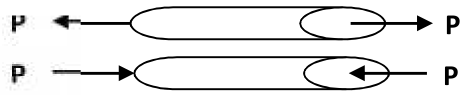
Lecture No. 1

-Review of strength of material-

1-1 Direct stress: -

1- Tensile stress.

2- Compression stress.



$\sigma_{axial} = \frac{P}{A}$ $\frac{N}{m^2}$ …………1.1

Where: -

σ_{axial} = axial stress.

P = axial load (N).

A= cross section area (m^2).

$\varepsilon = \frac{\sigma}{E}$ …………1.2

Where: -

ε= Strain

E= modulus of elasticity

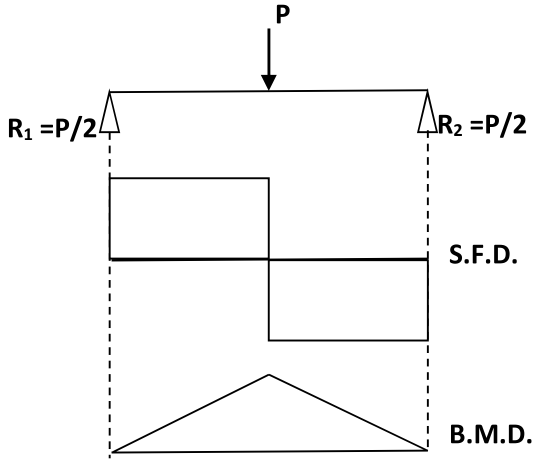
2-2 Bending stress: -

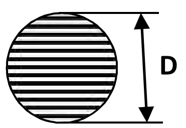
$\sigma_b = \frac{My}{I}$ …………1.3

Where: -

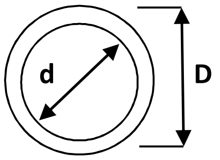
M= moment (N.m)

I= moment of inertia (m^4)

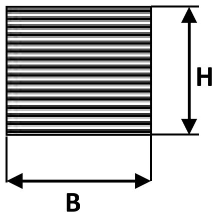




$I = \frac{\pi D^4}{64}$



$$I=\frac{\pi(D^4-d^4)}{64}$$



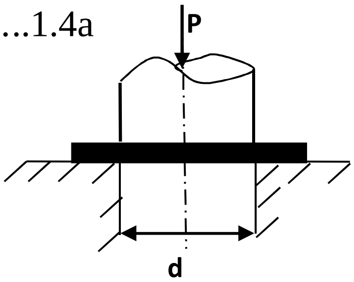
$$I=\frac{BH^3}{12}$$

2-3 Shear stresses: -

The shear stress may be : -

a- Direct shear: - This occurs over an area parallel to the applied load.

$$\tau_{avg}=\frac{P}{A} \quad \frac{N}{m^2} \qquad \text{........1.4a}$$



Where: -

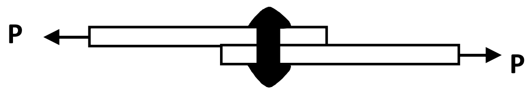
τ_{avg} =Average shear stress.

P = Shear force.

A = Area of the section.

b- Induced shear: - This occurs over section (or face shear).

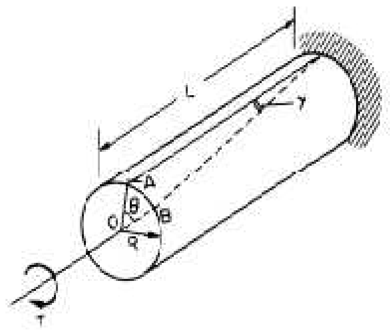
$$\tau_{avg}=\frac{P}{A} \quad \frac{N}{m^2} \qquad \text{.......1.4b}$$



Where: -

$$A_{rivet}=\frac{\pi d^2}{4}$$

c- Torsion: -



$$\tau_{max} = \frac{TR}{J} \qquad \text{……1.5}$$

$$\Theta = \frac{TL}{JG} \qquad \text{……1.6}$$

Where: -

T= Torque.

L= Length of shaft.

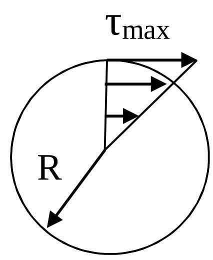
G= Shear modulus or Modulus of rigidity.

ϴ= Angle of twist.

J = Polar moment of inertia.

For solid shaft $J = \frac{\pi D^4}{32}$.

For hollow shaft $J = \frac{\pi (D^4 - d^4)}{32}$.



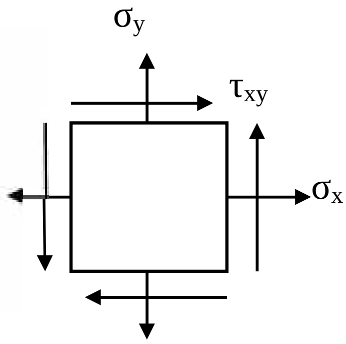
Shear stress distribution in circular shaft

2-4 Combined stresses: -

In sections 2.1, 2.2 and 2.3, we studied three basic types of loading, axial, flexural (bending), shear and torsion. Each of three types was discussed on the assumption that only one of these loadings was acting on a member at a time, but in actual condition in most cases one or more of these loading act simultaneously upon a member. There are four possible combinations of these loadings.

a- Axial and Flexural loadings.
b- Axial and Torsional loadings.
c- Torsional and Flexural loadings.
d- Axial, Torsional and Flexural loadings acting simultaneously.

Combined stresses → principal stresses (σ_1, σ_2, σ_3).
→ Max. Shear stress (τ_{max}).



σ_1 or $\sigma_{max} = \frac{\sigma_x+\sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}$ …1.7

σ_2 or $\sigma_{min} = \frac{\sigma_x+\sigma_y}{2} - \frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}$ …1.8

Where: -

$\sigma_1 > \sigma_2$

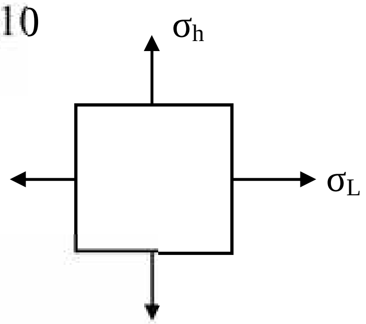
$\tau_{max} = \frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}$ …1.9

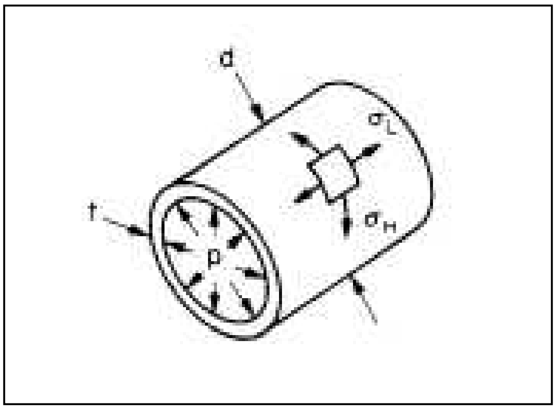
2-5 Thin wall cylinders subjected to internal pressure:-

Cylinders or spherical vessels are commonly used in industry to serve as boilers or tanks. When vessel under pressure, the material of which they are made is subjected to loading from all directions.

In general "thin wall" refers to a vessel having an inner radius to wall thickness ratio of 10 or more.

i.e. $\frac{r}{t} \geq 10$





$\sigma_h = \frac{Pd}{2t}$ (Hoop stress) ….1.10

$\sigma_L = \frac{Pd}{4t}$ (Longitudinal stress). … 1.11

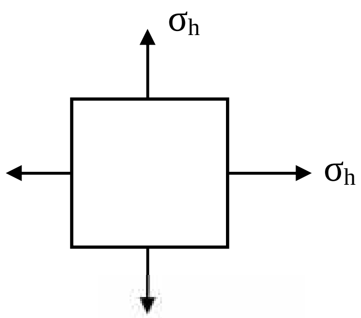
Where: -

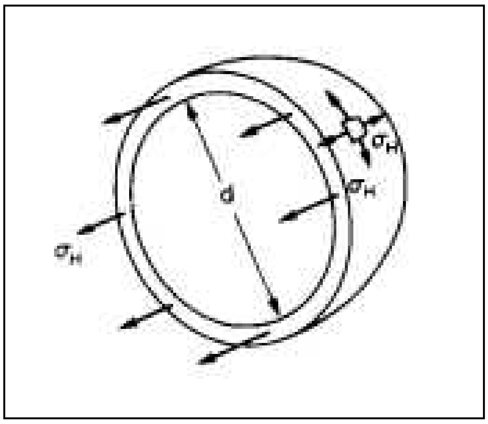
P= Internal pressure (Pa or $\frac{N}{m^2}$).

d= 2r= Internal diameter.

t= Cylinder thickness.

For spherical vessel: -





$$\sigma_h = \sigma = \frac{Pd}{4t} \qquad \ldots.1.12$$

Where: -

d=spherical radius (inner).

Prob.1 A steel bar ***ABCD*** consists of three sections: ***AB*** is of **20mm** diameter and 200 mm long, ***BC*** is **25 mm** square and **400 mm** long, and ***CD*** is of **12 mm** diameter and **200 mm** long. The bar is subjected to an axial compressive load which induces a stress of **30 MN/m²** on the largest cross-section. Determine the total decrease in the length of the bar when the load is applied. For steel **E = 210GN/m²**.

Prob.2 A beam AB, **1.2 m** long, is simply-supported at its ends A and Band carries two concentrated loads, one of **10 kN** at **C**, the other **15 kN** at **D**. Point **C** is **0.4 m** from **A**, point **D** is **1 m** from **A**. Draw the **S.F.** and **B.M.** diagrams for the beam inserting principal values.

Prob.3 An I-section girder, **200 mm** wide by **300 mm** deep, with flange and web of thickness **20 mm** is used as a simply supported beam over a span of **7 m**. The girder carries a distributed load of ***5* kN/m** and **a** concentrated load of **20 kN** at mid-span. Determine: (a) the second moment of area of the cross-section of the girder, (b) the maximum stress set-up.

Prob.4 Determine the dimensions of a hollow shaft with a diameter ratio of **3:4** which is to transmit **60 kW** at **200 rev/min**. The maximum shear stress in the shaft is limited to **70 MN/m^2** and the angle of twist to **3.8^o** in a length of **4 m**.
For the shaft material **G = 80 GN/m^2**.

Prob.5 A circular bar ***ABC,*** **3 m** long, is rigidly fixed at its ends ***A*** and ***C.*** The portion ***AB*** is **1.8 m** long and of **50 mm** diameter and ***BC*** is **1.2 m** long and of **25 mm** diameter. If a twisting moment of **680 N.m** is applied at ***B,*** determine the values of the resisting moments at ***A*** and ***C*** and the maximum stress in each section of the shaft. What will be the angle of twist of each portion?
For the material of the shaft **G = 80 GN/m^2**.

Prob.6 A hollow shaft is **460 mm** inside diameter and **25 mm** thick. It is subjected to an internal pressure of **2 MN/m^2**, a bending moment of **25 kN.m** and a torque of **40 kN.m**. Assuming the shaft may be treated as a thin cylinder, make a neat sketch of an element of the shaft, showing the stresses resulting from all three actions.
Determine the values of the principal stresses and the maximum shear stress.

……………….……………………..End………………………………………

Lecture No. 2

-Three dimensional stresses and Strains-

2-1 General.

Consider a cube of infinitesimal dimensions shown in figure (1), All stresses acting on this cube are identified on the diagram. The subscripts (τ) are the shear stress, associate the stress with a plane perpendicular to a given axis, the second designate the direction of the stress, i.e.

τ_{xy}

Face Direction

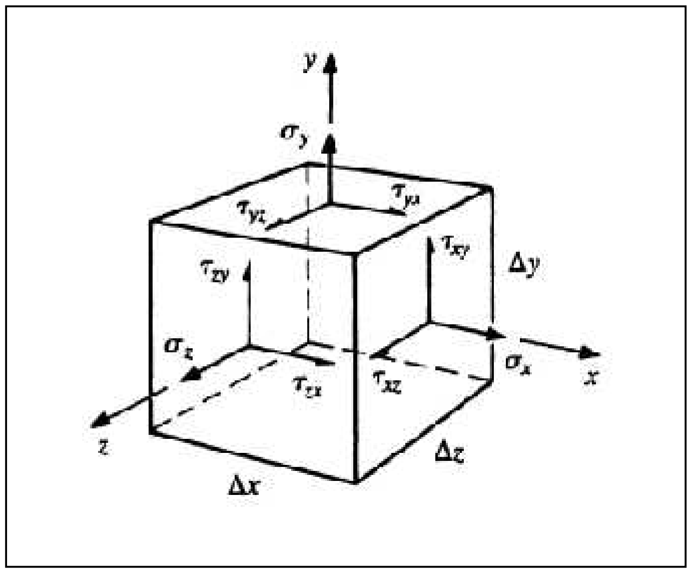


Figure (1)

The stress symbols in figure (1),

shows that three normal stresses: -

$\sigma_x = \tau_{xx}$, $\sigma_y = \tau_{yy}$, $\sigma_z = \tau_{zz}$

and six shearing stresses, τ_{xy} , τ_{xz} , τ_{yx} , τ_{yz} , τ_{zx} , τ_{zy}.

The force vector (P) has only three components P_x , P_y and P_z.

$$\begin{matrix} P_x \\ P_y \\ P_z \end{matrix}$$

And stress vector: -

$$\begin{matrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{matrix} = \begin{matrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{matrix} \qquad \ldots\ldots 2.1$$

This is a matrix representation of the stress tensor. It is a second –rank tensor requiring two indices to identify its elements or components. A vector is first- rank tensor, and scalar is a zero tensor.

Sometimes, for brevity, a stress tensor is written in identical natation as τ_{ij} , where i, j and k designations x, y and z.

