each wavelength band, we calculate

\[ \lambda T_1 = (4)(1273) = 5092 \mu \text{m} \cdot \text{K} \]
\[ \lambda T_3 = (4)(303) = 1212 \mu \text{m} \cdot \text{K} \]

Consulting Table 8-1, we find

\[ E_{b_1}(0 - 4 \mu \text{m}) = 0.6450 E_{b_3} = 96.021 \text{ W/m}^2 \]
\[ E_{b_3}(0 - 4 \mu \text{m}) = 0.00235 E_{b_3} = 1.123 \text{ W/m}^2 \]
\[ E_{b_3}(4 - \infty) = (1 - 0.6450) E_{b_3} = 52,849 \text{ W/m}^2 \]
\[ E_{b_3}(4 - \infty) = (1 - 0.00235) E_{b_3} = 476.7 \text{ W/m}^2 \]

We now apply these numbers to the network for the two wavelength bands, with unit areas. $0 < \lambda < 4 \mu \text{m}$ band:

\[
\frac{1}{F_{13} T_2} = 0.9 \quad \frac{1}{F_{32}(1 - T_2)} = 1.0 \quad \frac{1}{F_{12}(1 - T_2)} = 1.0
\]
\[
\frac{\rho_2}{e_2(1 - T_2)} = 0
\]
The net heat transfer from the network is then

\[ q = \frac{E_{i1} - E_{i3}}{R_{\text{equiv}}} = \frac{96.021 - 1.123}{1.0526} = 91.219 \text{ W/m}^2 \quad 0 < \lambda < 4 \mu \text{m} \]

\( 4 \mu \text{m} < \lambda < +\infty \) band:

\[ \frac{1}{F_{13}} = \infty \quad \frac{1}{F_{32}(1 - \tau_2)} = \frac{1}{F_{12}(1 - \tau_2)} = 1.0 \]

\[ \frac{\rho_2}{c_2(1 - \tau_2)} = \frac{0.2}{0.8} = 0.25 \]

The net heat transfer from the network is

\[ q = \frac{E_{i1} - E_{i3}}{1 + 0.25 + 0.25 + 1} = \frac{52.849 - 476.7}{2.5} = 20,949 \text{ W/m}^2 \quad 4 < \lambda < \infty \]

The total heat loss is then

\[ q_{\text{total}} = 91,219 + 20,949 = 112,168 \text{ W/m}^2 \quad \text{[35,560 Btu/h} \cdot \text{ft}^2\text{]} \]

With no window at all, the heat transfer would have been the difference in blackbody emissive powers,

\[ q - E_{i1} = 1.4887 \times 10^5 - 477.8 = 1.4839 \times 10^5 \text{ W/m}^2 \quad \text{[47,040 Btu/h} \cdot \text{ft}^2\text{]} \]

8-13 | FORMULATION FOR NUMERICAL SOLUTION

The network method that we have used to analyze radiation problems is an effective artifice for visualizing radiant exchange between surfaces. For simple problems that do not involve too many surfaces, the network method affords a solution that can be obtained quite easily. When many heat-transfer surfaces are involved, it is to our advantage to formalize the procedure for writing the nodal equations. For this procedure we consider only opaque, gray, diffuse surfaces. The reader should consult Reference 10 for information on transmitting and specular surfaces. The radiant-energy balance on a particular opaque surface can be written

Net heat lost by surface = energy emitted – energy absorbed

or on a unit-area basis with the usual gray-body assumptions,

\[ \frac{q}{A} = \epsilon E_b - \alpha G \]

Considering the \( i \)th surface, the total irradiation is the sum of all irradiations \( G_j \) from the other \( j \) surfaces. Thus, for \( \epsilon = \alpha_\epsilon \)

\[ \frac{q_i}{A_i} = \epsilon_i \left( E_{b_i} - \sum_j G_j \right) \quad \text{[8-97]} \]

But, the irradiations can be expressed by

\[ A_j J_j F_{ji} = G_j A_i \quad \text{[8-98]} \]
From reciprocity, we have

\[ A_i F_j = A_j F_i \]

so that we can combine the equations to give

\[ \frac{q_i}{A_i} = \epsilon_i \left(E_{i} - \sum_j F_{ij} J_j\right) \tag{8-99} \]

The heat transfer at each surface is then evaluated in terms of the radiosities \( J_i \). These parameters are obtained by recalling that the heat transfer can also be expressed as

\[ \frac{q_i}{A_i} = J_i - G_i = J_i - \sum_j F_{ij} J_j \tag{8-100} \]

Combining Equations (8-99) and (8-100) gives

\[ J_i - (1 - \epsilon_i) \sum_j F_{ij} J_j = \epsilon_i E_{i} \tag{8-101} \]

In the equations above it must be noted that the summations must be performed over all surfaces in the enclosure. For a three-surface enclosure, with \( i = 1 \), the summation would then become

\[ \sum_j F_{ij} J_j = F_{11} J_1 + F_{12} J_2 + F_{13} J_3 \]

Of course, if surface 1 is convex, \( F_{11} = 0 \) and some simplification could be effected.

