More Gates & Minterm-Maxterm
Outline

• Review: Duality Principle
• Mores gates: XOR, XNOR, NAND, NOR
• Design
  – Min term & Sum of Product
  – Max Term & Product of SUM
• Boolean function: Representation, Canonical form
• Karnough map simplification
Principle of Duality

• Dual of a statement S is obtained
  • By interchanging * and +
  • By interchanging 0 and 1
  • By interchanging for all x by x’ also valid
    (for an expression) only for simple demorgan law
• Dual of \((a \times 1) \times (0 + a')\) = 0 is \((a + 0) + (1 \times a') = 1\)
  \((a + b)' = a' \times b'\)
  \((a + b) \times 1 = (a' \times b') + 0\)
Duality examples

- $x + 0 = x$
- $x.1 = x$
- $X + x' = 1$
- $x.x' = 0$
- $A + B'C$
- $A. (B' + C)$
- $A'B' + AB$
- $(A' + B').(A + B)$
Logic Gates: XOR, XNOR

- **XOR**

- **XNOR**
Logic Gates: NAND, NOR

- NAND

- NOR
NAND & NOR are universal

- 
  
- 

- 

- 

\( (xx)' = x' \)

\( ((xy)' )' = xy \)

\( (x'y')' = x+y \)
NAND & NOR are universal

- \((x+x)' = x'\)
- \(((x+y)' )' = x+y\)
- \((x'+y')' = x.y\)
Multi-input gate

X1
X2
X3
Xn

X1 + X2 + X3 + ... + Xn

X1
X2
X3
Xn

X1 \cdot X2 \cdot X3 \cdot ... \cdot Xn
Boolean Functions: Terminology

\[ F(a,b,c) = a'bc + abc' + ab + c \]

- **Variable**
  - Represents a value (0 or 1), Three variables: \( a, b, \) and \( c \)

- **Literal**
  - Appearance of a variable, in true or complemented form
  - Nine literals: \( a', b, c, a, b, c', a, b, \) and \( c \)

- **Product term**
  - Product of literals, Four product terms: \( a'bc, abc', ab, c \)

- **Sum-of-products (SOP)**
  - Above equation is in sum-of-products form.
  - “\( F = (a+b)c + d \)” is not.
Representations of Boolean Functions

**English 1:** F outputs 1 when $a$ is 0 and $b$ is 0, or when $a$ is 0 and $b$ is 1.

**English 2:** F outputs 1 when $a$ is 0, regardless of $b$’s value.

**Equation 1:** $F(a,b) = a'b' + a'b$

**Equation 2:** $F(a,b) = a'$

- A function can be represented in different ways
  - Above shows seven representations of the same function $F(a,b)$, using four different methods: English, Equation, Circuit, and Truth Table.
Truth Table Representation of Functions

- Define value of $F$ for each possible combination of input values
  - 2-input function: 4 rows
  - 3-input function: 8 rows
  - 4-input function: 16 rows

- Q: Use truth table to define function $F(a,b,c)$ that is 1 when $abc$ is 5 or greater in binary
Converting among Representations

- Can convert from any representation to any other
- Common conversions
  - Equation to circuit
  - Truth table to equation
  - Equation to truth table
    - Easy -- just evaluate equation for each input combination (row)
    - Creating intermediate columns helps

Q: Convert to equation: \( F = a'b' + a'b \)

Q: Convert to truth table: \( F = a'b' + a'b \)

<table>
<thead>
<tr>
<th>Inputs</th>
<th>Outputs</th>
<th>Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>( F )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
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<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
F = ab'c + abc' + abc
\]
**Standard Representation: Truth Table**

- How to determine two functions are the same?
  - Use algebraic methods
  - But if we failed, does that prove *not* equal? No.

- Solution: Convert to truth tables
  - Only ONE truth table representation of given same functions: **Standard representation**

<table>
<thead>
<tr>
<th>F = ab + a’</th>
<th>F = a’b’ +a’b + ab</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>0</td>
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<td>1</td>
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</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Same
Canonical Form -- Sum of Minterms

• Truth tables too big for numerous inputs
• Use standard form of equation instead
  – Known as **canonical form**
  – Regular algebra: group terms of polynomial by power
    • $ax^2 + bx + c$  
      \[(3x^2 + 4x + 2x^2 + 3 + 1 \rightarrow 5x^2 + 4x + 4)\]
  – Boolean algebra: create sum of minterms
    • **Minterm**: product term with every function literal appearing exactly once, in true or complemented form
    • Just multiply-out equation until sum of product terms
    • Then expand each term until all terms are minterms

Determine if $F(a,b)=ab+a'$ is same function as $F(a,b) = a'b'+a'b+ab$ by to canonical form.

\[
F = ab+a' \text{ (already sum of products)}
\]
\[
F = ab + a'(b+b') \text{ (expanding term)}
\]
\[
F = ab + a'b + a'b' \text{ (it is canonical form)}
\]
Canonical form or Standard Form