The stress tensor is symmetric, i.e. $\tau_{ij} = \tau_{ji}$ or

$$\begin{aligned} \tau_{xy} &= \tau_{yx} \\ \tau_{xz} &= \tau_{zx} \\ \tau_{yz} &= \tau_{zy} \end{aligned} \qquad \ldots 2.2$$

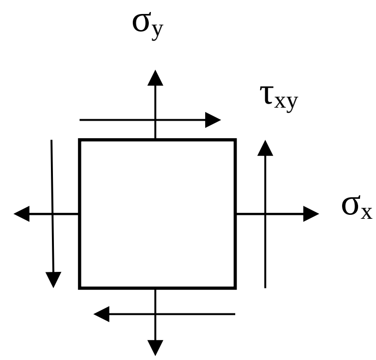
2-2 Two dimensional stress (Biaxial stress): -

For a two dimensional case of plane stress where (σ_3=0).

Where:-

σ = Normal stress

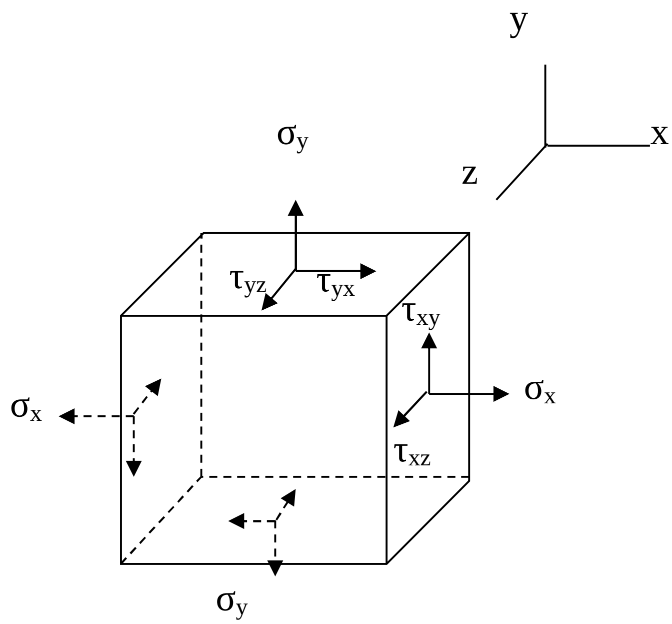
τ = Shear stress



$$\sigma_{ij} = \begin{matrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_y & 0 \\ 0 & 0 & 0 \end{matrix}$$

2-3 Three dimensional stress (Triaxial stress): -

There are nine components of stress. Moment equilibrium can be used to reduce the number of stress components to six.



$\tau_{xy} = \tau_{yx}$

$\tau_{xz} = \tau_{zx}$

$\tau_{yz} = \tau_{zy}$

Stresses in 3D is represented by vector tensor: -

$$\sigma_{ijk} = \begin{matrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{matrix}$$

A homogenous linear equation has a solution only if the determinant of the coefficient matrix is equal to zero this called Eigenvalue problem. Such as

$$\begin{matrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{matrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \quad \ldots 2.3$$

In case that the equation is in eigenvalue problem such as: -

$$\begin{matrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{matrix} = 0 \quad \ldots 2.4$$

The determinant can be expanded to yield the equation

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \quad . \quad ..2.5$$

Where I_1, I_2 and I_3 are the first, second and third invariants of the Cauchy stress tensor.

$$I_1 = \sigma_x + \sigma_y + \sigma_z \quad ..2.6$$

$$I_2 = \sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}{}^2 - \tau_{xz}{}^2 - \tau_{yz}{}^2 \qquad ...2.7$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - \sigma_x\tau_{yz}{}^2 - \sigma_y\tau_{xz}{}^2 - \sigma_z\tau_{xy}{}^2 \qquad ...2.8$$

There are three roots the characteristic equation 2.5, σ_1, σ_2 and σ_3. Each root is one of the principal stresses. The direction cosines can be found by substituting the principal stresses into the homogenous equation 2.3 and solving. The direction cosines define the principal direction or planes.

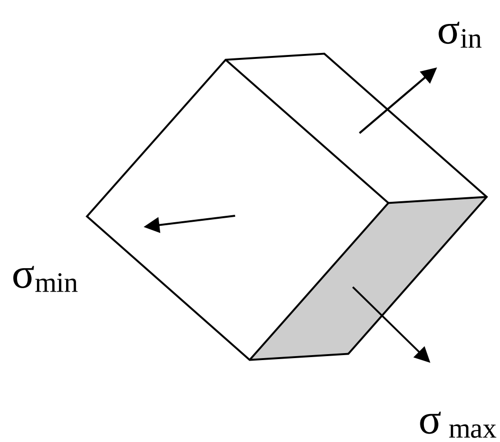
$\sigma_{max} \geq \sigma_{in} \geq \sigma_{min}$

$\sigma_1 \geq \sigma_2 \geq \sigma_3$

$$\tau_{12} = \frac{\sigma_1 - \sigma_2}{2} \qquad ......2.9$$

$$\tau_{23} = \frac{\sigma_2 - \sigma_3}{2} \qquad ......2.10$$

$$\tau_{13} = \frac{\sigma_1 - \sigma_3}{2} = \tau_{max} \qquad ......2.11$$



2-4 Poisson's Ratio (υ): -

Biaxial and Triaxial deformation another type of elastic deformation is the change in transverse dimensions accompanying axial tension or compression.

Experiments show that if a bar is lengthened by axial tension, there is a reduction in the transverse dimensions. Simeon D. Poisson showed in 1811 that the ratio of the unit dimensions or strain in these directions is constant for stresses within the proportional limit,

Poisson's Ratio (υ) = $-\frac{\textit{Lateral strain}}{\textit{Longitudinal Strain}}$ …2.12

$$=-\frac{\varepsilon_y}{\varepsilon_x}=-\frac{\varepsilon_z}{\varepsilon_x}$$

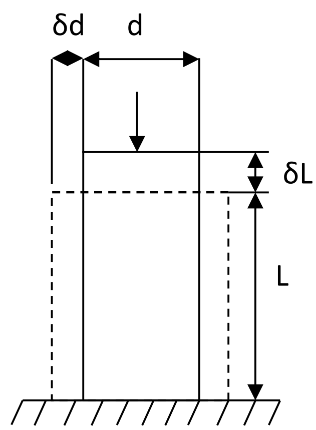
Where: -

$\varepsilon_x = Strain\ in\ x - direction.$

$\varepsilon_y = Strain\ in\ y - direction.$

$\varepsilon_z = Strain\ in\ z - direction.$

Note: - Minus sign indicates a decrease in transverse dimension position as in the case of tensile elongation.



$$\varepsilon_L = \frac{\delta L}{L}$$

$$\varepsilon_d = \frac{\delta d}{d}$$

$$\therefore \nu = -\frac{\delta d/d}{\delta L/L}$$

2-5 Three Dimensional Strains: -

For Tri-axial stress state,

If $\sigma_y, \sigma_z = 0 \rightarrow \varepsilon_x = \frac{\sigma_x}{E}$

If $\sigma_x, \sigma_z = 0 \rightarrow \varepsilon_y = \frac{\sigma_y}{E}$ $and\ \varepsilon_x = -\upsilon \frac{\sigma_y}{E}$

Now, If $\sigma_x, \sigma_y = 0 \rightarrow \varepsilon_z = \frac{\sigma_z}{E}$ $and\ \varepsilon_x = -\upsilon \frac{\sigma_z}{E}$

$$\boxed{\begin{aligned} \therefore \boldsymbol{\varepsilon_x} &= \frac{\boldsymbol{\sigma_x}}{\boldsymbol{E}} - \boldsymbol{\upsilon}\frac{\boldsymbol{\sigma_y}}{\boldsymbol{E}} - \boldsymbol{\upsilon}\frac{\boldsymbol{\sigma_z}}{\boldsymbol{E}} \\ \therefore \boldsymbol{\varepsilon_y} &= \frac{\boldsymbol{\sigma_y}}{\boldsymbol{E}} - \boldsymbol{\upsilon}\frac{\boldsymbol{\sigma_x}}{\boldsymbol{E}} - \boldsymbol{\upsilon}\frac{\boldsymbol{\sigma_z}}{\boldsymbol{E}} \\ \therefore \boldsymbol{\varepsilon_z} &= \frac{\boldsymbol{\sigma_z}}{\boldsymbol{E}} - \boldsymbol{\upsilon}\frac{\boldsymbol{\sigma_x}}{\boldsymbol{E}} - \boldsymbol{\upsilon}\frac{\boldsymbol{\sigma_y}}{\boldsymbol{E}} \end{aligned}} \quad \ldots 2.13$$

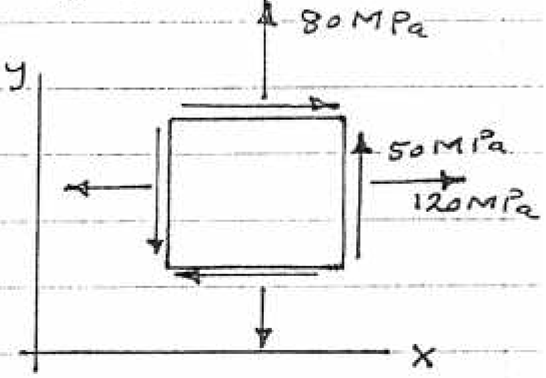
So, the principal strains in terms of principal stresses,

$$\boxed{\begin{aligned} \therefore \boldsymbol{\varepsilon_1} &= \frac{\boldsymbol{1}}{\boldsymbol{E}}(\boldsymbol{\sigma_1} - \boldsymbol{\nu\sigma_2} - \boldsymbol{\upsilon\sigma_3}) \\ \therefore \boldsymbol{\varepsilon_2} &= \frac{\boldsymbol{1}}{\boldsymbol{E}}(\boldsymbol{\sigma_2} - \boldsymbol{\nu\sigma_1} - \boldsymbol{\upsilon\sigma_3}) \\ \therefore \boldsymbol{\varepsilon_3} &= \frac{\boldsymbol{1}}{\boldsymbol{E}}(\boldsymbol{\sigma_3} - \boldsymbol{\nu\sigma_1} - \boldsymbol{\upsilon\sigma_2}) \end{aligned}} \quad \ldots 2.14$$

Example 6-1.

For the following state of stress, find the principle and critical values.

$$\sigma_{ij} = \begin{pmatrix} 120 & 50 & 0 \\ 50 & 80 & 0 \\ 0 & 0 & 0 \end{pmatrix} MPa$$



Solution:

Tensor shows that:

$\sigma_z = 0$ and $\tau_{xz} = \tau_{yz} = 0$

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$
$$= 120 + 80 + 0 = 200 \text{ MPa}$$
$$I_2 = \sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2$$
$$= 120*80 + 120*0 + 80*0 - (50)^2 - (0)^2 - (0)^2 = 7100$$
$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2$$
$$= 120*80*0 + 2*50*0*0 - 120*0^2 - 80*0^2 - 0*80^2 = 0$$

But $\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$

$$\sigma^3 - 200\sigma^2 + 7100\sigma - 0 = 0$$
$$\sigma(\sigma^2 - 200\sigma + 7100) = 0$$

either $\sigma = 0$ or $\sigma = 154$ or $\sigma = 45$

$\therefore \sigma_1 = 154$, $\sigma_2 = 45$, $\sigma_3 = 0$

$$\tau_{12} = \frac{\sigma_1 - \sigma_2}{2} = \frac{154-45}{2} = 54.5 \text{ MPa}$$

$$\tau_{23} = \frac{\sigma_2 - \sigma_3}{2} = \frac{45-0}{2} = 22.5 \text{ MPa}$$

$$\tau_{13} = \frac{\sigma_1 - \sigma_3}{2} = \frac{154-0}{2} = 77 \text{ MPa}$$

$\therefore \tau_{max} = 77$ MPa

Example 6-2.

Determine the maximum principal stresses for the following triaxial stress state.

$$\sigma = \begin{pmatrix} 20 & 40 & -30 \\ 40 & 30 & 25 \\ -30 & 25 & -10 \end{pmatrix} MPa$$

solution:

$$\sigma = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} = \begin{pmatrix} 20 & 40 & -30 \\ 40 & 30 & 25 \\ -30 & 25 & -10 \end{pmatrix}$$

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z = 20 + 30 - 10 = 40\ MPa$$

$$I_2 = \sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2$$

$$= 20*30 + 20*-10 + 30*-10 - (40)^2 - (-30)^2 - (25)^2$$

$$= -3025\ MPa$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2$$

$$= 20*30*-10 + 2*40*(-30)*25 - 20*25^2 - 30*(-30)^2 - (-10)*(40)^2$$

$$= -89500\ MPa$$

$$\sigma^3 - 40\sigma^2 - 3025\sigma + 89500 = 0$$

From which;

$$\sigma_1 = 65.3\ MPa$$

$$\sigma_2 = 26.5\ MPa$$

$$\sigma_3 = -51.8\ MPa$$

&

$$\tau_{max} = \frac{65.3 + 51.8}{2} = 58.5\ MPa$$

……………………………….End……………………………

Lecture No. 3

-Theories of failure-

3-1 Stress - Strain diagram.