The nodal equations for the radiosities may also be derived from the nodes in the network formulation, as indicated in Figure 8-62. At each \( J_i \) node an energy balance gives

\[ \frac{\epsilon_i}{1 - \epsilon_i} (E_{i} - J_i) + \sum_j F_{ij} (J_j - J_i) = 0 \tag{8-102} \]

Again, an equation will be obtained for each \( J_i \) that is entirely equivalent to Equation (8-101). Once all the equations are written out they can be expressed in the matrix form

\[ [A][J] = [C] \tag{8-103} \]

where

\[
[A] = \begin{bmatrix}
\alpha_{i1} & \alpha_{i2} & \cdots & \alpha_{ii} \\
\alpha_{i2} & \alpha_{i2} & \cdots & \alpha_{ij} \\
\cdots & \cdots & \cdots & \cdots \\
\alpha_{i1} & \alpha_{i2} & \cdots & \alpha_{ii}
\end{bmatrix}
\]

\[ [J] = \begin{bmatrix} J_1 \\ J_2 \\ \vdots \\ J_i \end{bmatrix} \]

\[ [C] = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_i \end{bmatrix} \]

The solution for the radiosities is found by obtaining the inverse to \([A]\) such that

\[ [J] = [A]^{-1}[C] \]

The inverse \([A]^{-1}\) is written as

\[
[A]^{-1} = \begin{bmatrix}
b_{i1} & b_{i2} & \cdots & b_{in} \\
b_{21} & \cdots & \cdots & b_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
b_{n1} & b_{n2} & \cdots & b_{nn}
\end{bmatrix}
\]
so that the unknown radiosities are written as
\[ J_i = b_{11}C_1 + b_{12}C_2 + \cdots + b_{1n}C_n \]
\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
\[ J_i = b_{11}C_1 + b_{12}C_2 + \cdots + b_{1n}C_n \]

Standard computer subroutines are available to obtain the inverse matrix and perform the final calculations of the \( J_i \). The heat-transfer rate at each \( \theta \)th surface having an area \( A_i \), is then calculated from
\[ \frac{q_i}{A_i} = \varepsilon_i (E_{b_i} - J_i) \]  \[8-104\]

In formulating the nodal equations one must take note of the consequence of Equation (8-104) for an insulated surface (i.e., one for which there is no net heat transfer). Equation (8-110) thus requires that
\[ E_{b_i} = J_i \quad \text{for insulated surface} \]  \[8-105\]

From a practical point of view, a Gauss-Seidel iteration scheme may be the most efficient numerical procedure to follow in solving the set of equations for the \( J_i \)'s. For the Gauss-Seidel scheme the above equations must be organized in explicit form for \( J_i \). Solving for \( J_i \) in Equation (8-101) and breaking out the \( F_{ii} \) term gives
\[ J_i = (1 - \varepsilon_i) \sum_{j \neq i} V_{ij} J_j + (1 - \varepsilon_i) F_{ii} J_i + \varepsilon_i E_{b_i} \]
\[ J_i = \frac{1}{1 - F_{ii}(1 - \varepsilon_i)} \left[ (1 - \varepsilon_i) \sum_{j \neq i} V_{ij} J_j + \varepsilon_i E_{b_i} \right] \]  \[8-106\]

For a surface in radiant equilibrium, \( q_i/A_i = 0 \) and \( J_i = E_{b_i} \) may be substituted into Equation (8-106) to give
\[ J_i = \frac{1}{1 - F_{ii}} \sum_{j \neq i} F_{ij} J_j \quad \text{for} \quad \frac{q_i}{A_i} = 0 \]  \[8-107\]

If the problem formulation is to include a specified heat flux \( q_i/A_i \) at one of the \( \theta \) surfaces, we can solve for \( E_{b_i} \) from Equation (8-104) to give
\[ E_{b_i} = J_i + \frac{1 - \varepsilon_i}{\varepsilon_i} \frac{q_i}{A_i} \]  \[8-108\]

Substituting this value into Equation (8-101) and then solving for \( J_i \) give
\[ J_i = \frac{1}{1 - F_{ii}} \left( \sum_{j \neq i} F_{ij} J_j + \frac{q_i}{A_i} \right) \]  \[8-109\]

In many cases the radiation solution must take conduction and convection into account at the \( \theta \)th surface. The appropriate energy balance is then, for steady state,

Heat conducted into surface + heat convected into surface
\[ = \text{radiant heat lost from surface} \]

or
\[ q_{\text{cond,}\theta} + q_{\text{conv,}\theta} = q_{i,\text{rad}} \]  \[8-110\]
This energy balance may then be used in conjunction with Equation (8-109) to obtain the proper nodal equation for $J_i$.

While the above formulations may appear rather cumbersome at first glance they are easily solved by computer, with either matrix inversion or iteration. For many practical radiation problems, the number of equations is small and programmable calculators may be employed for solution. In most cases one will not know the surface properties $(\epsilon, \sigma)$ within better than a few percent, so an iterative solution need not be carried out to unreasonable limits of precision.

In summary, we outline the computational procedure to be followed for numerical solution of radiation heat transfer between diffuse, gray surfaces.

1. Evaluate $F_{ij}$ and $\epsilon_i$ for all surfaces.
2. Evaluate $E_{b_i}$ for all surfaces with specified temperature.
3. Formulate nodal equations for the $J_i$ using:
   a. Equation (8-106) for surfaces with specified $T_i$.
   b. Equation (8-107) for surfaces in radiant balance $(J_i = E_{b_i})$.
   c. Equation (8-109) for surfaces with specified $q_i$.
4. Solve the equations for the $J_i$'s.
5. Compute the $q_i$'s and $T_i$'s, using:
   a. $q_i$ from Equation (8-104) for gray surfaces and Equation (8-99) for black surfaces with specified $T_i$.
   b. $T_i$ from $J_i = E_{b_i} = \alpha_i T_i^4$ for surfaces in radiant balance.
   c. $T_i$ using $E_{b_i}$ obtained from Equation (8-108) for surfaces with specified $q_i$.

Of course, the above equations may be put in the following form if direct matrix inversion is preferred over an iteration scheme:

$$J_i[1 - F_{ij}(1 - \epsilon_j)] - (1 - \epsilon_j) \sum_{j \neq i} F_{ij} J_j = \epsilon_i E_{b_i}$$  \hspace{0.5cm} [8-106a]

$$J_i(1 - F_{ij}) - \sum_{j \neq i} F_{ij} J_j = 0$$  \hspace{0.5cm} [8-107a]

$$J_i(1 - F_{ij}) - \sum_{j \neq i} F_{ij} J_j = \frac{q_i}{A_i}$$  \hspace{0.5cm} [8-109a]

Computation of the surface temperatures and heat transfers is the same as in step 5 above.

