Lecture No. 6

-Thick Cylinders-

6-1 Difference in treatment between thin and thick cylinders - basic assumptions:

The theoretical treatment of thin cylinders assumes that the hoop stress is constant across the thickness of the cylinder wall (Fig. 6.1), and also that there is no pressure gradient across the wall. Neither of these assumptions can be used for thick cylinders for which the variation of hoop and radial stresses is shown in (Fig. 6.2), their values being given by the Lame equations:

\[
\sigma_H = A + \frac{B}{r^2} \quad \ldots 6.1
\]

\[
\sigma_r = A - \frac{B}{r^2} \quad \ldots 6.2
\]

Where:

\[\sigma_H\] = Hoop stress \(\frac{N}{m^2} = Pa\).

\[\sigma_r\] = Radial stress \(\frac{N}{m^2} = Pa\).

\[r\] = Radius (m). \(A\) and \(B\) are Constants.

Figure 6.1: - Thin cylinder subjected to internal pressure.
Consider now the thick cylinder shown in (Fig. 6.3) subjected to an internal pressure $P$, the external pressure being zero.

![Figure 6.3: - Cylinder cross section.](image)

The two known conditions of stress which enable the Lame constants $A$ and $B$ to be determined are:

At $r = R_1$, \( \sigma_r = -P \) and at $r = R_2$, \( \sigma_r = 0 \)

**Note:** The internal pressure is considered as a negative radial stress since it will produce a radial compression (i.e. thinning) of the cylinder walls and the normal stress convention takes compression as negative.

Substituting the above conditions in eqn. (6.2),
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\[ \sigma_r = A - \frac{B}{r^2} \]

\[-P = A - \frac{B}{R_1^2} \quad \text{and} \quad 0 = A - \frac{B}{R_2^2} \]

Then \[ A = \frac{PR_1^2}{(R_2^2 - R_1^2)} \] and \[ B = \frac{PR_1^2 R_2^2}{(R_2^2 - R_1^2)} \]

Substituting A and B in equations 6.1 and 6.2,

\[
\sigma_r = \frac{PR_1^2}{(R_2^2 - R_1^2)} \left[ 1 - \frac{R_2^2}{r^2} \right] \quad \ldots 6.3
\]

\[
\sigma_H = \frac{PR_1^2}{(R_2^2 - R_1^2)} \left[ 1 + \frac{R_2^2}{r^2} \right] \quad \ldots 6.4
\]

6-3 Longitudinal stress: -

Consider now the cross-section of a thick cylinder with closed ends subjected to an internal pressure \( P_1 \) and an external pressure \( P_2 \), (Fig. 6.4).

For horizontal equilibrium:

\[ P_1^2 \pi R_1^2 - P_2^2 \pi R_2^2 = \sigma_L \pi [R_2^2 - R_1^2] \]

Where \( \sigma_L \) is the longitudinal stress set up in the cylinder walls,
Longitudinal stress,

\[ \sigma_L = \frac{P_1 R_1^2 - P_2 R_2^2}{(R_2^2 - R_1^2)} \]  \[\ldots 6.5\]

But for \( P_2 = 0 \) (no external pressure),

\[ \sigma_L = \frac{P_1 R_1^2}{(R_2^2 - R_1^2)} = A, \text{ constant of the Lame equations.} \]  \[\ldots 6.6\]

6-4 Maximum shear stress: -

It has been stated in section 6.1 that the stresses on an element at any point in the cylinder wall are principal stresses.

It follows, therefore, that the maximum shear stress at any point will be given by equation of Tresca theory as,

\[ \frac{\sigma_y}{2} = \tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \]  \[\ldots 6.7\]

\[ \tau_{max} = \frac{\sigma_H - \sigma_r}{2} \]  \[\ldots 6.8\]

\[ \tau_{max} = \frac{1}{2} \left[ (A + \frac{B}{r^2}) - (A - \frac{B}{r^2}) \right] \]  \[\ldots 6.9\]

\[ \tau_{max} = \frac{B}{r^2} \]  \[\ldots 6.10\]

6-5 Change of diameter: -

It has been shown that the diametral strain on a cylinder equals the hoop or circumferential strain.

Change of diameter = diametral strain x original diameter.

= circumferential strain x original diameter.
With the principal stress system of hoop, radial and longitudinal stresses, all assumed tensile, the circumferential strain is given by

\[
\varepsilon_H = \frac{1}{E} (\sigma_H - \nu \sigma_r - \nu \sigma_L) \quad \text{....6.11}
\]

\[
\delta D = \frac{D}{E} (\sigma_H - \nu \sigma_r - \nu \sigma_L) \quad \text{....6.12}
\]

Similarly, the change of length of the cylinder is given by,

\[
\delta L = \frac{L}{E} (\sigma_L - \nu \sigma_r - \nu \sigma_H) \quad \text{....6.13}
\]

6-6 Comparison with thin cylinder theory: -

In order to determine the limits of D/t ratio within which it is safe to use the simple thin cylinder theory, it is necessary to compare the values of stress given by both thin and thick cylinder theory for given pressures and D/t values. Since the maximum hoop stress is normally the limiting factor, it is this stress which will be considered.