The properties of materials such as stiffness, hardness, toughness and ductility are determined by making tests on materials and comparing the results with established standards. One of these tests is the tension test for mild steel as an example:

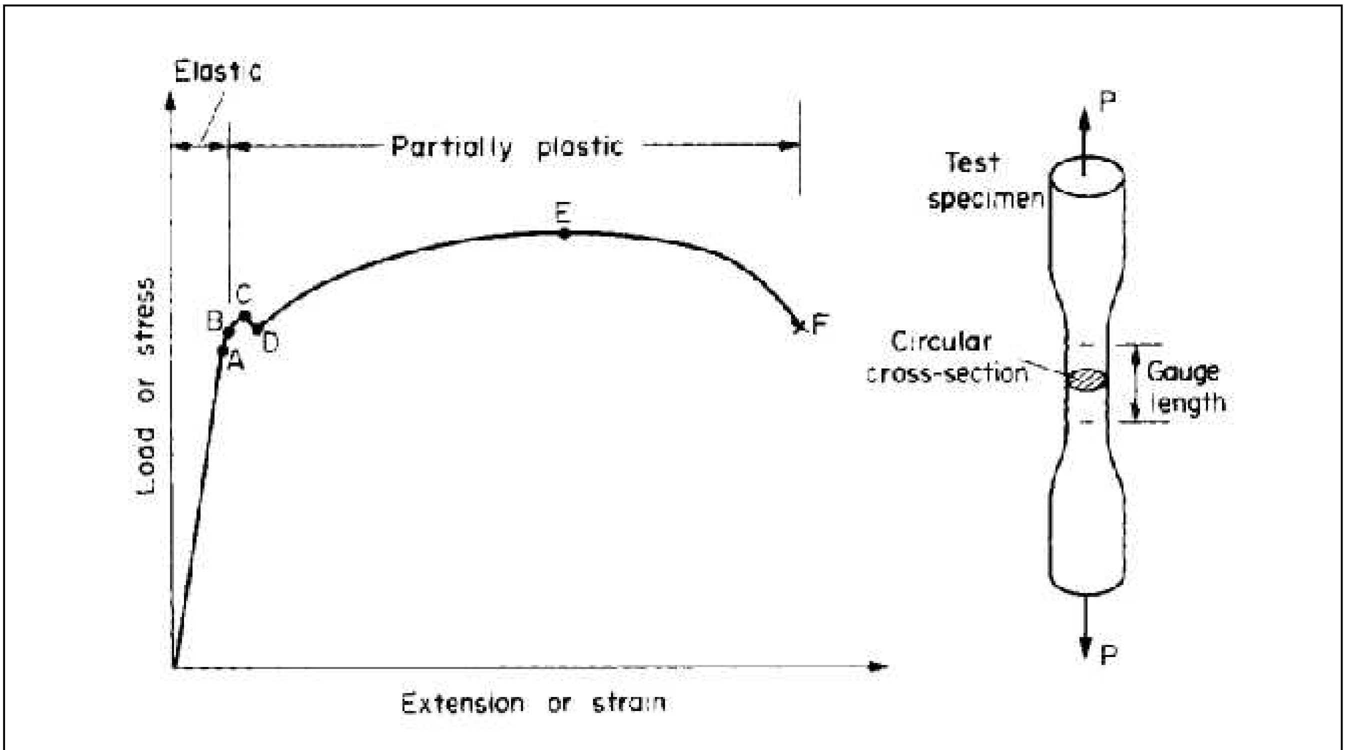


Fig. 3.1. Typical tensile test curve for mild steel.

For the first part of the test it will be observed that Hooke’s law is obeyed, i.e. the material behaves elastically and stress is proportional to strain, giving the straight-line graph indicated. Some point A is eventually reached, however, when the linear nature of the graph ceases and this point is termed the limit of proportionality.

For a short period beyond this point the material may still be elastic in the sense that deformations are completely recovered when load is removed (i.e. strain returns to zero) but Simple Stress and Strain Hooke's law does not apply. The limiting point B for this condition is termed the elastic limit.

For most practical purposes it can often be assumed that points A and B are coincident. Beyond the elastic limit plastic deformation occurs and strains are not totally recoverable.

There will thus be some permanent deformation or permanent set when load is removed.

After the points C, termed the upper yield point, and D, the lower yield point, relatively rapid increases in strain occur without correspondingly high increases in load or stress. The graph thus becomes much shallower and covers a much greater portion of the strain axis than does the elastic range of the material.

For certain materials, for example, high carbon steels and non-ferrous metals, it is not possible to detect any difference between the upper and lower yield points and in some cases no yield point exists at all. In such cases a proof stress is used to indicate the onset of plastic strain or as a comparison of the relative properties with another similar material. This involves a measure of the permanent deformation produced by a loading cycle; the 0.1 % proof stress, for example, is that stress which, when removed, produces a permanent strain or "set" of 0.1 % of the original gauge length-see Fig. 3.2(a).

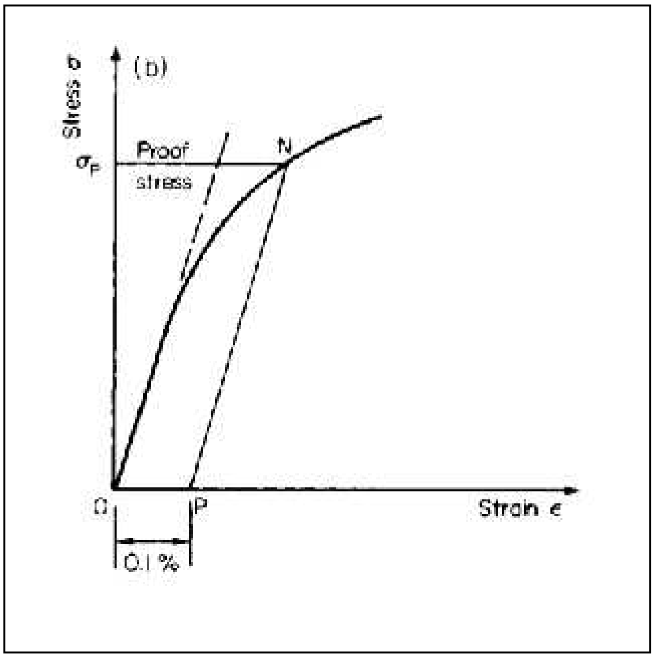


Fig 3.2a Determination of 0.1% proof stress

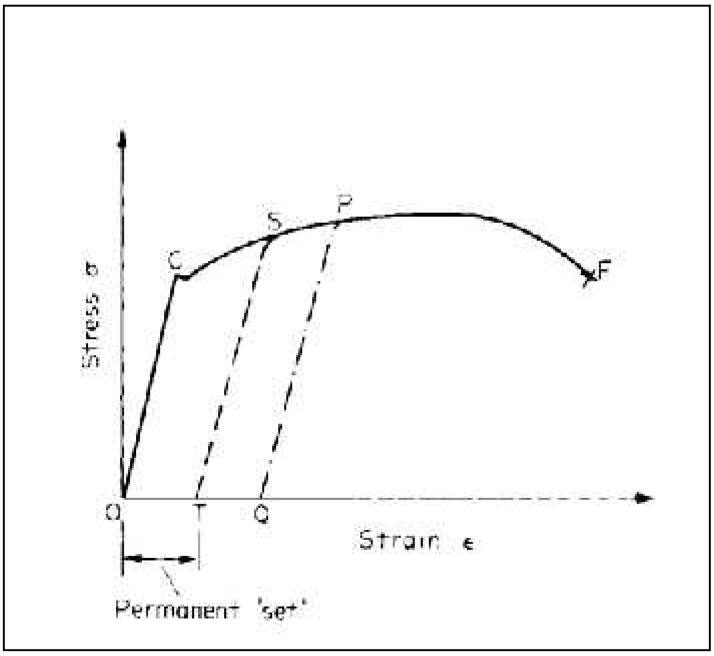


Fig 3.2b Permanent deformation or "set" after straining beyond the yield point.

Beyond the yield point some increase in load is required to take the strain to point E on the graph. Between D and E the material is said to be in the elastic-plastic state, some of the section remaining elastic and hence contributing to recovery of the original dimensions if load is removed, the remainder being plastic. Beyond E the cross-sectional area of the bar begins to reduce rapidly over a relatively small length of the bar and the bar is said to neck. This necking takes place whilst the load reduces, and fracture of the bar finally occurs at point F.

The nominal stress at failure, termed the maximum or ultimate tensile stress, is given by the load at E divided by the original cross-sectional area of the bar. (This is also known as the tensile strength of the material of the bar.)

3-2 Theories of failure: -

Various theories of failure have been proposed, their purpose being to establish from the behavior of a material subjected to simple tension or compression test, the point at which failure will occur under any type of combined loading. By failure we mean either yielding or actual rupture, whichever occurs first.

The following are some of these theories: -

1- The maximum shear stress theory (Tresca): -

This theory states that failure can be assumed to occur when the maximum shear stress in the complex stress system becomes equal to that at the yield point in the simple tensile test.

Since the maximum shear stress is half the greatest difference between two principal stresses the criterion of failure becomes: -

$$\frac{1}{2}(\sigma_{max} - \sigma_{min}) = \frac{1}{2}\sigma_{yield} \qquad \text{......3.1}$$

2- The maximum stress theory: -

The maximum stress theory is proposed by Rankine is the oldest as well as the simplest of all of theories. It is based on the assumption that failure occurs when the max. Principal stress on an element reaches a limiting value, the limit being the yield point in a simple tension test (or ultimate strength if the material is brittle). The theory disregards the effect of possible other principal stresses and shearing stresses on the other through the element.

$$\sigma_{max} = \sigma_{yield} \qquad \text{for ductile material} \qquad \text{.....3.2a}$$

$$\sigma_{max} = \sigma_{ultimate} \qquad \text{for brittle material} \qquad \text{....3.2b}$$

3- Maximum shear strain energy per unit volume(Von- Mises) : -

This theory assumes that yielding can occur in a general three dimensional state of stress when the root mean square of the differences between the principal stresses is equal to the same value in a tensile test.

If $\sigma_1 \geq \sigma_2 \geq \sigma_3$ are the principal stresses and σ_y is the yield stress in simple tension, this concept gives: -

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y{}^2 \qquad \text{…..3.3}$$

For a state of 2D-stress, where $\sigma_3 = 0$

$$\sigma_1{}^2 - \sigma_1\sigma_2 + \sigma_2{}^2 = \sigma_y{}^2 \qquad \text{…..3.4}$$

4- Mohr's modified shear stress theory for brittle materials: -

Brittle materials in general show little ability to deform plastically and hence will usually fracture at, or very near to, the elastic limit. Any of the so-called "yield criteria" introduced above, therefore, will normally imply fracture of a brittle material. It has been stated previously, however, that brittle materials are usually considerably stronger in compression than in tension and to allow for this Mohr has proposed a construction based on his stress circle in the application of the maximum shear stress theory.

$$\frac{\sigma_1}{\sigma_{yt}} + \frac{\sigma_2}{\sigma_{yc}} = 1$$

Where: -

σ_{yt} and σ_{yc} the yield strengths of the brittle material in tension and compression respectively.

3-3 Summery: -

Experimental work shows best agreement with the Mises theory when applied to ductile material but the Tresca maximum shear stress theory is the most conservative (i.e. the safest) theory and this, together with its easily applied and simple formula, probably explains its widespread use in industry.

For brittle materials the maximum principal stress or Mohr "internal friction" theories are most suitable.

Examples: -

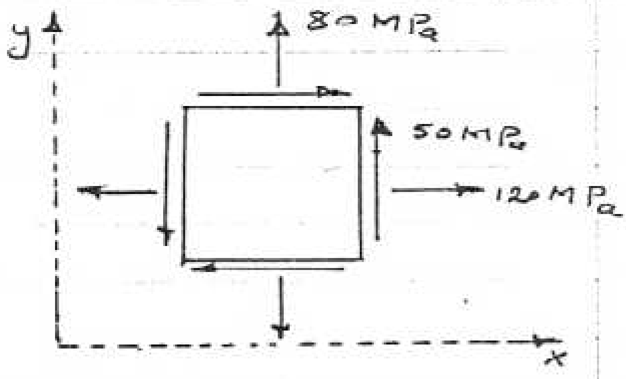
Example).

Determine if failure will occur for the following complex stress state, given the material has a tensile yield strength of 250 MPa and an ultimate tensile strength

σ_y 300 MPa.

$$\sigma_{ij} = \begin{pmatrix} 120 & 50 & 0 \\ 50 & 80 & 0 \\ 0 & 0 & 0 \end{pmatrix} MPa$$

Solution:



Tensor shows that:

$\sigma_z = 0$ &

$\tau_{xz} = \tau_{yz} = 0$

By using 3D stress analysis

$\sigma_3 = 0$ MPa,

$\sigma_2 = 45$ MPa,

$\sigma_1 = 154$ MPa

$\tau_{max} = 77$ MPa

1. The max. shear theory

$\sigma_{yield}/2 = 250/2 = 125$ MPa

$\tau_{max} = 77 < 125$ MPa, Safe: FS = 1.62

2. Max. stress theory

$\sigma_{ultimate} = 350$ MPa

$\sigma_{max} = \sigma_1 = 154 < 350$ MPa, Safe; FS = 2.27

3. Von-Mises

$$\sigma_{yield} = \frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}}\left[(154-45)^2 + (45-0)^2 + (0-154)^2\right]^{1/2} = 137 \text{ MPa}$$

$\therefore$ 137 < 250 MPa, Safe; FS = 250/137 = 1.82

Example: -

The cast iron used in the manufacture of an engineering component has tensile and compressive strengths of 400 MN/m^2 and 1.20 GN/m^2 respectively.

If the maximum value of the tensile principal stress is to be limited to one-quarter of the tensile strength, determine the maximum value and nature of the other principal stress using Mohr's modified yield theory for brittle materials.

Solution: -

Maximum principal stress = $\frac{400}{4} = 100$MN/m^2

According to Mohr's theory

$$\frac{\sigma_1}{\sigma_{yt}} + \frac{\sigma_2}{\sigma_{yc}} = 1$$

$$\frac{100*10^6}{400*10^6} + \frac{\sigma_2}{-1.2*10^9} = 1$$

$$\sigma_2 = -900\ MPa$$

……………………………End…………………………

Lecture No. 4

-Fatigue-

4-1 Repeated Loading (Cyclic Loading): -

Many machine parts are subjected to varying stresses caused by repeated loading and unloading. Parts subjected to such loading frequently fail at a stress much smaller than the ultimate strength determined by a static tensile test. Such type of failure of a material is knows as Fatigue.

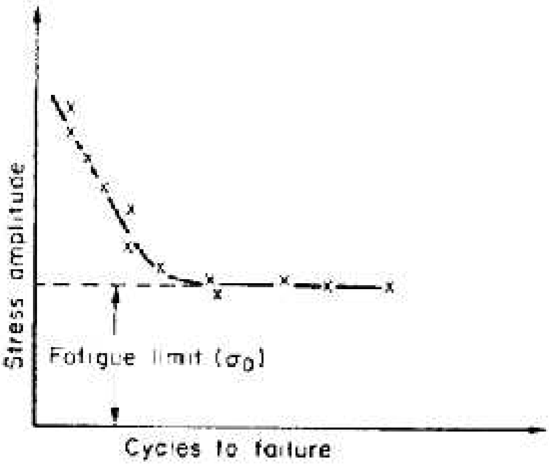
The failure is caused by means of a progressive crack formation which is usually fine and microscopic size. The failure may occur even without out any prior indication.