**Insulated Surfaces and Surfaces with Large Areas**

We have seen in the application of the network method that an insulated surface acts as if it were perfectly reflective with $\epsilon \rightarrow 0$, and thus $J = E_b$. We may note that if one takes $\epsilon = 0$ in Equation (8-106a), Equation (8-107a) will result. When the system of equations is solved for the $J_i$'s, one may thus obtain the temperature of an insulated surface from $T = (E_b/\sigma)^{1/4} = (J/\sigma)^{1/4}$. When a surface with a very large area compared to other surfaces in the system is involved, it behaves like a blackbody with $\epsilon \rightarrow 1,0$ because of its low surface resistance. If such a large surface is concave, it will behave as if $F_{ii} \rightarrow 1,0$ and all the $F_{ij} \rightarrow 0$. 
Numerical Solution for Enclosure

The geometry of Example 8-5 is used for radiant exchange with a large enclosure. Surface 2 is diffuse with \(\varepsilon = 0.5\) while surface 1 is perfectly insulated. \(T_2 = 1000\) K, and \(T_3 = 300\) K. Calculate the heat lost to the large room per unit length of surface 2, using the numerical formulation. Also calculate the temperature of the insulated surface.

**Solution**

For unit length we have:

\[
E_{b2} = \sigma T_2^4 = 5.669 \times 10^4 \\
E_{b3} = \sigma T_3^4 = 459 \\
A_1 = (4)(0.2) = 0.8 \text{ m}^2/\text{m} \\
A_2 = \pi(0.60)/2 = 0.94 \text{ m}^2/\text{m}
\]

We will use the numerical formulation. We find from Example 8-5, using the nomenclature of the example, \(F_{11} = 0.314, F_{12} = 0.425, F_{13} = 0.261, F_{21} = 0.5, F_{22} = 0, F_{23} = 0.5, F_{31} \rightarrow 0, F_{32} \rightarrow 0, F_{33} \rightarrow 1.0\). We now write the equations. Surface 1 is insulated so we use Equation (8-107a):

\[J_1(1 - 0.314) - 0.425J_2 - 0.261J_3 = 0\]

Surface 2 is constant temperature so we use Equation (8-106a):

\[J_2(1 - 0) - (1 - 0.5)(0.5J_1 + 0.5J_3) = (0.5)(56,690)\]

Because surface 3 is so large,

\[J_3 = E_{b3} = 459 \text{ W/m}^2\]

Rearranging the equations gives

\[
0.686J_1 - 0.425J_2 = 119.8 \\
-0.25J_1 + J_2 = 28,460
\]

which have the solutions

\[
J_1 = 21,070 \text{ W/m}^2 \\
J_2 = 33,727 \text{ W/m}^2
\]

The heat transfer is thus

\[
q = \frac{E_{b2} - J_2}{1 - \varepsilon_2} = \frac{56,690 - 33,727}{(1 - 0.5)/(0.5)(4)(0.2)} = 18,370 \text{ W/m length}
\]

Because surface 1 is insulated, \(J_1 = E_{b1}\), and we could calculate the temperature as

\[
T_1 = \left(\frac{21,070}{5.669 \times 10^{-8}}\right)^{1/4} = 781 \text{ K}
\]

For this problem a solution using the network method might be simpler because it involves only a simple series–parallel circuit.

Numerical Solutions for Parallel Plates

Two 1-m-square surfaces are separated by a distance of 1 m with \(T_1 = 1000\) K, \(T_2 = 400\) K, \(\varepsilon_1 = 0.8, \varepsilon_2 = 0.5\). Obtain the numerical solutions for this system when (a) the plates are surrounded by a large room at 300 K and (b) the surfaces are connected by a re-radiating wall perfectly insulated on its outer surface. Part (a) of this example is identical in principle to the problem that is solved by the network method in Example 8-6.
Solution

Consulting Figure 8-12, we obtain

\[ F_{12} = 0.2 \quad F_{21} = 0.2 \quad F_{11} = 0 = F_{22} \]
\[ F_{13} = 0.8 \quad F_{23} = 0.8 \]
\[ A_1 = A_2 = 1 \text{ m}^2 \]

(surface 3 is the surroundings or insulated surface). For part (a)

\[ E_{b1} = \sigma T_1^4 = 56.69 \text{ kW/m}^2 \quad [17,970 \text{ Btu/h} \cdot \text{ft}^2] \]
\[ E_{b2} = \sigma T_2^4 = 1.451 \text{ kW/m}^2 \]
\[ E_{b3} = \sigma T_3^4 = 0.459 \text{ kW/m}^2 \]

Because \( A_3 \to \infty \), \( F_{31} \) and \( F_{32} \) must approach zero since \( A_1 F_{13} = A_3 F_{31} \) and \( A_2 F_{23} = A_3 F_{32} \). The nodal equations are written in the form of Equation (8-107):

**surface 1:**

\[ J_1 - (1 - \varepsilon_1)(F_{11} J_1 + F_{12} J_2 + F_{13} J_3) = \varepsilon_1 E_{b1} \]

**surface 2:**

\[ J_2 - (1 - \varepsilon_2)(F_{21} J_1 + F_{22} J_2 + F_{23} J_3) = \varepsilon_2 E_{b2} \quad [a] \]

**surface 3:**

\[ J_3 - (1 - \varepsilon_3)(F_{31} J_1 + F_{32} J_2 + F_{33} J_3) = \varepsilon_3 E_{b3} \]

Because \( F_{31} \) and \( F_{32} \) approach zero, \( F_{33} \) must be 1.0.