• Canonical forms
  – Sum of minterms (SOM)
  – Product of maxterms (POM)

• Standard forms (may use less gates)
  – Sum of products (SOP)
  – Product of sums (POS)

F = ab+a’ (already sum of products: SOP)
F = ab + a’(b+b’) (expanding term)
F = ab + a’b + a’b’ (it is canonical form: SOM)
Canonical Forms

• It is useful to specify Boolean functions in a form that:
  – Allows comparison for equality.
  – Has a correspondence to the truth tables

• Canonical Forms in common usage:
  – Sum of Minterms (SOM)
  – Product of Maxterms (POM)
Minterms

- **Product term** is a term where literals are ANDed.
  - Example: $x'y'$, $xz$, $xyz$, ...

- **Minterm**: A product term in which all variables appear exactly once, in normal or complemented form.
  - Example: $F(x,y,z)$ has 8 minterms
    - $x'y'z'$, $x'y'z$, $x'yz'$, ...
  - Function with $n$ variables has $2^n$ minterms

- A minterm equals 1 at exactly one input combination and is equal to 0 otherwise.
  - Example: $x'y'z' = 1$ only when $x=0$, $y=0$, $z=0$

- A minterm is denoted as $m_i$ where $i$ corresponds the input combination at which this minterm is equal to 1.
2 Variable Minterms

- Two variables (X and Y) produce $2 \times 2 = 4$ combinations
  - $XY$ (both normal)
  - $XY'$ (X normal, Y complemented)
  - $X'Y$ (X complemented, Y normal)
  - $X'Y'$ (both complemented)
Maxterms

- **Maxterms** are OR terms with every variable in true or complemented form.
  - \(X+Y\) (both normal)
  - \(X+Y'\) (x normal, y complemented)
  - \(X'+Y\) (x complemented, y normal)
  - \(X'+Y'\) (both complemented)
Maxterms and Minterms

• Two variable minterms and maxterms.

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<tr>
<th>Index</th>
<th>Minterm</th>
<th>Maxterm</th>
</tr>
</thead>
<tbody>
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<td>x 'y</td>
<td>x + y'</td>
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<tr>
<td>2</td>
<td>x y'</td>
<td>x' + y</td>
</tr>
<tr>
<td>3</td>
<td>x y</td>
<td>x' + y'</td>
</tr>
</tbody>
</table>

• The index above is important for describing which variables in the terms are true and which are complemented.
# Minterms

## Minterms for Three Variables

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>Product Term</th>
<th>Symbol</th>
<th>m₀</th>
<th>m₁</th>
<th>m₂</th>
<th>m₃</th>
<th>m₄</th>
<th>m₅</th>
<th>m₆</th>
<th>m₇</th>
</tr>
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<tbody>
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<td>0</td>
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<td>\overline{X} \overline{Y} Z</td>
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<td>0</td>
<td>\overline{X} Y \overline{Z}</td>
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</tr>
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<td>1</td>
<td>X \overline{Y} \overline{Z}</td>
<td>m₃</td>
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<td>\overline{X} Y \overline{Z}</td>
<td>m₄</td>
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<td>0</td>
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</tr>
</tbody>
</table>

- \( m_i \) indicated the \( i \)th minterm
- \( m_i \) indicates the binary combination
- \( m_i \) is equal to 1 for ONLY THAT combination

**Variable complemented if 0**

**Variable uncomplemented if 1**
Maxterms

• **Sum term**: A term where literals are ORed.
  • Example: \(x'+y', x+z, x+y+z, \ldots\)

• **Maxterm**: A sum term in which all variables appear exactly once, in normal or complemented form.
  • Example: \(F(x,y,z)\) has 8 maxterms
    \((x+y+z), (x+y+z'), (x+y'+z), \ldots\)

• Function with \(n\) variables has \(2^n\) maxterms

• A maxterm equals 0 at exactly one input combination and is equal to 1 otherwise.
  • Example: \((x+y+z) = 0\) only when \(x=0, y=0, z=0\)

• A maxterm is denoted as \(M_i\) where \(i\) corresponds to the input combination at which this maxterm is equal to 0.
# Maxterms

## Maxterms for Three Variables

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>Sum Term</th>
<th>Symbol</th>
<th>M₀</th>
<th>M₁</th>
<th>M₂</th>
<th>M₃</th>
<th>M₄</th>
<th>M₅</th>
<th>M₆</th>
<th>M₇</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$X + Y + Z$</td>
<td>$M₀$</td>
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<tr>
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<td>0</td>
<td>1</td>
<td>$X + Y + \overline{Z}$</td>
<td>$M₁$</td>
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<tr>
<td>0</td>
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<td>0</td>
<td>$X + \overline{Y} + Z$</td>
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<tr>
<td>0</td>
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<td>1</td>
<td>$X + \overline{Y} + \overline{Z}$</td>
<td>$M₃$</td>
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<td>$\overline{X} + Y + Z$</td>
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</tr>
</tbody>
</table>

$M_i$ indicated the $i^{th}$ maxterm $i$ indicates the binary combination $M_i$ is equal to 0 for ONLY THAT combination.