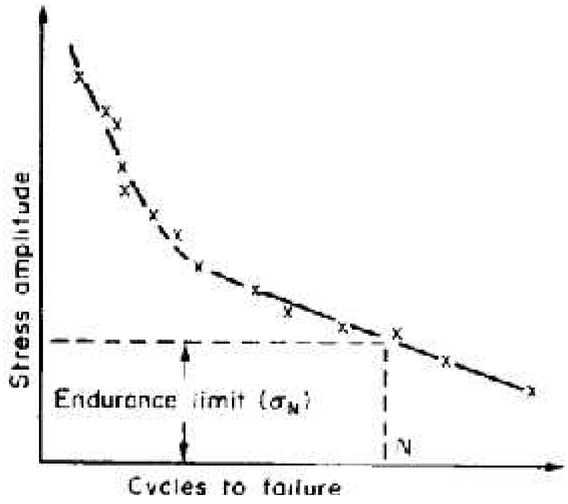
The fatigue of material is effected by: -

1- The size of the component.
2- Relative magnitude of static and fluctuating loads.
3- The number of load reversal.

In order to properly design members that are subjected to stress reversal, it is necessary to know the stress that can safely be carried an infinite number of times (or a somewhat higher stress that can be carried safely for a limited number of reversals, as when a machine is used only occasionally and may therefore have a long life).

Testing to determine these values is called fatigue testing.





-a- **-b-**

Fig 4.1 S-N curve showing **(a)** Fatigue limit. **(b)** Endurance limit

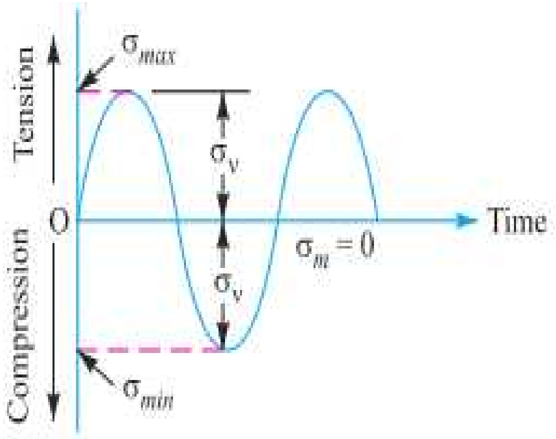
It is shown from figure 4.1 that if the stress is kept below a certain value, as shown by dotted line, the material will not fail whatever may be the number of cycles. This stress represented by the dotted line is known as Endurance or Fatigue limit (σ_e).It is defines as a maximum value of the completely reversal bending stress which a polished standard specimens can withstand without failure for infinite life number of cycles usually 10^7 cycles.

Generally, ferrous metal specimens often produce S/N curves which exhibit fatigue limits as indicated in Fig. 4.1a. The ‘Fatigue strength” or “endurance limit”, is the stress condition under which a specimen would have a fatigue life of N cycles as shown in Fig. 4.1b. Non-ferrous metal specimens show this type of curve and hence components made from aluminum, copper and nickel, etc., must always be designed for a finite life.

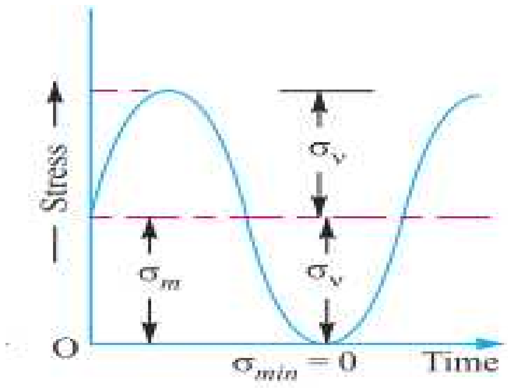
4-2 Types of Stresses: -

Stresses

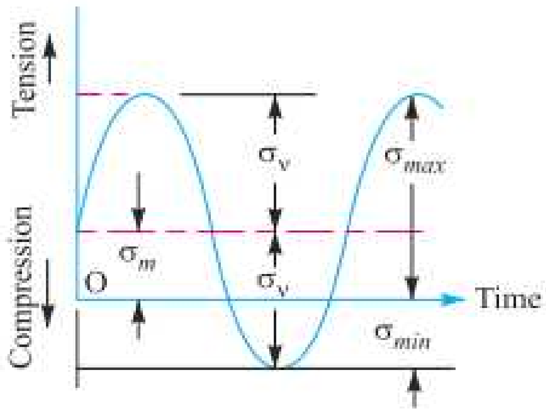
- **Static**
- **Dynamic**
  - **Impact (shock)**
  - **Fatigue**
    - **Completely Reversed**
    - **repeated**
    - **Fluctuating**



Completely reversed stress.



Repeated stress.



Fluctuating stress.

Fig.4.2 Types of fatigue stresses

Mean or average stress $\sigma_m = \frac{\sigma_{max}+\sigma_{min}}{2}$ ...4.1

Amplitude stress (variable) $\sigma_v = \frac{\sigma_{max}-\sigma_{min}}{2}$ 4.2

Stress Ratio $R = \frac{\sigma_{min}}{\sigma_{max}}$ ...4.3

Notes: -

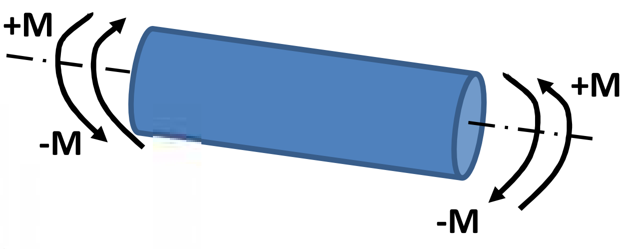
1- For reversal loading R= -1.
2- For repeated loading R=0.
3- σ_e is for completely reversal loading.
4- For any stress range, the endurance limit $\sigma_e{}' = \frac{3\sigma_e}{2-R}$ ...4.4

4.3 Effect of loading on Endurance limit –

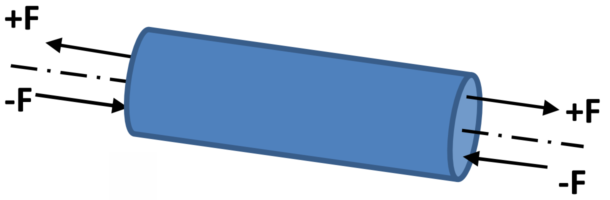
The endurance limit σ_e of a material as determined by the rotation beam method is for reversal bending load. There are many machines members which are subjected to loads other than reversal bending loads, thus endurance limit will also be different for different types of loading.

The endurance limit depending upon the types of loading may be modified as: -

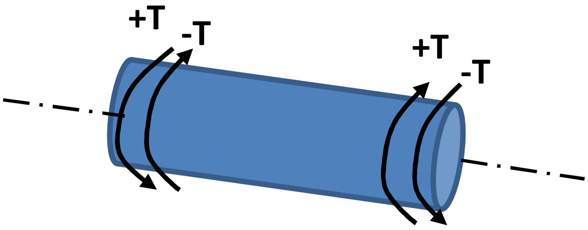
Endurance limit for bending, $(\sigma_e)_b = \sigma_e$ ...4.5



Endurance limit for reversal axial load, $(\sigma_e)_a = 0.8\sigma_e$ ...4.6



Endurance limit for reversal shear, $(\sigma_e)_a = 0.55\sigma_e$ ….4.7



4-4 Factors effecting on Endurance Limit: -

1- Surface finish - reduce σ_e ……….K_{sur}.

2- Size factor - reduce σ_e ……….K_{size}.

3- Hardness -reduces σ_e …….. K_{hard}.

4- Temperature - reduce σ_e ………. K_{temp}.

5- Notch or stress concentration reduces σ_e ……… K_t.

$K_f = 1 + q\,(K_t - 1)$ ...4.8

Where: -

K_f = fatigue stress concentration factor.

q= notch sensitivity.

K_t = theoretical stress concentration factor.

$$K_{over} = \frac{K_{sur}K_{size}K_{hard}K_{temp}}{K_f}$$ …4.9

For reversal bending $\sigma_e' = \sigma_e * K_{over}$ …4.10

For reversal axial load $\sigma_{ea}{}' = \sigma_{ea} * K_{over}$ ...4.11

For reversal shear $\sigma_{e\tau}{}' = \sigma_{e\tau} * K_{over}$ ...4.12

4-5 Relationship between Endurance limit and σ_{ult}: -

For: -

Steel, $\sigma_e = 0.5\sigma_{ult}$

Cast steel, $\sigma_e = 0.4\sigma_{ult}$

Cast iron, $\sigma_e = 0.35\sigma_{ult}$

Non ferrous metals and alloys, $\sigma_e = 0.3\sigma_{ult}$

4-6 Fatigue Theories: -

There are many theories of fatigue; the following are some of them.

a- Goodman theory: -

$$\boxed{\sigma_v = \sigma_e\left(\frac{1}{S.F} - \frac{\sigma_m}{\sigma_{ult}}\right)}$$

b- Soderberg theory: -

$$\boxed{\sigma_v = \sigma_e\left(\frac{1}{S.F} - \frac{\sigma_m}{\sigma_{yield}}\right)}$$

c- Gerber theory: -

$$\boxed{\sigma_v = \sigma_e\left[\frac{1}{S.F} - \left(\frac{\sigma_m}{\sigma_{ult}}\right)^2\right]}$$

When the dimensions of the element increase the stress decrease.

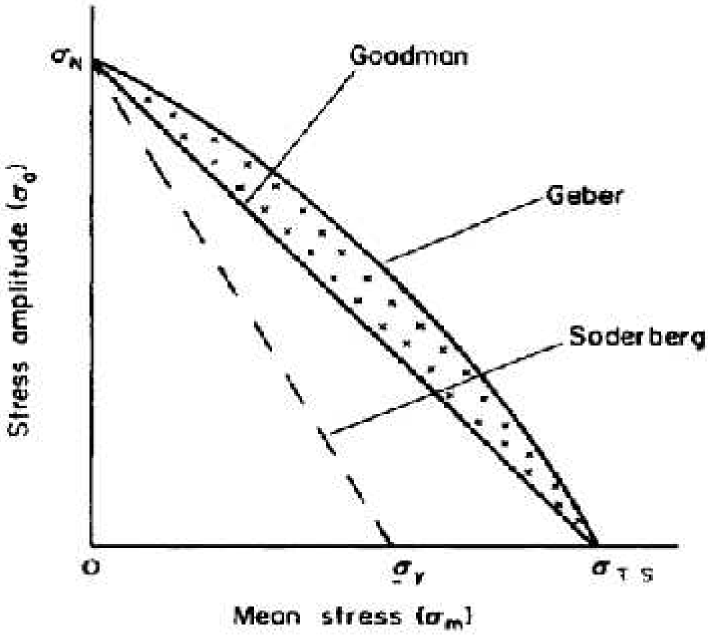


Fig 4.3 Amplitude-mean stress relationships as per Goodman, Gerber and Soderberg**.**

4-7 Combined variable normal stress and variable shear stress:-

When a machine part is subjected to both normal stress and a variable shear stress; then it is designed by using the following two theories of combined stresses:-

1- Maximum shear stress theory.
2- Maximum normal stress theory.

We have discussed in sec. 4-6, that according to soderberg's formula,

$$\frac{1}{S.F} = \frac{\sigma_m}{\sigma_y} + \frac{\sigma_v}{\sigma_{eb}} \quad \text{.... For reversed bending load.}$$

Multiplying throughout by σ_y, we get

$$\frac{\sigma_y}{S.F} = \sigma_m + \frac{\sigma_v \sigma_y}{\sigma_{eb}}$$

The term on the right hand side of the above expression is known as equivalent normal stress due to reversed bending.

Equivalent normal stress due to reversed bending,

$$\sigma_{neb} = \sigma_m + \frac{\sigma_v \sigma_y}{\sigma_{eb}}$$

Similarly, equivalent normal stress due to reversed axial loading,

$$\sigma_{nea} = \sigma_m + \frac{\sigma_v \sigma_y}{\sigma_{ea}}$$

And total equivalent normal stress,

$$\boxed{\sigma_{ne} = \sigma_{neb} + \sigma_{nea} = \frac{\sigma_y}{S.F}}$$

For reversed torsional or shear loading,

$$\frac{1}{S.F} = \frac{\tau_m}{\tau_y} + \frac{\tau_v}{\tau_e}$$

Multiplying throughout by τ_y,

$$\frac{\tau_y}{S.F} = \tau_m + \frac{\tau_v \tau_y}{\tau_e}$$

The term on the right hand side of the above expression is known as equivalent shear stress.

Equivalent shear stress due to reversed torsion or shear loading,

$$\tau_{es} = \tau_m + \frac{\tau_v \tau_y}{\tau_e}$$

The maximum shear stress theory is used in designing machine parts of ductile material.

According to this theory, maximum equivalent shear stress,

$$\boxed{\tau_{es(\max)} = \frac{1}{2}\sqrt{(\sigma_{ne})^2 + 4(\tau_{es})^2} = \frac{\tau_y}{S.F}}$$

The maximum normal stress theory is used in designing machine parts of brittle materials.

According this theory, maximum equivalent normal stress,

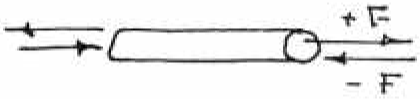
$$\boxed{\sigma_{ne(max)} = \frac{1}{2}\sigma_{ne} + \frac{1}{2}\sqrt{(\sigma_{ne})^2 + 4(\tau_{es})^2} = \frac{\sigma_y}{S.F}}$$

Example (-1).