Inserting the numerical values for the various terms, we have

\[ J_1 - (1 - 0.8)(0.2)J_1 + (0.2)J_2 + (0.8)J_3 = (0.8)(56.69) \]
\[ J_2 - (1 - 0.5)(0.2)J_1 + (0)J_2 + (0.8)J_3 = (0.5)(1.451) \quad [b] \]
\[ J_3 - (1 - 0.3)(0)J_1 + (0)J_2 + (1.0)J_3 = \varepsilon_3(0.459) \]

The third equation yields \( J_3 = 0.459 \text{ kW/m}^2 \cdot \text{K} \). Because the room is so large it acts like a hohlraum, or blackbody. But it does not have zero heat transfer.

Finally, the equations are written in compact form as

\[ J_1 - 0.04 J_2 - 0.16 J_3 = 45.532 \]
\[ -0.1 J_1 + J_2 - 0.4 J_3 = 0.7255 \quad [c] \]
\[ J_3 = 0.459 \]

Of course, there only remain two unknowns, \( J_1 \) and \( J_2 \), in this set.

For part (b), \( A_3 \) for the enclosing wall is 4.0 \text{ m}^2, and we set \( J_3 = E_{b3} \) because surface 3 is insulated. From reciprocity we have

\[ A_1 F_{13} = A_3 F_{31} \quad F_{31} = \begin{pmatrix} 1.0 \end{pmatrix}(0.8) = 0.2 \]
\[ A_2 F_{23} = A_3 F_{32} \quad F_{32} = \begin{pmatrix} 1.0 \end{pmatrix}(0.8) = 0.2 \]

Then from \( F_{31} + F_{32} + F_{33} = 1.0 \) we have \( F_{33} = 0.6 \).

The set of equations in (a) still applies, so we insert the numerical values to obtain (with \( J_3 = E_{b3} \))

\[ J_1 - (1 - 0.8)(0)J_1 + (0.2)J_2 + (0.8)J_3 = (0.8)(56.69) \]
\[ J_2 - (1 - 0.5)(0.2)J_1 + (0)J_2 + (0.8)J_3 = (0.5)(1.451) \quad [d] \]
\[ J_3 - (1 - \varepsilon_3)(0.2)J_1 + (0.2)J_2 + (0.6)J_3 = \varepsilon_3 J_3 \]

Notice that the third equation of set (d) can be written as

\[ J_3(1 - \varepsilon_3) - (1 - \varepsilon_3)(0.2)J_1 + (0.2)J_2 + (0.6)J_3 = 0 \]
so that the $1 - \epsilon_3$ term drops out, and we obtain our final set of equations as:

$$J_1 - 0.04 J_2 - 0.16 J_3 = 45.352$$

$$-0.1 J_1 + J_2 - 0.4 J_3 = 0.7255$$

$$-0.2 J_1 - 0.2 J_2 + 0.4 J_3 = 0$$

To obtain the heat transfers the set of equations is first solved for the radiances. For set (e),

$$J_1 = 45.644 \text{ kW/m}^2$$

$$J_2 = 5.474 \text{ kW/m}^2$$

$$J_3 = 0.459 \text{ kW/m}^2$$

The heat transfers are obtained from Equation (8-104):

$$q_1 = \frac{A_1 r_1}{1 - \epsilon_1} (E_{t_1} - J_1) = \frac{(1.0)(0.8)}{1 - 0.8}(56.69 - 45.644) = 44.184 \text{ kW} \quad [150,760 \text{ Btu/h}]$$

$$q_2 = \frac{A_2 r_2}{1 - \epsilon_2} (E_{t_2} - J_2) = \frac{(1.0)(0.5)}{1 - 0.5}(1.451 - 5.474) = -4.023 \text{ kW} \quad [-13,730 \text{ Btu/h}]$$

The net heat absorbed by the room is the algebraic sum of $q_1$ and $q_2$ or

$$q_{3, \text{ absorbed}} = 44.184 - 4.023 = 40.161 \text{ kW} \quad [137,030 \text{ Btu/h}]$$

For part (b) the solutions to set (e) are

$$J_1 = 51.956 \text{ kW/m}^2 \quad J_2 = 20.390 \text{ kW/m}^2 \quad J_3 = 36.173 \text{ kW/m}^2$$

The heat transfers are

$$q_1 = \frac{A_1 r_1}{1 - \epsilon_1} (E_{t_1} - J_1) = \frac{(1.0)(0.8)}{1 - 0.8}(56.69 - 51.965) = 18.936 \text{ kW}$$

$$q_2 = \frac{A_2 r_2}{1 - \epsilon_2} (E_{t_2} - J_2) = \frac{(1.0)(0.5)}{1 - 0.5}(1.451 - 20.390) = -18.936 \text{ kW}$$

Of course, these heat transfers should be equal in magnitude with opposite sign because the insulated wall exchanges no heat. The temperature of the insulated wall is obtained from

$$J_3 = E_{t_3} = \alpha T_3^4 = 36.173 \text{ kW/m}^2$$

and

$$T_3 = 894 \text{ K} \quad [621^\circ \text{C}, 1150^\circ \text{F}]$$

Radiation from a Hole with Variable Radiosity

EXAMPLE 8-17

To further illustrate the radiation formulation for numerical solution we consider the circular hole 2 cm in diameter and 3 cm deep, as shown in Figure Example 8-17. The hole is machined in a large block of metal, which is maintained at 1000°C and has a surface emissivity of 0.6. The temperature of the large surrounding room is 20°C. A simple approach to this problem would assume the radiosity uniform over the entire heated internal surface. In reality, the radiosity varies over the surface, and we break it into segments 1 (bottom of the hole), 2, 3, and 4 (sides of the hole) for analysis.