Thus for various D/t ratios the stress values from the two theories may be plotted and compared; this is shown in (Fig. 6.5).

Also indicated in (Fig. 6.5) is the percentage error involved in using the thin cylinder theory.

It will be seen that the error will be held within 5 % if D/t ratios in excess of 15 are used.
From the sketch of the stress distributions in Figure 6.6 it is evident that there is a large variation in hoop stress across the wall of a cylinder subjected to internal pressure. The material of the cylinder is not therefore used to its best advantage. To obtain a more uniform hoop stress distribution, cylinders are often built up by shrinking one tube on to the outside of another. When the outer tube contracts on cooling the inner tube is brought into a state of compression. The outer tube will conversely be brought into a state of tension. If this compound cylinder is now subjected to internal pressure the resultant hoop stresses will be the algebraic sum of those resulting from internal pressure and those resulting from shrinkage as
drawn in Fig. 6.6; thus a much smaller total fluctuation of hoop stress is obtained. A similar effect is obtained if a cylinder is wound with wire or steel tape under tension.

![Figure 6.6: - Compound cylinders-combined internal pressure and shrinkage effects.](image)

The method of solution for compound cylinders constructed from similar materials is to break the problem down into three separate effects:

(a) shrinkage pressure only on the inside cylinder.

(b) shrinkage pressure only on the outside cylinder.

(c) internal pressure only on the complete cylinder.

For each of the resulting load conditions there are two known values of radial stress which enable the Lamé constants to be determined in each case

**condition (a) shrinking - internal cylinder:**

At \( r = R_1 \), \( \sigma_r = 0 \)

At \( r = R_c \), \( \sigma_r = -p \) (compressive since it tends to reduce the wall thickness)

**condition (b) shrinking - external cylinder:**

At \( r = R_2 \), \( \sigma_r = 0 \)
At \( r = R_c \), \( \sigma_r = -p \)

**condition (c) internal pressure - compound cylinder:**

At \( r = R_2 \), \( \sigma_r = 0 \)

At \( r = R_1 \), \( \sigma_r = -P_1 \)

Thus for each condition the hoop and radial stresses at any radius can be evaluated and the principle of superposition applied, i.e. the various stresses are then combined algebraically to produce the stresses in the compound cylinder subjected to both shrinkage and internal pressure. In practice this means that the compound cylinder is able to withstand greater internal pressures before failure occurs or, alternatively, that a thinner compound cylinder (with the associated reduction in material cost) may be used to withstand the same internal pressure as the single thick cylinder it replaces.

**Figure 6.7:** - Distribution of hoop and radial stresses through the walls of a compound cylinder.
**Example 6-1:** - A thick cylinder of 100 mm internal radius and 150 mm external radius is subjected to an internal pressure of 60 MN/m$^2$ and an external pressure of 30 MN/m$^2$. Determine the hoop and radial stresses at the inside and outside of the cylinder together with the longitudinal stress if the cylinder is assumed to have closed ends.

**Solution:** -

At $r=0.1$ m, $\sigma_r=-60$ MPa.

$r=0.15$ m, $\sigma_r=-30$ MPa.

So,

-60 = A - 100B \hspace{1cm} \ldots 1

-30 = A - 44.5B \hspace{1cm} \ldots 2

By solving equations 1 and 2,

$A=-6$ and $B=0.54$

Therefore at $r=0.1$ m

$$\sigma_H = A + \frac{B}{r^2} = -6 + \frac{0.54}{(0.1)^2} = 48$$ MPa.

At $r=0.15$ m,

$$\sigma_H = A + \frac{B}{r^2} = -6 + \frac{0.54}{(0.15)^2} = 18$$ MPa

$$\sigma_L = \frac{p_1 R_1^2 - p_2 R_2^2}{(R_2^2 - R_1^2)} = \frac{60(0.1)^2 - 30(0.15)^2}{(0.15^2 - 0.1^2)} = -6$$ MPa i.e. compression.
Example 6-2: - An external pressure of 10 MN/m$^2$ is applied to a thick cylinder of internal diameter 160 mm and external diameter 320 mm. If the maximum hoop stress permitted on the inside wall of the cylinder is limited to 30 MN/m$^2$, what maximum internal pressure can be applied assuming the cylinder has closed ends? What will be the change in outside diameter when this pressure is applied? $E = 207$ GN/m$^2$, $v = 0.29$.