A steel rod is subjected to reversed axial load of 180 KN. Find the diameter of the rod for a factor of safty of 2. The material has an ultimate tensile strength of 1070 N/mm^2 & yield stress of 910 N/mm^2?, The endurance limit in reversed bending may be assumed to be one-half the ultimate tensile strength. The other corrections may be taken as:

for machined surface (k_{sur}) = 0.8
for size (k_{size}) = 0.85
for stress concentration (k_t) = 1.0

Solution:



$\sigma_{ea} = \sigma_{eb} * 0.8$

but $\sigma_{eb} = 0.5\ \sigma_{mt} = 0.5 * 1070 = 535\ N/mm^2$

$\therefore \sigma_{ea} = 535 * 0.8 = 428\ N/mm^2$

$\sigma'_{ea} = 428 * k_{sur} * k_{size}$

$= 428 * 0.8 * 0.85 = 291\ N/mm^2$

$A = \frac{\pi}{4} d^2 = 0.7854\ d^2\ mm^2$

$F_m = \frac{F_{max} + F_{min}}{2} = \frac{180 + (-180)}{2} = 0 \Rightarrow \sigma_m = 0$

$F_a = \frac{F_{max} - F_{min}}{2} = \frac{180 - (-180)}{2} = 180\ KN$

$\sigma'_a = \frac{F_a}{A} = \frac{180 * 10^3}{0.7854\ d^2} = \frac{291 * 10^3}{d^2}$

using sodbery theory

$\sigma_a = \sigma_e \left(\frac{1}{S.F} - \frac{\sigma_m}{\sigma_{yield}} \right)$ or

$\frac{1}{S.F} = \frac{\sigma_m}{\sigma_{yield}} + \frac{\sigma_a}{\sigma_{ea}}$

$$\frac{1}{2} = \frac{0}{910} + \frac{229 * 10^3 / d^2}{291}$$

$\therefore d = 40$ mm

for static load

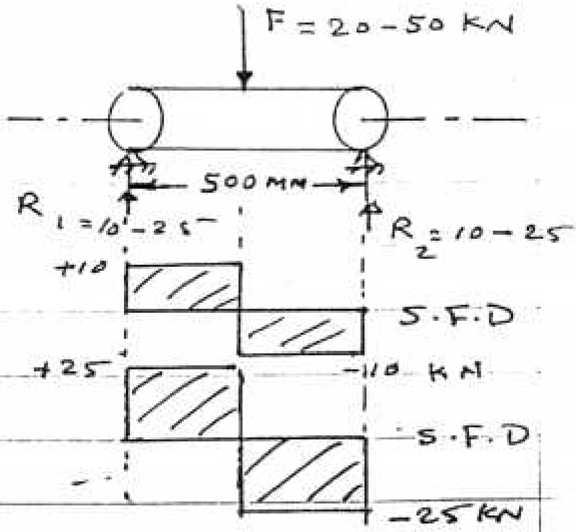
$$\frac{\sigma_{yield}}{S.F} = \frac{F}{\frac{\pi}{4} d^2}$$

$$\frac{910}{2} = \frac{180 * 10^3}{\frac{\pi}{4} d^2} \Rightarrow d = 23 \text{ mm}$$

$\therefore$ The rod diameter = 40 mm .... Ans

Example (-2).

A circular bar of 500 mm length is supported freely at its two ends. It is acted upon by a central concentrated cyclic load having a minimum value of 20 kN and a maximum value of 50 kN. Determine the diameter of bar by taking a factor of safety of 1.5, size effect of 0.85, surface finish factor of 0.9. The material properties of bar are given by: ultimate strength of 650 N/mm², yield strength of 500 N/mm² & Endurance strength of 350 N/mm² ?



Solution:

$$M_{max} = 250 * 25000$$
$$= 625 * 10^4 \ N-mm$$

$$M_{min} = 250 * 10000$$
$$= 250 * 10^4 \ N-mm$$

Mean or average bending moment

$$M_m = \frac{M_{max.} + M_{min}}{2}$$

$$M_m = \frac{625 * 10^4 + 250 * 10^4}{2} = 437.5 * 10^4 \ N\text{-}mm$$

Variable bending moment, $M_a = \dfrac{M_{max} - M_{min}}{2}$

$$= \frac{(625 - 250) * 10^4}{2} = 187.5 * 10^4 \ N\text{-}mm$$

$$\sigma_{bending} = \frac{M * y}{I} = \frac{M}{Z} \quad , Z = \text{section modulus} = I/y$$

$$Z = \frac{\pi}{32} d^3 = 0.0982\, d^3 \ mm^3$$

mean average bending stress

$$\sigma_m = \frac{M_m}{Z} = \frac{437.5 * 10^4}{0.0982\, d^3} = \frac{4.45 * 10^7}{d^3} \ N/mm^2$$

variable bending stress

$$\sigma_a = \frac{M_a}{Z} = \frac{187.5 * 10^4}{0.0982\, d^3} = \frac{1.91 * 10^7}{d^3} \ N/mm^2$$

according to Goodman theory

$$\frac{1}{S.F} = \frac{\sigma_m}{\sigma_{ult}} + \frac{\sigma_a}{\sigma_e'}$$

$$\sigma_e' = \sigma_e * \frac{K_{sur} * K_{siz}}{K_f} = 350 * \frac{0.9 * 0.85}{1}$$

$$\frac{1}{1.5} = \frac{4.45 * 10^7}{d^3 * 650} + \frac{1.91 * 10^7}{d^3 * 350 * 0.9 * 0.85}$$

$d = 59.3$ mm ...... Ans.

or by Sodbery Theory.

$$\frac{1}{S.F} = \frac{\sigma_m}{\sigma_{yield}} + \frac{\sigma_a}{\sigma_e}$$

$$\frac{1}{1.5} = \frac{4.45 * 10^6}{d^3 * 500} + \frac{1.91 * 10^7}{d^3 * 350 * 0.9 * 0.85}$$

$d = 49.37$ mm .... Ans

The diameter of the shaft is 59.3 mm

Example (-3).

A 5 cm diameter shaft is made from carbon steel have ultemate tensile strength of 630 MPa and yield stress of 510 MPa. It is subjected to a torque which flactuates between 20×10^5 N-mm to -8×10^5 N-mm. Using sodbery theory, find the S.F. Assume $K_{sur} = 0.87$, $K_{size} = 0.85$, $K_f = 1$, & $K_{temp} = 0.9$

Solution: $\tau_{max} = \dfrac{Td}{2J}$ But $J = \dfrac{\pi d^4}{32}$

$$\therefore \tau_{max} = \frac{16T}{\pi d^3}$$

$$\tau_{max} = \frac{20 \times 10^5 \times 16}{\pi d^3} = \frac{32 \times 10^6}{\pi d^3}$$

$$\tau_{min} = \frac{-8 \times 10^5 \times 16}{\pi d^3} = \frac{-12.8 \times 10^6}{\pi d^3}$$

$$\tau_a = \frac{\tau_{max} - \tau_{min}}{2} = \frac{22.4 \times 10^6}{\pi (50)^3} = 57.04$$

shear

$$\sigma_e = 0.5 (\sigma_{ult} \times K_{sur} \times K_{size} \times K_{fn.} \times K_{temp}) / K_{con\ of\ f}$$

$$= 0.5 \times 630 \times 0.87 \times 0.85 \times 1 \times 0.9 = 209.65$$

0.55

$$\tau_e = 0.5\, \sigma_e = 0.5 \times 209.65 = 104.82$$

$$\tau_a = \tau_e \left(\frac{1}{S.F} - \frac{\tau_m}{\tau_{ult}} \right) \quad \ldots\ldots *$$

$Y_s = 0.5Y$

$$\tau_m = \frac{\tau_{max} + \tau_{min}}{2} = \frac{9.6 \times 10^5}{\pi (50)^3} = 24.45$$

sub. the values of τ_a, τ_e, τ_m, & τ_{ult} into equation(*)

$$57.04 = 104.82 \left(\frac{1}{S.F} - \frac{24.45}{630} \right) \rightarrow 0.5\,\sigma_y$$

$\therefore$ S.F = 1.71 ...... Ans.

Example (4):- A steel cantilever is 200 mm long. It is subjected to an axial load which varies from 150 N compression to 450 N tension and also a transverse load at its free end which varies 80 N up to 120 N down. The cantilever is of circular cross section. It is of diameter 2d for the first 50 mm and of diameter d for the remaining length. Determine its diameter taking a factor of safety of (2). Assume the following values.

Yield stress = 330 MPa.

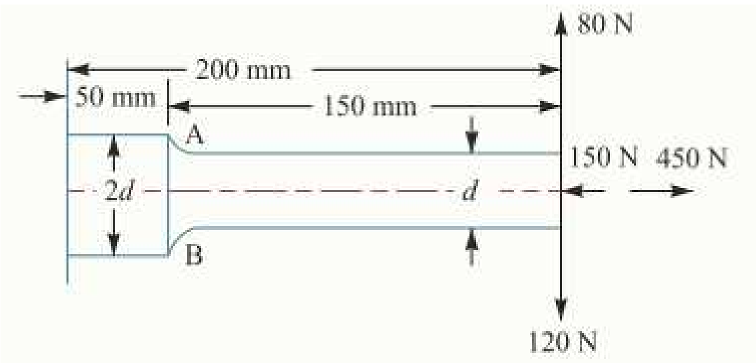
Endurance limit in reversed loading=300 MPa.

Correction factors = 0.8 in reversed axial loading.

= 1 in reversed bending.

K_t= 1.44 for bending, 1.64 for axial loading.

K_{size}= 0.85, $K_{surface}$ = 0.9, Notch sensitivity index = 0.9



Sol.

For axial load,

$$W_m = \frac{W_{a(\max)} + W_{a(min)}}{2} = \frac{450 + (-150)}{2} = 150N$$

$$W_v = \frac{W_{a(\max)} - W_{a(min)}}{2} = \frac{450 - (-150)}{2} = 300N$$

$$\sigma_m = \frac{W_m}{A} = \frac{150(4)}{\pi d^2} = \frac{191}{d^2}\ N.mm^2$$

$$\sigma_v = \frac{W_v}{A} = \frac{300(4)}{\pi d^2} = \frac{382}{d^2}\ N.mm^2$$

$$K_{fa} = 1 + q(K_{ta} - 1) = 1 + 0.9(1.64 - 1) = 1.576$$

σ_{ea}=300*0.8*0.85*0.9/1.576=300*0.388= 116.497MPa.

$$\sigma_{nea} = \sigma_m + \frac{\sigma_v \sigma_y}{\sigma_{ea}} = \frac{191}{d^2} + \frac{382 * 330}{d^2 * 116.497}$$

$$= \frac{191}{d^2} + \frac{1082.08}{d^2} = \frac{1273.08}{d^2}$$

Now for bending due to transverse load,

$$W_m = \frac{W_{t(\max)} + W_{t(min)}}{2} = \frac{120 + (-80)}{2} = 20N$$

$$W_v = \frac{W_{t(\max)} - W_{t(min)}}{2} = \frac{120 - (-80)}{2} = 100N$$

Mean bending moment at point A=20(200-50) =3000N.mm

And variable bending moment at point A=100(200-50) =15000N.mm

Mean of bending stress = $\frac{M_m Y}{I} = \frac{30550}{d^3}$ N/mm^2

Variable of bending stress = $\frac{M_v Y}{I} = \frac{152750}{d^3}$ N/mm^2

$$K_{fb} = 1 + q(K_{tb} - 1) = 1 + 0.9(1.44 - 1) = 1.396$$

σ_{ea}=300*1*0.85*0.9/1.396=300*0.515= 164.398MPa

$$\sigma_{nea} = \sigma_m + \frac{\sigma_v \sigma_y}{\sigma_{ea}} = \frac{30550}{d^3} + \frac{152750 * 330}{d^3 * 164.398}$$

$$= \frac{30550}{d^3} + \frac{306618.69}{d^3} = \frac{337168.69}{d^3}$$

$$\sigma_{ne} = \sigma_{neb} + \sigma_{nea} = \frac{1273.08}{d^2} + \frac{337168.69}{d^3} = \frac{\sigma_y}{S.F}$$

$$\frac{1273.08}{d^2} + \frac{337168.69}{d^3} = \frac{330}{2} = 165$$

$$1273.08d + 337168.69 = 165d^3$$

d=12.9 mm   Ans.

Example(5):- A hot rolled steel shaft is subjected to a torsional moment that varies from 330 N.m clockwise to 110 N.m counterclockwise and an applied bending moment at a critical section varies from 440 N.m to -220 N.m. The shaft is of uniform cross section and no key way is present at the critical section. Determine the required shaft diameter. The material has an ultimate strength of 550 MN/m^2 and yield strength of 410 MN/m^2. Take the endurance limit as half the ultimate strength, factor of safety of 2, size factor of 0.85and a surface finish factor of 0.62.

Sol.

For torsional moment,

$$T_m = \frac{T_{(\max)} + T_{(min)}}{2} = \frac{330 + (-110)}{2} = 110N.m$$

$$T_v = \frac{T_{(\max)} - T_{(min)}}{2} = \frac{330 - (-110)}{2} = 220N.m$$

Mean shear stress,$\tau_m = \frac{16T_m}{\pi d^3} = \frac{16(110)}{\pi d^3} = \frac{560}{d^3}$ N/m^2

and variable shear stress, $\tau_v = \frac{16T_v}{\pi d^3} = \frac{16(220)}{\pi d^3} = \frac{1120}{d^3}$ N/m^2

Since the endurance limit in shear (τ_e) is $0.55\sigma_e$ and yield strength in shear is $0.5\sigma_y$

$$\tau_e = 0.55(0.5 * 550)(0.62)(0.85) = 79.708\ MP\ .$$

$$\tau_y = 0.5(410) = 205\ MPa.$$

$$\tau_{es} = \tau_m + \frac{\tau_v \tau_y}{\tau_e} = \frac{560}{d^3} + \frac{1120(205)}{d^3(79.708)} = \frac{3440}{d^3}\ \frac{N}{m^2}$$

For bending moment,

$$M_m = \frac{M_{(\max)} + M_{(min)}}{2} = \frac{440 + (-220)}{2} = 110N.m$$

$$M_v = \frac{M_{(\max)} - M_{(min)}}{2} = \frac{440 - (-220)}{2} = 330N.m$$

Mean bending stress,$\sigma_m = \frac{32M_m}{\pi d^3} = \frac{32(110)}{\pi d^3} = \frac{1120}{d^3}$ N/m^2

and variable bending stress, $\sigma_v = \frac{326TM_v}{\pi d^3} = \frac{32(330)}{\pi d^3} = \frac{3360}{d^3}$ N/m^2

σ_{eb} =0.5(550)(0.62)(0.85)=144.925 MPa

$$\sigma_{neb} = \sigma_m + \frac{\sigma_v \sigma_y}{\sigma_{eb}} = \frac{1120}{d^3} + \frac{3360 * 410}{d^3 * 144.925}$$

$$\sigma_{neb} = \frac{1120}{d^3} + \frac{9506}{d^3} = \frac{10626}{d^3} \text{ Pa.}$$

$$\sigma_{ne} = \sigma_{neb} = \frac{10626}{d^3} \text{ Pa.}$$

But,

$$\tau_{es(\max)} = \frac{1}{2}\sqrt{(\sigma_{ne})^2 + 4(\tau_{es})^2} = \frac{\tau_y}{S.F}$$

$$= \frac{1}{2}\sqrt{(\frac{10626}{d^3})^2 + 4(\frac{3440}{d^3})^2} = \frac{205 * 10^6}{2}$$

d=39.5mm Ans.