The large room acts like a blackbody at 20°C, so for analysis purposes we can assume the hole is covered by an imaginary black surface 5 at 20°C. We set the problem up for a numerical solution for the radiances and then calculate the heat-transfer rates. After that, we shall examine an insulated-surface case for this same geometry.
Figure Example 8-17

Solution
All the shape factors can be obtained with the aid of Figure 8-13 and the imaginary disk surfaces 6 and 7. We have
\[ E_{b1} = \sigma T_1^4 = (5.669 \times 10^{-8})(1273)^4 = 1.48874 \times 10^5 \text{ W/m}^2 \]
\[ E_{b2} = E_{b3} = E_{b4} \]
\[ E_{b5} = \sigma T_2^4 = (5.669 \times 10^{-8})(293)^4 = 417.8 \text{ W/m}^2 \]
\[ \epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0.6 \quad \epsilon_5 = 1.0 \]
\[ A_1 = A_5 = \pi (1)^2 = \pi \text{ cm}^2 = A_6 = A_7 \]
\[ A_2 = A_3 = A_4 = \pi (2)(1) = 2\pi \]
\[ F_{11} = F_{55} = 0 \quad F_{16} = 0.37 \quad F_{17} = 0.175 \quad F_{15} = 0.1 \]
\[ F_{12} = 1 - F_{16} = 0.63 = F_{54} \]
\[ F_{13} = F_{16} - F_{17} = 0.195 = F_{53} \]
\[ F_{14} = F_{17} - F_{15} = 0.075 = F_{52} \]
\[ F_{21} = F_{26} = F_{16} \frac{A_1}{A_2} = 0.315 = F_{45} = F_{36} = F_{37} \]
\[ F_{22} = 1 - F_{21} - F_{26} = 0.37 = F_{33} = F_{44} \]
\[ F_{31} = F_{13} \frac{A_1}{A_3} = 0.0975 \]
\[ F_{32} = F_{36} - F_{31} = 0.2175 = F_{34} = F_{43} = F_{23} \]
\[ F_{27} = F_{26} - F_{23} = F_{21} - F_{23} = 0.0975 = F_{46} \]
\[ F_{41} = F_{14} \frac{A_1}{A_4} = 0.0375 = F_{25} \]
\[ F_{42} = F_{46} - F_{41} = 0.06 = F_{24} \]

The equations for the radiosities are now written in the form of Equation (8-106), noting that \( F_{11} = 0 \) and \( J_2 = E_{b5} \):
\[ J_1 = (1 - \epsilon_1)(F_{12} J_2 + F_{13} J_3 + F_{14} J_4 + F_{15} E_{b5}) + \epsilon_1 E_{b1} \]
\[ J_2 = \frac{1}{1 - F_{22} (1 - \epsilon_2)}[(1 - \epsilon_2) (F_{21} J_1 + F_{23} J_3 + F_{24} J_4 + F_{25} E_{b5}) + \epsilon_2 E_{b2}] \]
\[ J_3 = \frac{1}{1 - \epsilon_3(1 - \epsilon_3)} \left( (1 - \epsilon_3)(F_{31}J_1 + F_{32}J_2 + F_{34}J_4 + F_{35}E_{35}) + \epsilon_3 E_{33} \right) \]
\[ J_4 = \frac{1}{1 - \epsilon_4(1 - \epsilon_4)} \left( (1 - \epsilon_4)(F_{41}J_1 + F_{42}J_2 + F_{43}J_3 + F_{45}E_{35}) + \epsilon_4 E_{34} \right) \]

When all the numerical values are inserted, we obtain
\[ J_1 = 0.252J_2 + 0.078J_3 + 0.03J_4 + 89.341 \]
\[ J_2 = 0.1479J_1 + 0.1021J_3 + 0.02817J_4 + 104.848 \]
\[ J_3 = 0.04577J_1 + 0.1021J_2 + 0.1021J_4 + 104.859 \]
\[ J_4 = 0.01761J_1 + 0.02817J_2 + 0.1021J_3 + 104.902 \]

These equations may be solved to give
\[ J_1 = 1.4003 \times 10^5 \text{ W/m}^2 \]
\[ J_2 = 1.4326 \times 10^5 \text{ W/m}^2 \]
\[ J_3 = 1.3872 \times 10^5 \text{ W/m}^2 \]
\[ J_4 = 1.2557 \times 10^5 \text{ W/m}^2 \]

The heat transfers can be calculated from Equation (8-104):
\[ q_1 = \frac{\epsilon_1(A_1)}{1 - \epsilon_1} (E_{13} - J_1) \]
\[ q_1 = \frac{(0.6)(\pi \times 10^{-4})}{1 - 0.6} (1.4887 - 1.4003 \times 10^5) = 4.1658 \text{ W} \]
\[ q_2 = \frac{(0.6)(2\pi \times 10^{-4})}{1 - 0.6} (1.4887 - 1.4326 \times 10^5) = 5.2873 \text{ W} \]
\[ q_3 = \frac{(0.6)(2\pi \times 10^{-4})}{1 - 0.6} (1.4887 - 1.3872 \times 10^5) = 9.5661 \text{ W} \]
\[ q_4 = \frac{(0.6)(2\pi \times 10^{-4})}{1 - 0.6} (1.4887 - 1.2557 \times 10^5) = 21.959 \text{ W} \]

The total heat transfer is the sum of these four quantities or
\[ q_{\text{total}} = 40.979 \text{ W} \quad [139.8 \text{ Btu/h}] \]