Solution: -

At $r = 0.08$ m, $\sigma_r = -P$, $\frac{1}{r^2} = 156$

At $r = 0.16$ m, $\sigma_r = -10$, $\frac{1}{r^2} = 39$

And at $r = 0.08$ m, $\sigma_H = 30$ MPa

- $10 = A - 39B$ … (1)  
- $30 = A + 156B$ … (2)

Subtracting (1) from (2), $A = -2$ and $B = 0.205$

Therefore, at $r = 0.08$, $\sigma_r = -P = A - 156B = -2 - 156(0.205) = -34$ MPa.

i.e. the allowable internal pressure is 34 MN/m$^2$.

The change in diameter is given by

$$\delta D = \frac{D}{E} (\sigma_H - v\sigma_r - v\sigma_L) \quad \ldots (3)$$

But $\sigma_r = -10$ MN/m$^2$, $\sigma_H = A + \frac{B}{r^2} = -2 + 39 \times 0.205 = 6$ MN/m$^2$
And finally, \( \sigma_L = \frac{p_1 R_1^2 - p_2 R_2^2}{(R_2^2 - R_1^2)} = \frac{(34 + 0.08^2 - 10*0.16^2)}{(0.16^2 - 0.08^2)} = -1.98 \text{ MPa} \) i.e compressive.

Substitute \( \sigma_r, \sigma_H \) and \( \sigma_L \) in eqn. 3,

\[
\delta D = \frac{0.32}{207 \times 10^9} [6 - 0.29(-10) - 0.29(-1.98)] 10^6 = 14.7 \mu \text{m}
\]

**Example 6-3:** - A compound cylinder is formed by shrinking a tube of 250 mm internal diameter and 25 mm wall thickness onto another tube of 250 mm external diameter and 25 mm wall thickness, both tubes being made of the same material. The stress set up at the junction owing to shrinkage is 10 MN/m\(^2\). The compound tube is then subjected to an internal pressure of 80 MN/m\(^2\). Compare the hoop stress distribution now obtained with that of a single cylinder of 300 mm external diameter and 50 mm thickness subjected to the same internal pressure.

A solution is obtained as described before by considering the effects of shrinkage and internal pressure separately and combining the results algebraically.

**Shrinkage only - outer tube,**

At \( r = 0.15, \sigma_r = 0 \) and at \( r = 0.125, \sigma_r = -10 \text{ MN/m}^2 \)

\[
0 = A - \frac{B}{(0.15^2)} = A - 44.5B
\]

\[
-10 = A - \frac{B}{(0.125^2)} = A - 64B
\]

\[
\therefore \quad B = 0.514, \quad A = 22.85
\]

Hoop stress at 0.15 m radius = \( A + 44.5B = 45.7 \text{ MPa} \).
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hoop stress at 0.125 m radius: $A + 64B = 55.75 \text{MPa}$.

**Shrinkage only - inner tube,**

At $r = 0.10$, $\sigma_r = 0$ and at $r = 0.125$, $\sigma_r = -10 \text{ MN/m}^2$

$0 = A - \frac{B}{(0.1^2)} = A - 100B$

$-10 = A - \frac{B}{(0.125^2)} = A - 64B$

$\therefore B = -0.278, \quad A = -27.8$

hoop stress at 0.125 m radius: $A + 64B = -45.6 \text{ MPa}$.

hoop stress at 0.10 m radius: $A + 100B = -55.6 \text{ MPa}$.

**Considering internal pressure only (on complete cylinder)**

At $r = 0.15$, $\sigma_r = 0$ and at $r = 0.10$, $\sigma_r = -80$

$0 = A - \frac{B}{(0.15^2)} = A - 44.5B$

$-80 = A - \frac{B}{(0.1^2)} = A - 100B$

$\therefore B = 1.44, \quad A = 64.2$

At $r = 0.15 \text{ m}$, \quad $\sigma_H = A + 44.5B = 128.4 \text{ MN/m}^2$

$r= 0.125 \text{ m}$, \quad $\sigma_H = A + 64B = 156.4 \text{ MN/m}^2$

$r= 0.1 \text{ m}$, \quad $\sigma_H = A + 100B = 208.2 \text{ MN/m}^2$

The resultant stresses for combined shrinkage and internal pressure are then:

outer tube: $r = 0.15 \quad \sigma_H = 128.4 + 45.7 = 174.1 \text{ MN/m}^2$. 

\[ r = 0.125 \quad \sigma_H = 156.4 + 55.75 = 212.15 \text{ MN/m}^2. \]

inner tube: \[ r = 0.125 \quad \sigma_H = 156.4 - 45.6 = 110.8 \text{ MN/m}^2. \]

\[ r = 0.1 \quad \sigma_H = 208.2 - 55.6 = 152.6 \text{ MN/m}^2. \]

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