…………………………End…………………………

Lecture No. 5

-Transmission Systems-

5-1 A machine: -

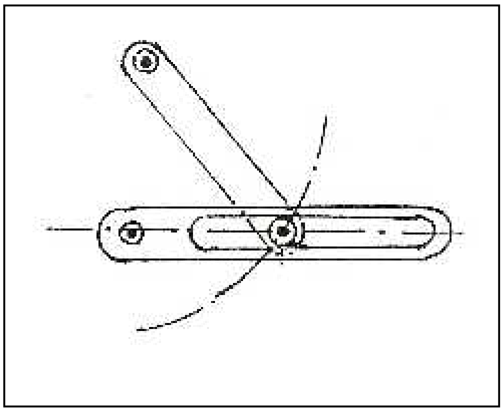
Is a combination of resistant bodies so arranged that by their means the mechanical forces of nature can be compelled to produce some effect or works accompanied with certain determinant motions. In general, it may properly be said that a machine is an assemblage of parts interposed between the source of power and the work; each of pieces in a machine either moves or helps to guide some of the other pieces in their motion.

5-2 Transmission of motion (or power): -

a- Direct contact: -

When the driving member of a mechanism is in direct contact with the driven piece the bodies constituting the driver and driven member are either in pure rolling contact or there must be sliding between the surfaces in contact.

Figure 5-1: -Sliding Contact



b- Cams: -

A cam is plate, cylinder, or other solid with a surface of contact so designed to cause or modify the motion of second piece, or of the cam itself. Either the cam or the other piece or both may be moving. They have a curved outline or a groove, which rotates about a fixed axis and by its rotation imports motion to a piece in contact with it, known as the follower. A cam and it's follower from an application of principle of transmitting motion by direct sliding contact.

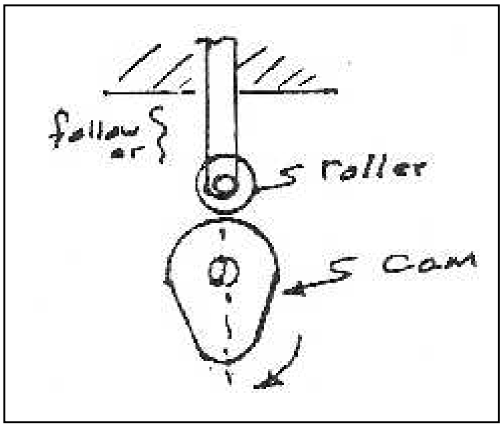


Figure 5-2: - Cam

c- Pure rolling contact: -

Pure rolling contact consists of such a relative motion of two lines or surfaces that the consecutive points or elements of one come successively into contact with those of the other in their order

<u>Case One:</u> - Cylinders roll together without slipping, external contact (Figure 5-3): -

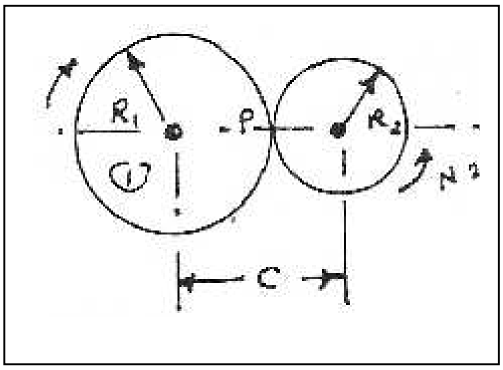


Figure 5-3: -Cylinder rolls in external contact.

$C=R_1+R_2$ **….5.1**

The surfaces will touch at point (P).

Surface speed of cylinder 1 = $2\pi R_1 N_1$.

Surface speed of cylinder 2= $2\pi R_2 N_2$.

Therefore, if the surface speeds of (1) equals to the surface speed of (2), (without slipping).

$2\pi R_1 N_1 = 2\pi R_2 N_2$ or $\frac{N_2}{N_1} = \frac{R_1}{R_2}$ **.....5.2**

Case Two: - Cylinders roll together without slipping, internal contact

$C = R_1 - R_2$ **....5.3**

Also,

$$\frac{N_2}{N_1} = \frac{R_1}{R_2}$$ **....5.4**

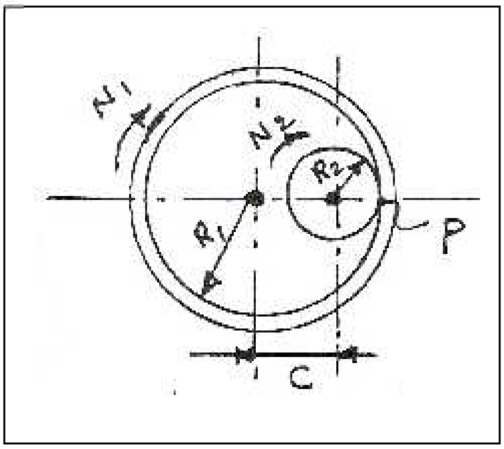


Figure 5-4: - Cylinder rolls in internal contact.

Case Three: - Cones roll together without slipping.

$$\frac{N_2}{N_1} = \frac{R_1}{R_2} \qquad \text{....5.5}$$

But R_2 = OP sinβ and

R_1 = OP sinα, therefore

$$\frac{R_2}{R_1} = \frac{OPsin\beta}{OPsin\alpha} = \frac{sin\beta}{sin\alpha} \qquad \text{....5.6}$$

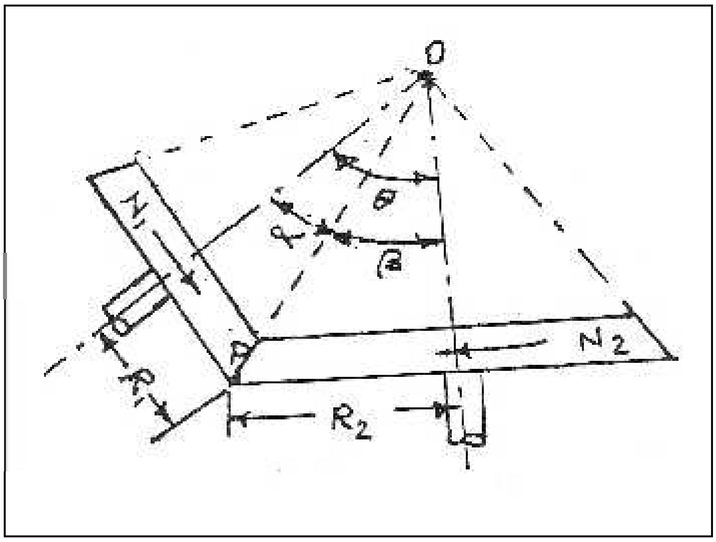


Figure 5-5: - Cones rolls without slipping

Where: -

Ɵ=Is the angle between shafts or shafts angle.

α= is the cone or center angle of the cone 1.

β= is the cone angle 2.

N= rolling speed rpm.

The cone angles when the angle between the axis and speed ratio are known.

$$\frac{N_1}{N_2} = \frac{\sin\beta}{\sin\alpha} = \frac{\sin\beta}{\sin(\theta-\beta)} = \frac{sin\beta}{sin\theta cos\beta - \cos\theta\sin\beta}$$

$$= \frac{\left({sin\beta}/{\cos\beta}\right)}{\sin\theta - {\cos\theta sin\beta}/{cos\beta}} = \frac{\tan\beta}{\sin\theta - \cos\theta tan\beta}$$

Then $$\tan\beta = \frac{\sin\theta}{\left(\frac{N_2}{N_1} + \cos\theta\right)} \qquad \ldots 5.7$$

In similar manner: -

$$\tan\alpha = \frac{\sin\theta}{\left(\frac{N_1}{N_2}+\cos\theta\right)} \qquad \ldots 5.8$$

d –Gears: -

Gears are toothed wheels, as the gears turn, the teeth of one gear slide on the other but are so designed that the angular speeds of the gears are the same as those of the rolling bodies which they replace. Gears may be classified according to the case for which they are designed. and as follows, Spur gears, Bevel gears and Screw gears.

5-3 Power transmission by Belts, Ropes and Chains: -

When the distance between the driving shaft and the driven shaft is too great to be connected by gears, a flexible connector is used. Flexible connectors may be divided into three general classes: -

1- Chains are composed of metallic links jointed together and run on either sprockets or drums grooved, notched or toothed to fit the links of the chains.

   Chains are usually used for connecting shafts that are less than 5 meter a part. The speed of the chain will depend upon the type of chain.

2- Ropes made of manilahemp, cotton or wire is nearly circular in section and run on either grooved pulleys or drums with flanges. Rope may be used for connecting shafts up to 30 m apart and should operate at a speed of less than 180 m/min.

3- Belts made of leather, rubber or woven fabrics are flat, round, V-type toothed or without toothed, and run on pulley. Belts may be run economically at speed as high as 1500 m/min.

5-4 Speed ratio and directional relation of shafts connected by a belt.

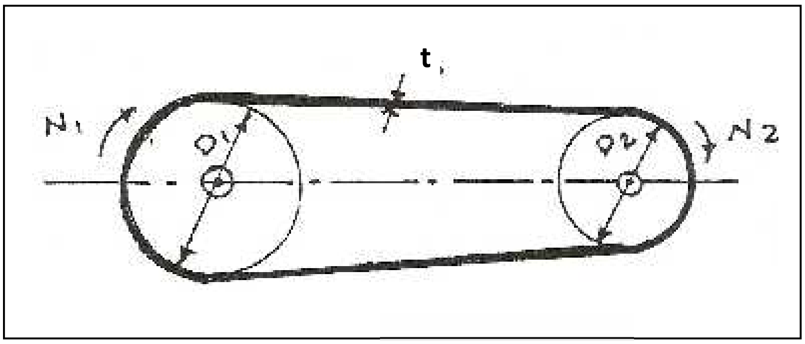


Figure 5-6: -Open Type Belt.

Linear speed of pitch surface of 1 $=\pi N_1(D_1 + t)$

Linear speed of pitch surface of 2 $=\pi N_2(D_2 + t)$

If the belt speed is supposed to be equal to the speed of the pitch surfaces of the pulleys then,

$$\pi N_1(D_1 + t) = \pi N_2(D_2 + t)$$

Then,

$$\frac{N_1}{N_2} = \frac{(D_2+t)}{(D_1+t)} \qquad \text{....5.9}$$

Where t= thickness of belt.

The belt shown in figure 5-6, is known as an open type belt and the pulleys turn in the same direction as suggest by the arrows. The belt shown in figure 5-7 is known as a crosses type belt and the pulleys turn in opposite directions.

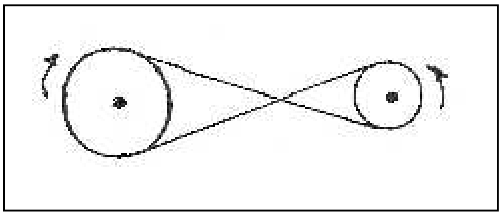


Figure 5-7: -Cross Type Belt.

a- Length of belt connecting parallel axes, Open Type: -

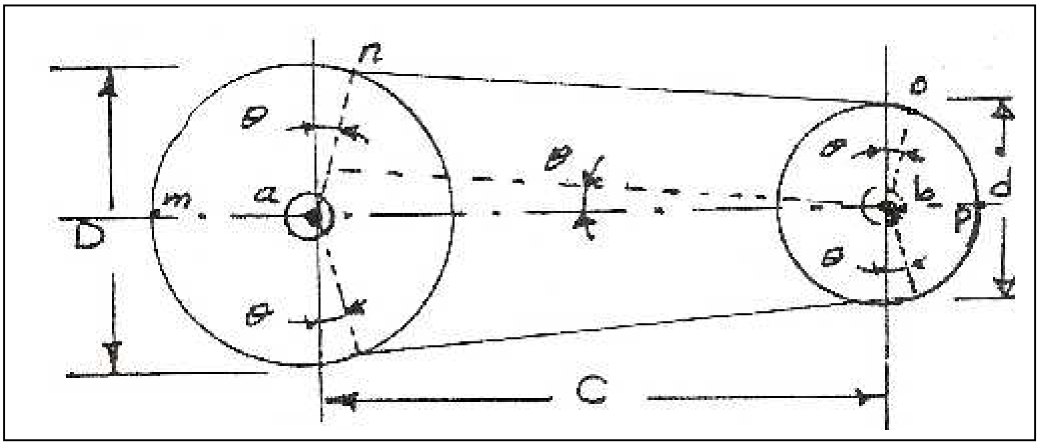


Figure 5-8:- Dimension's parameters for open type belt.

Where:-

C = the distance between the pulley axes.

L = the length of the belt.

L =2(mn+no+op)

$$= \left(\frac{\pi}{2} + \theta\right) D + 2C\cos\theta + \left(\frac{\pi}{2} - \theta\right) d$$

$$\boxed{L = \frac{\pi}{2}(D + d) + 2C + \frac{(D-d)^2}{4C}}$$ for Open type belt …..5.10

b- Length of belt connecting parallel axes, Cross Type: -

$$L = \frac{\pi}{2}(D + d) + 2C + \frac{(D+d)^2}{4C}$$ for Cross type belt …5.11

c- Angle of Contact: -

- For open belt,

$\alpha = (\pi - 2\theta)$ Angle of contact for open type belt … 5.12

$\sin\theta = \frac{r_1 - r_2}{C}$ but $\sin\theta \cong \theta$ where θ is very small.