It is of interest to compare this heat transfer with the value we would obtain by assuming uniform radiosity on the hot surface. We would then have a two-body problem with
\[ A_1 = \pi + 3(2\pi) = 7\pi \text{ cm}^2 \quad A_2 = \pi \quad I_{\beta_1} = 1.0 \quad \epsilon_1 = 0.6 \quad \epsilon_2 = 1.0 \]

The heat transfer is then calculated from Equation (8-43), with appropriate change of nomenclature:
\[ q = \frac{(E_{13} - E_{35})A_3}{1/\epsilon_3 + (A_3/A_1)(1/\epsilon_1 - 1)} = \frac{(\pi \times 10^{-4})(1.4887 \times 10^5 - 417.8)}{1 + (\frac{1}{7})(1/0.6 - 1)} = 42.581 \text{ W} \quad [145.3 \text{ Btu/h}] \]

Thus, the simple assumption of uniform radiosity gives a heat transfer that is 3.9 percent above the value obtained by breaking the hot surface into four parts for the calculation. This indicates that the uniform-radiosity assumption we have been using is a rather good one for engineering calculations.
Let us now consider the case where surface 1 is still radiating at 1000°C with $\epsilon = 0.6$ but the side walls 2, 3, and 4 are insulated. The radiation is still to the large room at 20°C. The nodal equation for $J_1$ is the same as before but now the equations for $J_2$, $J_3$, and $J_4$ must be written in the form of Equation (8-107). When that is done and the numerical values are inserted, we obtain

$$
J_1 = 0.252 J_2 + 0.078 J_3 + 0.03 J_4 + 89.341 \\
J_2 = 0.5 J_1 + 0.3452 J_3 + 0.09524 J_4 + 24.869 \\
J_3 = 0.1548 J_1 + 0.3452 J_2 + 0.3452 J_4 + 64.66 \\
J_4 = 0.05952 J_1 + 0.09524 J_2 + 0.3452 J_3 + 208.9
$$

When these equations are solved, we obtain

$$
J_1 = 1.1532 \times 10^5 \text{ W/m}^2 \quad [36.560 \text{ Btu/h} \cdot \text{ft}^2] \\
J_2 = 0.81019 \times 10^5 \text{ W/m}^2 \\
J_3 = 0.57885 \times 10^5 \text{ W/m}^2 \\
J_4 = 0.34767 \times 10^5 \text{ W/m}^2
$$

The heat transfer at surface 1 is

$$
q_1 = \frac{J_1}{\epsilon} = \frac{(0.6) 1.1532 \times 10^5}{1 - 0.6} = 15.81 \text{ W} \quad [53.95 \text{ Btu/h}]
$$

The temperatures of the insulated surface elements are obtained from

$$
T_2 = 1093 \text{ K} = 820^\circ \text{C} \quad [1508^\circ \text{F}] \\
T_3 = 1005 \text{ K} = 732^\circ \text{C} \quad [1350^\circ \text{F}] \\
T_4 = 895 \text{ K} = 612^\circ \text{C} \quad [1134^\circ \text{F}]
$$

It is of interest to compare the heat transfer calculated above with that obtained by assuming surfaces 2, 3, and 4 uniform in temperature and radiosity. Equation (8-41) applies for this case:

$$
q = \frac{A_1 E_{T_1} - E_{T_2}}{A_5 - A_1 A_5 / A_5 A_5 / A_5} + \left( \frac{1}{\epsilon_1 - 1} \right) + \frac{A_1}{A_5} \left( \frac{1}{\epsilon_5 - 1} \right)
$$

and

$$
q = \frac{(\pi \times 10^{-4}) (1.1532 \times 10^5 - 417.8)}{\pi + \pi - 2 \pi (0.1)^2} + \frac{1}{0.6 - 1} = 18.769 \text{ W} \quad [64.04 \text{ Btu/h}]
$$

In this case the assumption of uniform radiosity at the insulated surface gives an overall heat transfer with surface 1 (bottom of hole) that is 18.7 percent too high.

**EXAMPLE 8-18**

**Heater with Constant Heat Flux and Surrounding Shields**

In Figure Example 8-18, an electric heater is installed in surface 1 such that a constant heat flux of 100 kW/m² is generated at the surface. The four surrounding surfaces are in radiant
balance with surface 1 and a large surrounding room at 20°C. The surface properties are \( \epsilon_1 = 0.8 \) and \( \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = 0.4 \). Determine the temperatures of all surfaces. The back side of surface 1 is insulated. Repeat the calculation assuming surfaces 2, 3, 4, and 5 are just one surface uniform in temperature.

**Figure Example 8-18**

![Diagram of a rectangular object with dimensions labeled: 30 cm, 30 cm, and 60 cm. Lines indicate surfaces 1, 2, 3, 4, and 5 with a dashed line through surfaces 3 and 4.]

### Solution

In reality, surfaces 2, 3, 4, and 5 have *two* surfaces each: an inside and an outside surface. We thus have *nine* surfaces plus the room, so a 10-body problem is involved. Of course, from symmetry we can see that \( T_2 = T_3 \) and \( T_1 = T_5 \), but we set up the problem in the general numerical formulation. We designate the large room as surface 6 and it behaves as if \( \epsilon_6 = 1.0 \). So, it is as if the opening were covered with a black surface at 20°C. The shape factors of the inside surfaces are obtained from Figures 8-12 and 8-14:

\[
F_{16} = F_{61} = 0.285 \\
F_{12} = F_{14} = 0.115 \\
F_{35} = F_{53} = 0.285 \\
F_{25} = F_{23} = F_{45} = F_{33} = F_{21} = F_{41} = F_{26} = F_{46} = 0.23 \\
F_{11} = F_{22} = F_{33} = F_{34} = F_{55} = 0
\]

For the *outside* surfaces,

\[
F'_{26} = F'_{36} = F'_{46} = F'_{56} = 1.0
\]

where the primes indicate the outside surfaces. We shall also use primes to designate the radiosities of the outside surfaces. For the room, \( J_6 = E_{6e} = (5.669 \times 10^{-8})(293)^4 = 417.8 \text{ W/m}^2 \).