Variable complemented if 1  
Variable not complemented if 0
Expressing Functions with Minterms

- Boolean function can be expressed algebraically from a given truth table
- Forming sum of all the minterms that produce 1 in the function

**Example**: Consider the function defined by the truth table

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th>m</th>
<th>F</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>m₆</td>
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<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>m₇</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ F(X, Y, Z) = X'Y'Z' + X'YZ' + XY'Z + XYZ \]

\[ = m₀ + m₂ + m₅ + m₇ \]

\[ = \sum m(0, 2, 5, 7) \]
Expressing Functions with Maxterms

- Boolean function: Expressed algebraically from a given truth table
- By forming logical product (AND) of ALL the maxterms that produce 0 in the function

**Example:**
Consider the function defined by the truth table

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>M</th>
<th>F</th>
<th>F'</th>
</tr>
</thead>
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<td>M₀</td>
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<td>M₇</td>
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</tr>
</tbody>
</table>

F(X,Y,Z) = Π M(1,3,4,6)

Applying DeMorgan

F' = m₁ + m₃ + m₄ + m₆ = Σ m(1,3,4,6)
F = F'' = [m₁ + m₃ + m₄ + m₆]’
= m₁’.m₃’.m₄’.m₆’
= M₁.M₃.M₄.M₆
= Π M(1,3,4,6)

Note the indices in this list are those that are missing from the previous list in Σm(0,2,5,7)
Sum of Minterms vs Product of Maxterms

- A function can be expressed algebraically as:
  - The sum of minterms
  - The product of maxterms
- Given the truth table, writing F as
  - $\sum m_i$ – for all minterms that produce 1 in the table, or
  - $\prod M_i$ – for all maxterms that produce 0 in the table
- Minterms and Maxterms are complement of each other.
Example: minterm & maxterm

- Write $E = Y' + X'Z'$ in the form of $\sum m_i$ and $\Pi M_i$?
- Method 1
  - First construct the Truth Table as shown
  - $E = \sum m(0,1,2,4,5)$, and
  - $E = \Pi M(3,6,7)$

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>m</th>
<th>M</th>
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</table>
Example (Cont.)

Solution: **Method2 a**
\[ E = Y' + X'Z' \]
\[ = Y'(X+X')(Z+Z') + Z'(Y+Y') \]
\[ = (XY'+X'Y')(Z+Z') + X'YZ' + X'Z'Y' \]
\[ = Y'Z+X'Y'Z+XY'Z'+X'Y'Z' + X'YZ'+X'Z'Y' \]
\[ = m_5 + m_1 + m_4 + m_0 + m_2 + m_0 \]
\[ = m_0 + m_1 + m_2 + m_4 + m_5 \]
\[ = \sum m(0,1,2,4,5) \]
To find the form \( \Pi M_i \), consider the remaining indices
\[ E = \Pi M(3,6,7) \]

Solution: **Method2 b**
\[ E = Y' + X'Z' \]
\[ = Y(X+Z) \]
\[ = YX + YZ \]
\[ = YX(Z+Z') + YZ(X+X') \]
\[ = XYZ+XYZ'+X'YZ \]
\[ E = \sum m(0,1,2,4,5) \]
\[ = (X'+Y'+Z')(X'+Y'+Z)(X+Y'+Z') \]
\[ = M_7 . M_6 . M_3 \]
\[ = \Pi M(3,6,7) \]
To find the form \( S_{m_i} \), consider the remaining indices
\[ E = \sum m(0,1,2,4,5) \]
Canonical Forms

• The sum of minterms and the product of maxterms forms are known as the **canonical forms** of a function.
Standard Forms

• Sum of Products (SOP) and Product of Sums (POS) are also standard forms
  • $AB + CD = (A+C)(B+C)(A+D)(B+D)$
• The sum of minterms is a special case of the SOP form, where all product terms are minterms
• The product of maxterms is a special case of the POS form, where all sum terms are maxterms
**SOP and POS Conversion**

### SOP → POS

\[
F = AB + CD
\]
\[
= (AB+C)(AB+D)
\]
\[
= (A+C)(B+C)(AB+D)
\]
\[
= (A+C)(B+C)(A+D)(B+D)
\]

**Hint 1:** Use id15: \( X+YZ=(X+Y)(X+Z) \)

**Hint 2:** Factor

### POS → SOP

\[
F = (A'+B)(A'+C)(C+D)
\]
\[
= (A'+BC)(C+D)
\]
\[
= A'C+A'D+BCC+BCD
\]
\[
= A'C+A'D+BC+BCD
\]
\[
= A'C+A'D+BC
\]

**Hint 1:** Use id15 \((X+Y)(X+Z)=X+YZ\)

**Hint 2:** Multiply
Thanks