Then $\theta = \frac{r_1 - r_2}{C}$ ……5.13

- For cross belt,

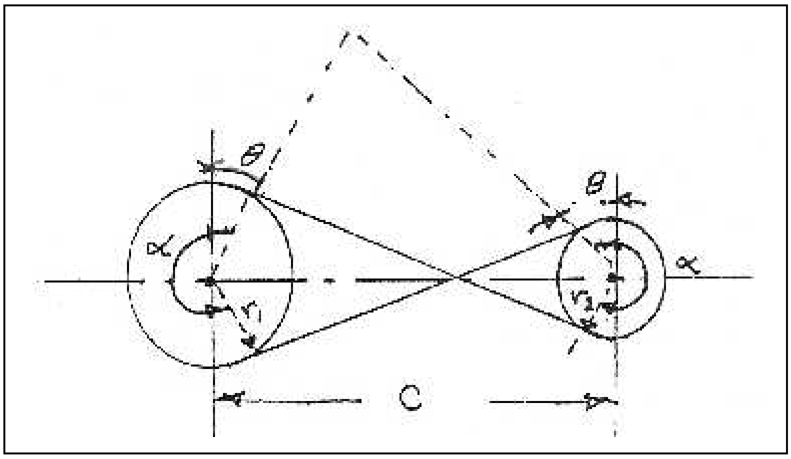


Figure 5-9: - Dimension's parameters for Cross type belt.

$\alpha = (\pi + 2\theta)$ Angle of contact for cross type belt … 5.14

$\sin\theta = \frac{r_1 + r_2}{C}$ But $\sin\theta \cong \theta$ for θ is very small.

Then,

$\theta = \frac{r_1 + r_2}{C}$ ... 5.15

d- Belt tension: -

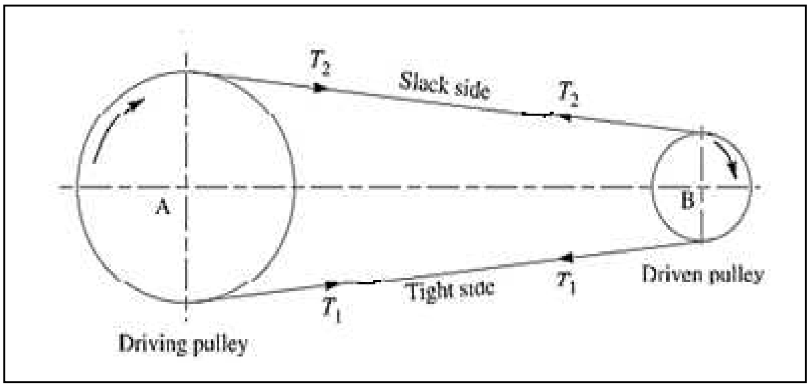


Figure 5.10: - Tension in belt.

Power = torque (N.m) *angular velocity (rad/sec)

P = T * w

hp =746 watt.

Torque = $T_1r-T_2r = r(T_1-T_2)$

Power = $wr(T_1 - T_2)$

$$\boxed{P = V*(T_1 - T_2)} \quad \text{…5.16}$$

Where: -

T_1 = Tension in belt in tight side (N).

T_2= Tension in belt in slack side (N).

V = Belt linear velocity (m/s).

$T_1 > T_2$

$$\boxed{T_{max} = T_1 + T_c} \quad \text{… 5.17}$$

$$\boxed{T_c = mV^2} \quad \text{… 5.18}$$

Where: -

m= mass of belt per unit length=ρA

ρ = Belt density (kg/m^3).

A = Belt cross section area (m^2).

T_c = Centrifugal Tension (N).

$$\boxed{\frac{T_1}{T_2} = e^{\mu\alpha}} \quad \text{… 5.19}$$

Where: -

μ = Coefficient of friction between belt and pulley.

α = Angle of contact between the **smallest pulley** and belt in radian.

$$\boxed{\sigma_{tension} = \frac{T_{max}}{A}} \quad \text{… 5.20}$$

Where: -

$\sigma_{tension}$ = Tension stress in belt which depend upon the material of the belt (σ_{yield}).

Initial tension $\boxed{T_o = \frac{T_1+T_2}{2}}$ … 5.21

Example (-1) .

The driver pulley has the following specification power output = 10 hp, diameter of pulley = 1000 mm, the driven pulley has a diameter of 250mm. The distance between the pulleys centers = 2000 mm, the belt is flat type has a width of 155 mm, thickness = 3 mm & density = 1100 kg/m³. The coefficient of friction between belt and pulleys = 0.2. Calculate the max. tension stress for the belt, if the connection is open type. and V=7.8 m/sec

Solution:

$$\theta = \frac{r_1 - r_2}{c} = \frac{0.5 - 0.25/2}{2} = 0.1875 \text{ rad.}$$

$$\alpha = \pi - 2\theta = 3.14 - 2 * 0.1875 = 2.7 \text{ rad.}$$

$$P = v(t_1 - t_2)$$

$$10 * 746 = 7.8(t_1 - t_2)$$

or

$$(t_1 - t_2) = 956.41 \quad \ldots\ldots (1)$$

$$\frac{t_1}{t_2} = e^{\mu\alpha}$$

$$\therefore \frac{t_1}{t_2} = e^{0.2 * 2.7}$$

$$t_1 = 1.716\, t_2 \quad \ldots\ldots (2)$$

From equation (1) & (2)

$$t_2 = 1335.77 \quad \& \quad t_1 = 2292.178$$

$$t_c = m * v^2 = \rho * A * v^2$$
$$= 1100 * 3 * 10^{-3} * 155 * 10^{-3} * (7.8)^2$$
$$= 81.1197$$

$$t_{max.} = t_1 + t_c = 2292.178 + 31.1197$$
$$= 2323.9$$

$$\sigma_{max} = \frac{t_{max}}{A} = \frac{2323.9}{3*10^{-3}*155*10^{-3}}$$
$$= 4.99 \ MPa \quad ----- \ Ans.$$

For the above e ample calculate the length of the belt.

Solution:

$$L = \frac{\pi}{2}(D+d) + 2C + \frac{(D-d)^2}{4C}$$

$D = 1$ meter, $d = 0.25$ m, & $C = 2$ m.

$$L = \frac{\pi}{2}(1+0.25) + 2*2 + \frac{(2-0.25)^2}{4*2}$$
$$= \frac{\pi}{2}*1.25 + 4 + \frac{3.0625}{8}$$
$$= 6.346 \text{ meter} \quad ---- \ Ans.$$

Example (-2).

An electric motor drive a fan through an open type belt the system has the following specification:

motor power $= 8$ HP, $N_1 = 1200$ rpm

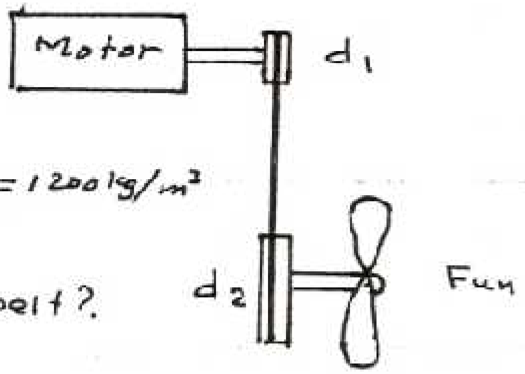
$d_1 = 100$ mm, $d_2 = 300$ mm

$C = 1300$ mm, $\mu = 0.32$

Belt thickness $= 4$ mm & density $= 1200$ kg/m^3

σ_{max} (belt) $= 16$ MPa

Calculate the width of the belt?



Solution:

$$V_1 = \omega * r_1 = \left(\frac{1200}{60} * 2\pi\right) * \frac{0.1}{2} = 6.28 \ m/s$$

$$\theta = \frac{r_1 - r_2}{c} = \frac{0.1/2 - 0.3/2}{1.3} = 0.077 \ rad.$$

$$\alpha = (\pi - 2\theta) = (3.14 - 2 * 0.0077) = 2.986 \ rad.$$

$$P = (t_1 - t_2) V$$

$$8 * 746 = (t_1 - t_2) * 6.28 \quad \text{from which}$$

$$t_1 - t_2 = 950.3 \quad \ldots \ (1)$$

$$\frac{t_1}{t_2} = e^{\mu\alpha} \Rightarrow \frac{t_1}{t_2} = e^{0.32 * 2.986} \quad \ldots (2)$$

From (1) & (2) $t_1 = 1544.2375$

$t_2 = 593.9375$

$$t_c = m v^2 = \rho * A * V^2 = \rho (t * w) v^2$$

$$= 1200 * 0.004 * w * 6.28^2 = 13.48 \ w$$

$$\sigma_{max} = \frac{t_{max}}{A}$$

$$16 * 10^6 = \frac{13.48 \ w + 1544.2375}{w * 0.004}$$

$w = \cancel{30.567}$ mm .... Ans.

24.1

Example No. (-3)

A leather belt 9 mm * 250 mm is used to drive a cast iron pulley 900 mm in diameter at 336 rpm. If the active arc on the smaller pulley is 120°, and the stress in the tight side is 2 N/mm². Find the power capacity of the belt, assume the leather density is 980 kg/m³ & the coefficient of friction is 0.35.

Solution:

$$V = N * r * \frac{2\pi}{60} = 336 * \frac{0.9}{2} * \frac{2\pi}{60}$$
$$= 15.8 \ m/s$$

$$A = b * t = 250 * 9 = 2250 \ mm^2$$

$$T_c = m \cdot v^2 = 0.009 * 0.25 * 980 * 15.8^2$$
$$= 550 \ N$$

$$\sigma_{max} = \frac{t_{max}}{A} \quad \text{or} \quad t_{max} = \sigma_{max} * A$$
$$= 2 * 2250 = 4500 \ N$$

$$t_{max} = t_1 + t_c$$
$$4500 = t_1 + 550 \qquad \therefore t_1 = 3950 \ N$$

$$\alpha = 120^\circ = 120 * \frac{\pi}{180} = 2.1 \ rad$$

$$\frac{t_1}{t_2} = e^{\mu \alpha}$$

$$\frac{3950}{t_2} = e^{0.35 * 2.1} \qquad \therefore t_2 = 1895 \ N$$

$$P = (t_1 - t_2) * v = (3950 - 1895) * 15.8$$
$$= 32469 \ watt = 34.5 \ hp \ \text{--- Ans.}$$

……………………End…………………………

Lecture No. 6

-Thick Cylinders-

6-1 Difference in treatment between thin and thick cylinders - basic assumptions:

The theoretical treatment of thin cylinders assumes that the hoop stress is constant across the thickness of the cylinder wall (Fig. 6.1), and also that there is no pressure gradient across the wall. Neither of these assumptions can be used for thick cylinders for which the variation of hoop and radial stresses is shown in (Fig. 6.2), their values being given by the Lame equations: -

$$\sigma_H = A + \frac{B}{r^2} \qquad \text{...6.1}$$

$$\sigma_r = A - \frac{B}{r^2} \qquad \text{...6.2}$$

Where: -

σ_H= Hoop stress ($\frac{N}{m^2} = Pa$).

σ_r= Radial stress ($\frac{N}{m^2} = Pa$).

r= Radius (m). A and B are Constants.

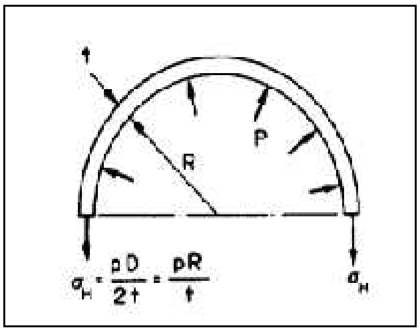


Figure 6.1: - Thin cylinder subjected to internal pressure.

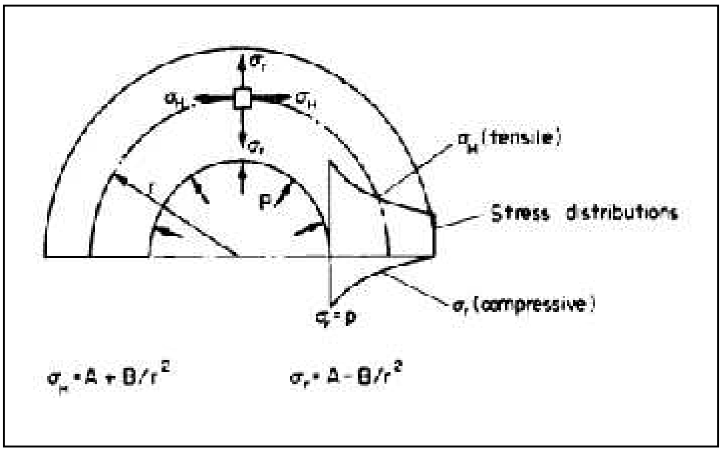


Figure 6.2: - Thick cylinder subjected to internal pressure.

6-2 Thick cylinder- internal pressure only: -

Consider now the thick cylinder shown in (Fig. 6.3) subjected to an internal pressure P, the external pressure being zero.

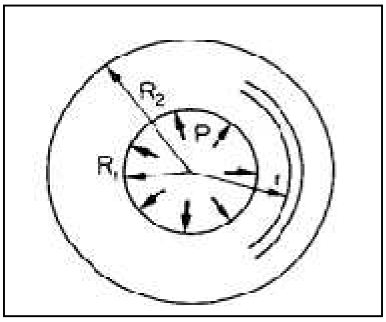


Figure 6.3: - Cylinder cross section.

The two known conditions of stress which enable the Lame constants A and B to be determined are:

At $r = R_1$, $\sigma_r = -P$ and at $r = R_2$, $\sigma_r = 0$

Note: -The internal pressure is considered as a negative radial stress since it will produce a radial compression (i.e. thinning) of the cylinder walls and the normal stress convention takes compression as negative.

Substituting the above conditions in eqn. (6.2),

$$\sigma_r = A - \frac{B}{r^2}$$

$$-P = A - \frac{B}{{R_1}^2} \text{ and } 0 = A - \frac{B}{{R_2}^2}$$

Then $A = \frac{P{R_1}^2}{({R_2}^2 - {R_1}^2)}$ and $B = \frac{P{R_1}^2{R_2}^2}{({R_2}^2 - {R_1}^2)}$

Substituting A and B in equations 6.1 and 6.2,

$$\boxed{\sigma_r = \frac{P{R_1}^2}{({R_2}^2 - {R_1}^2)}\left[1 - \frac{{R_2}^2}{r^2}\right]} \qquad ...6.3$$

$$\boxed{\sigma_H = \frac{P{R_1}^2}{({R_2}^2 - {R_1}^2)}\left[1 + \frac{{R_2}^2}{r^2}\right]} \qquad ...6.4$$

6-3 Longitudinal stress: -

Consider now the cross-section of a thick cylinder with closed ends subjected to an internal pressure P_1 and an external pressure P_2, (Fig. 6.4).