For surface 1 with constant heat flux, we use Equation (8.109a) and write

\[
J_1 = (F_{12} J_2 + F_{13} J_3 + F_{14} J_4 + F_{15} J_5 + F_{16} J_6) = 1.0 \times 10^5 \quad [a]
\]

Because of the radiant balance condition we have

\[
(J_2 - E_{1e}) \frac{\epsilon_2 A_2}{1 - \epsilon_2} = (E_{1e} - J_2') \frac{\epsilon_2 A_2}{1 - \epsilon_2}
\]

and

\[
E_{1e} = \frac{J_2 + J_2'}{2} \quad [b]
\]
where the prime designates the outside radiosity. A similar relation applies for surfaces 3, 4, and 5. Thus, we can use Equation (8-106a) for inside surface 2
\[
J_2' = (1 - \varepsilon_2) / (t_2' + J_3 + J_4 + J_5 + J_6) = \frac{t_2'}{2} (J_2 + J_2') \tag{c}
\]
and for the outside surface 2
\[
J_2'' = (1 - \varepsilon_2) / (t_2' + J_6) = \frac{t_2'}{2} (J_2 + J_2'') \tag{d}
\]
Equations like (c) and (d) are written for surfaces 3, 4, and 5 also, and with the shape factors and emissivities inserted the following set of equations is obtained:
\[
\begin{align*}
J_1 & = 0.115 J_2 - 0.24 J_3 - 0.115 J_4 - 0.24 J_5 = 1.0012 \times 10^5 \\
-0.138 J_1 + 0.8 J_2 - 0.2 J_3 - 0.138 J_4 - 0.0408 J_5 & = 57.66 \\
0.2 J_2 - 0.8 J_5 & = -250.68 \\
-0.144 J_1 - 0.069 J_2 + 0.8 J_3 - 0.2 J_4 - 0.05 J_5 & = 60.16 \\
0.2 J_3 - 0.8 J_4 & = -250.68 \\
-0.138 J_1 - 0.0408 J_2 - 0.138 J_3 + 0.8 J_4 & = 57.66 \\
0.2 J_4 - 0.8 J_5 & = -250.68 \\
-0.144 J_1 - 0.069 J_2 - 0.057 J_3 - 0.069 J_4 + 0.8 J_5 & = 60.16 \\
0.2 J_5 & = -250.68
\end{align*}
\]
We thus have nine equations and nine unknowns, which may be solved to give
\[
\begin{align*}
J_1 & = 1.24887 \times 10^5 \text{ W/m}^2 \\
J_2 & = J_4 = 37,549 \\
J_3 & = J_5 = 9701 \\
J_6 & = J_5 = 33,605 \\
J_7 & = J_8 = 8714
\end{align*}
\]
The temperatures are thus computed from Equation (b):
\[
\begin{align*}
E_2 / J_2 & = \frac{37,549 + 9701}{2} = 23,625 \quad T_2 = T_4 = 803.5 \text{ K} \\
E_3 / J_3 & = \frac{33,605 + 8714}{2} = 21,160 \quad T_3 = T_8 = 781.6 \text{ K}
\end{align*}
\]
For surface 1 we observed that
\[
\frac{q}{A_1} = \frac{e}{1 - \varepsilon} (E_{T_1} - J_1)
\]
so that
\[
E_{T_1} = \frac{(1.0 \times 10^5)(1 - 0.8)}{0.8} + 1.24887 \times 10^5 = 1.49887 \times 10^5
\]
and
\[
T_1 = 1275 \text{ K}
\]
We note again that we could have observed the symmetry of the problem and set \( J_2 = J_4, J_2' = J_4' \), and so on. By so doing, we could have had only five equations with five unknowns.

\section*{Surfaces 2, 3, 4, and 5 as one surface}
We now go back and take surfaces 2, 3, 4, and 5 as one surface, which we choose to call surface 7. The shape factors are then
\[
\begin{align*}
F_{16} & = F_{61} = 0.285 \quad F_{17} = 1 - 0.285 = 0.715 \\
A_1 & = 2.0 \quad A_7 = 6.0
\end{align*}
\]
Thus

\[ F_{\gamma 1} = (0.715) \left( \frac{3}{6} \right) = 0.2383 = F_{\gamma 6} \]

\[ F_{\gamma 7} = 1 - (2)(0.2383) = 0.5233 \quad F_{\gamma 6} = 1.0 \]

Then for surface 1 we use Equation (8-109a) to obtain

\[ J_1 = \left( F_{\gamma 1} J_7 + F_{\gamma 6} J_6 \right) = 1.0 \times 10^5 \]

Using \( F_{\gamma 7} = \left( J_7 + J_7^* \right)/2 \), we have for the inside of surface 7

\[ J_7[1 - F_{\gamma 7}(1 - \epsilon_7)] - (1 - \epsilon_7)(F_{\gamma 1} J_1 + F_{\gamma 6} J_6) = \frac{\epsilon_7}{2}(J_7 + J_7^*) \]

while for the outside we have

\[ J_7^* - (1 - \epsilon_7) F_{\gamma 6} J_6 = \frac{\epsilon_7}{2}(J_7 + J_7^*) \]

When the numerical values are inserted, we obtain the set of three equations:

\[ J_1 = 0.715 J_7 = 1.0012 \times 10^5 \]

\[ -0.143 J_1 + 0.486 J_7 - 0.2 J_7^* = 59.74 \]

\[ 0.2 J_7 - 0.8 J_7^* = -250.68 \]

which has the solution

\[ J_1 = 1.31054 \times 10^5 \text{ W/m}^2 \]

\[ J_7 = 43,264 \]

\[ J_7^* = 11,129 \]

The temperatures are then calculated as before:

\[ F_{\gamma 7} = \frac{43,264 + 11,129}{2} = 27,197 \quad T_7 = 832.2 \text{ K} \]

\[ F_{\gamma 1} = \frac{\left( 1.0 \times 10^5 \right)(1 - 0.8)}{0.8} + 1.31054 \times 10^5 = 1.65054 \times 10^5 \]

\[ T_1 = 1306 \text{ K} \]

So, there is about a 30 K temperature difference between the two methods.

**Comment**

With such a small difference between the solutions we may conclude that the extra complexity of choosing each surface at a different emissivity is probably not worth the effort, particularly when one recognizes the uncertainties that are present in the surface emissivities. This points out that our assumptions of uniform irradiation and emissivity, though strictly not correct, give answers that are quite satisfactory.

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**Numerical Solution for Combined Convection and Radiation (Nonlinear System)**

**EXAMPLE 8-19**

A 0.5 by 0.5 m plate is maintained at 1300 K and exposed to a convection and radiation surrounding at 300 K. Attached to the top are two radiation shields 0.5 by 0.5 m as shown in Figure Example 8-19(a). The convection heat-transfer coefficient for all surfaces is 50 W/m²·K, and \( \epsilon_1 = 0.8, \epsilon_2 = 0.3, \epsilon_3 = 0.3 \). Determine the total heat lost from the 1300 K surface and the temperature of the shields.
**Figure Example 8-19**

(a)

(b)

**Solution**

This example illustrates how it is possible to handle convection-radiation problems with the numerical formulation and an iterative computational procedure. Nomenclature is shown in the figure. Using Figures 8-12 and 8-14, we can evaluate the shape factors as

\[ F_{12} = F_{13} = 0.2 \quad F_{14} = 1 - 0.2 - 0.2 = 0.6 \]
\[ F_{23} = F_{32} = 0.2 \quad F_{24L} = F_{34R} = 1.0 \]
\[ F_{21} = F_{12} = F_{31} = 0.2 \quad F_{24R} = F_{34L} = 0.6 \]
\[ F_{11} = F_{22} = F_{33} = 0 \]
\[ J_{2R} = J_{3L} \quad J_{2L} = J_{3R} \quad \text{from symmetry} \]
\[ J_4 = E_{14} \]

We now use Equation (8-105) to obtain a relation for \( J_1 \):

\[ J_1 = (1 - \epsilon_1) [F_{12}J_{2R} + F_{13}J_{3L} + F_{14}J_4] + \epsilon_1 E_{14} \]
But \( J_{2R} = J_{3L} \) and \( F_{12} = F_{13} \) so that
\[
J_1 = (1 - \varepsilon_1)(2 F_{13} J_{2R} + F_{14} J_4) + \varepsilon_1 F_{11} \]  

We use Equation (8-108) for the overall energy balance on surface 2:
\[
2b(T_\infty - T_2) = \frac{\varepsilon_2}{1 - \varepsilon_2}(F_{12} - J_{2R}) + \frac{\varepsilon_2}{1 - \varepsilon_2}(F_{12} - J_{2L})
\]
\[
= \frac{\varepsilon_2}{1 - \varepsilon_2} \left( 2 F_{12} - J_{2R} - J_{2L} \right) \]  

Equation (8-105) is used for surface \( J_{2R} \):
\[
J_{2R} = (1 - \varepsilon_2)(F_{21} J_1 + F_{23} J_{3L} + F_{24R} J_4) + \varepsilon_2 F_{12}
\]
But \( J_{2R} = J_{3L} \) so that
\[
J_{2R} = \frac{1}{1 - (1 - \varepsilon_2) F_{23}} \left[ (1 - \varepsilon_2)(F_{21} J_1 + F_{23} J_{3L} + F_{24R} J_4) + \varepsilon_2 F_{12} \right] \]  

For surface \( J_{2L} \), the equation is
\[
J_{2L} = (1 - \varepsilon_2)(F_{24L} J_4) + \varepsilon_2 F_{12} \]  

Equation (b) is nonlinear in \( F_{12} \), so we must use an iterative method to solve the set. Such procedures are described in Reference 34. Applying the iteration technique, we obtain the final solution set as
\[
J_1 = 1.3135 \times 10^5, J_{2R} = 22.051
\]
\[
J_{2L} = 710, F_{12} = 1275, T_2 = 386.6 \text{ K}
\]
The total heat flux lost by surface 1 is
\[
\frac{q_1}{A_1} = b(T_1 - T_\infty) + (F_{11} - J_1) \frac{\varepsilon_1}{1 - \varepsilon_1}
\]
\[
= 1.7226 \times 10^5 \text{ W/m}^2 \]  

For a 0.5 by 0.5 m surface the heat lost is thus
\[
q_1 = (1.7226 \times 10^5) (0.5)^2 = 43.065 \text{ W}
\]
Other cases may be computed, and the influence that \( b \) and \( \varepsilon_2 \) have on the results is shown in the accompanying figure.

**Comment**
This example illustrates how nonlinear equations resulting from combined convection and radiation can be solved with an iterative procedure.

### 8-14 | SOLAR RADIATION

Solar radiation is a form of thermal radiation having a particular wavelength distribution. Its intensity is strongly dependent on atmospheric conditions, time of year, and the angle of incidence for the sun’s rays on the surface of the earth. At the outer limit of the atmosphere the total solar irradiation when the earth is at its mean distance from the sun is 1395 W/m². This number is called the **solar constant** and is subject to modification upon collection of more precise experimental data.

Not all the energy expressed by the solar constant reaches the surface of the earth, because of strong absorption by carbon dioxide and water vapor in the atmosphere. The