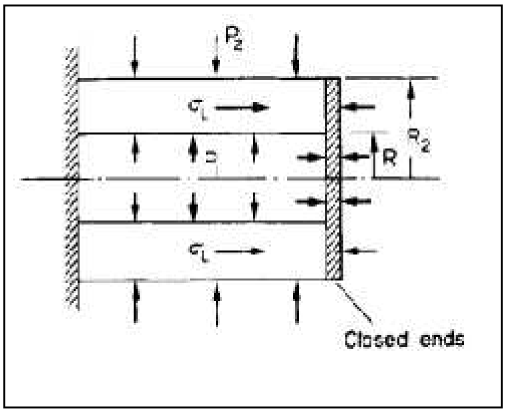


Figure 6.4: - Cylinder longitudinal section.

For horizontal equilibrium:

$$P_1{}^2 * \pi {R_1}^2 - P_2{}^2 * \pi {R_2}^2 = \sigma_L * \pi[{R_2}^2 - {R_1}^2]$$

Where σ_L is the longitudinal stress set up in the cylinder walls, Longitudinal stress,

$$\boxed{\sigma_L = \frac{P_1{R_1}^2 - P_2{R_2}^2}{({R_2}^2 - {R_1}^2)}} \qquad \text{....6.5}$$

But for P_2 =0 (no external pressure),

$$\sigma_L = \frac{P_1{R_1}^2}{({R_2}^2 - {R_1}^2)} = \text{A, constant of the Lame equations.} \qquad \text{....6.6}$$

6-4 Maximum shear stress: -

It has been stated in section 6.1 that the stresses on an element at any point in the cylinder wall are principal stresses.

It follows, therefore, that the maximum shear stress at any point will be given by equation of Tresca theory as,

$$\frac{\sigma_y}{2} = \tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \qquad \text{....6.7}$$

$$\tau_{max} = \frac{\sigma_H - \sigma_r}{2} \qquad \text{....6.8}$$

$$\tau_{max} = \frac{1}{2}\left[\left(A + \frac{B}{r^2}\right) - \left(A - \frac{B}{r^2}\right)\right] \qquad \text{....6.9}$$

$$\boxed{\tau_{max} = \frac{B}{r^2}} \qquad \text{....6.10}$$

6-5 Change of diameter: -

It has been shown that the diametral strain on a cylinder equals the hoop or circumferential strain.

Change of diameter = diametral strain x original diameter.

= circumferential strain x original diameter.

With the principal stress system of hoop, radial and longitudinal stresses, all assumed tensile, the circumferential strain is given by

$$\boxed{\epsilon_H = \frac{1}{E}(\sigma_H - \upsilon\sigma_r - \upsilon\sigma_L)} \quad \text{....6.11}$$

$$\delta D = \frac{D}{E}(\sigma_H - \upsilon\sigma_r - \upsilon\sigma_L) \quad \text{....6.12}$$

Similarly, the change of length of the cylinder is given by,

$$\boxed{\delta L = \frac{L}{E}(\sigma_L - \upsilon\sigma_r - \upsilon\sigma_H)} \quad \text{....6.13}$$

6-6 Comparison with thin cylinder theory: -

In order to determine the limits of D/t ratio within which it is safe to use the simple thin cylinder theory, it is necessary to compare the values of stress given by both thin and thick cylinder theory for given pressures and D/t values. Since the maximum hoop stress is normally the limiting factor, it is this stress which will be considered.

Thus for various D/t ratios the stress values from the two theories may be plotted and compared; this is shown in (Fig. 6.5).

Also indicated in (Fig. 6.5) is the percentage error involved in using the thin cylinder theory.

It will be seen that the error will be held within 5 % if D/t ratios in excess of 15 are used.

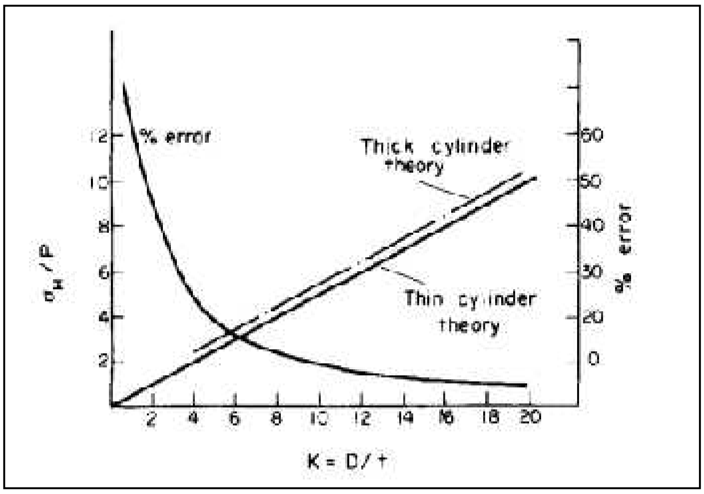


Figure 6.5: - Comparison of thin and thick cylinder theories for various diameter/thickness ratios.

6-7 Compound cylinders:-

From the sketch of the stress distributions in Figure 6.6 it is evident that there is a large variation in hoop stress across the wall of a cylinder subjected to internal pressure. The material of the cylinder is not therefore used to its best advantage. To obtain a more uniform hoop stress distribution, cylinders are often built up by shrinking one tube on to the outside of another. When the outer tube contracts on cooling the inner tube is brought into a state of compression. The outer tube will conversely be brought into a state of tension. If this compound cylinder is now subjected to internal pressure the resultant hoop stresses will be the algebraic sum of those resulting from internal pressure and those resulting from shrinkage as drawn in Fig. 6.6; thus a much smaller total fluctuation of hoop stress is obtained. A similar effect is obtained if a cylinder is wound with wire or steel tape under tension.

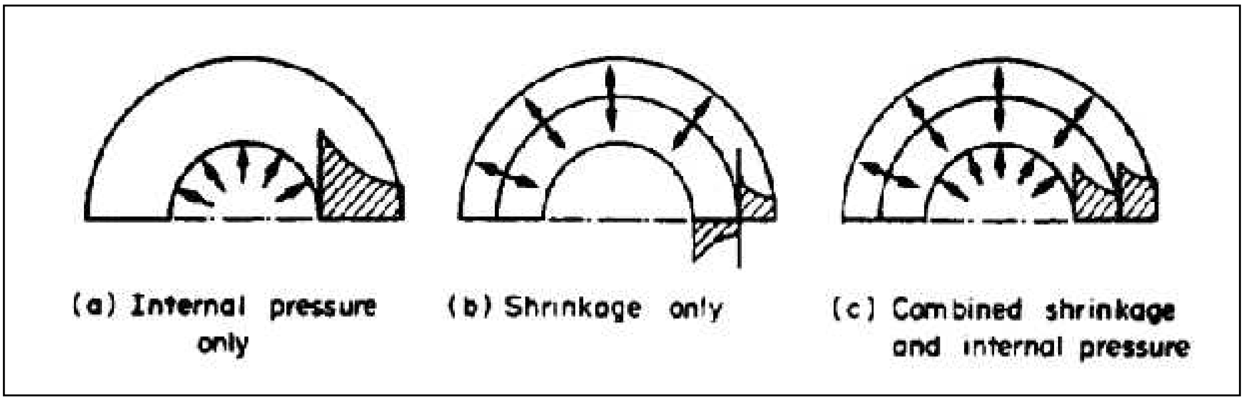


Figure 6.6: - Compound cylinders-combined internal pressure and shrinkage effects.

The method of solution for compound cylinders constructed from similar materials is to break the problem down into three separate effects:

(a) shrinkage pressure only on the inside cylinder.

(b) shrinkage pressure only on the outside cylinder.

(c) internal pressure only on the complete cylinder.

For each of the resulting load conditions there are two known values of radial stress which enable the Lame constants to be determined in each case

condition (a) shrinkage - internal cylinder:

At $r = R_1$, $\sigma_r = 0$

At $r = R_c$, $\sigma_r = -p$ (compressive since it tends to reduce the wall thickness)

condition (b) shrinkage - external cylinder:

At $r = R_2$, $\sigma_r = 0$

At $r = R_c$, $\sigma_r = - p$

condition (c) internal pressure - compound cylinder:

At $r = R_2$, $\sigma_r = 0$

At $r = R_1$, $\sigma_r = -P_1$

Thus for each condition the hoop and radial stresses at any radius can be evaluated and the principle of superposition applied, i.e. the various stresses are then combined algebraically to produce the stresses in the compound cylinder subjected to both shrinkage and internal pressure. In practice this means that the compound cylinder is able to withstand greater internal pressures before failure occurs or, alternatively, that a thinner compound cylinder (with the associated reduction in material cost) may be used to withstand the same internal pressure as the single thick cylinder it replaces.

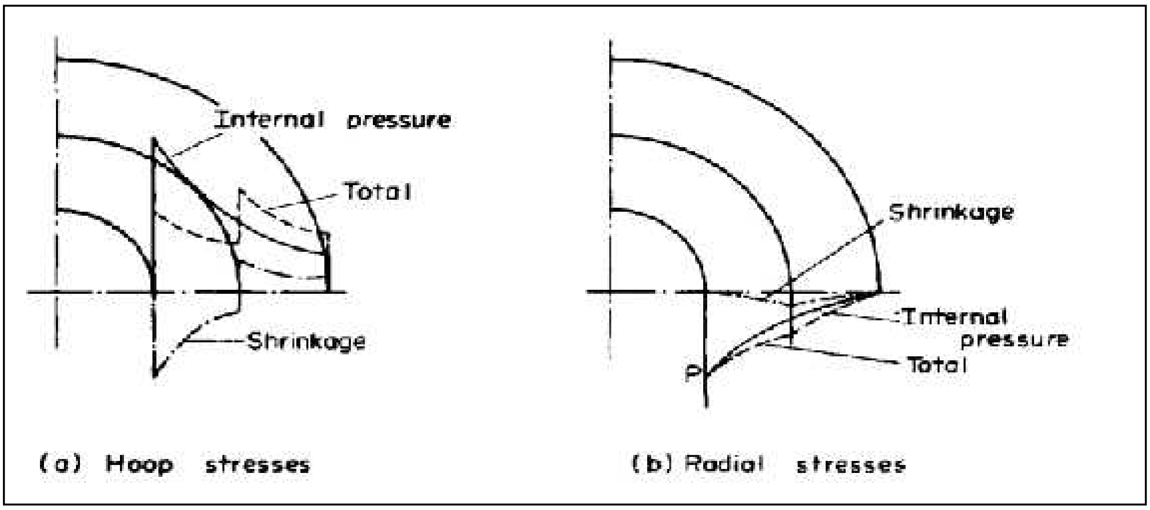
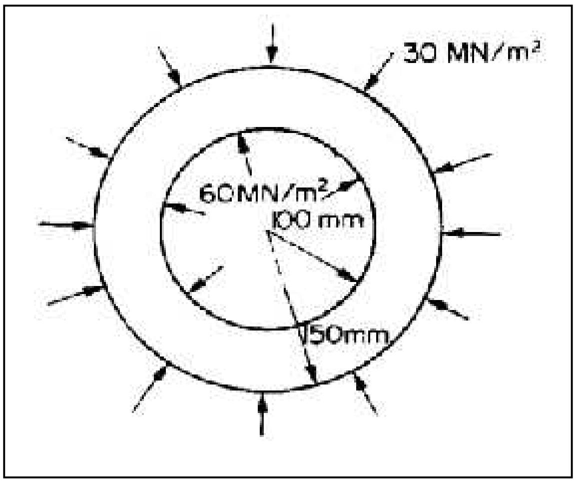


Figure 6.7: - Distribution of hoop and radial stresses through the walls of a compound cylinder.

Example 6-1: - A thick cylinder of 100 mm internal radius and 150 mm external radius is subjected to an internal pressure of 60 MN/m^2 and an external pressure of 30 MN/m^2. Determine the hoop and radial stresses at the inside and outside of the cylinder together with the longitudinal stress if the cylinder is assumed to have closed ends.



Solution: -

At r= 0.1m, σ_r=-60MPa.

r= 0.15 m, σ_r=-30 MPa.

So,

-60=A-100B ….1

-30=A-44.5B ….2

By solving equations 1 and 2,

A=-6 and B=0.54

Therefore at r=0.1m

$$\sigma_H = A + \frac{B}{r^2} = -6 + \frac{0.54}{(0.1)^2} = 48\text{MPa.}$$

At r=0.15m,

$$\sigma_H = A + \frac{B}{r^2} = -6 + \frac{0.54}{(0.15)^2} = 18\text{MPa}$$

$$\sigma_L = \frac{P_1 {R_1}^2 - P_2 {R_2}^2}{({R_2}^2 - {R_1}^2)} = \frac{60(0.1)^2 - 30(0.15)^2}{(0.15^2 - 0.1^2)} = -6\text{MPa i.e. compression.}$$

Example 6-2: - An external pressure of 10 MN/m^2 is applied to a thick cylinder of internal diameter 160 mm and external diameter 320 mm. If the maximum hoop stress permitted on the inside wall of the cylinder is limited to 30 MN/m^2, what maximum internal pressure can be applied assuming the cylinder has closed ends? What will be the change in outside diameter when this pressure is applied? E = 207 GN/m^2, v = 0.29.

Solution: -

At r =0.08m, σ_r=-P, $\frac{1}{r^2}=156$

At r = 0.16 m, σ_r= -10, $\frac{1}{r^2}=39$

And at r =0.08m, σ_H= 30MPa

- 10 = A - 39B ….(1)   30 = A + 156B ….(2)

Subtracting (1) from (2), A=-2 and B=0.205

Therefore, at r = 0.08, σ_r = - p = A -156B= -2-156*0.205 = -34MPa.

i.e. the allowable internal pressure is 34 MN/m^2.

The change in diameter is given by

$$\delta D = \frac{D}{E}(\sigma_H - \upsilon\sigma_r - \upsilon\sigma_L) \quad \dots (3)$$

But σ_r = - 10 MN/m^2, $\sigma_H = A + \frac{B}{r^2}$=-2+39*0.205=6 MN/m^2

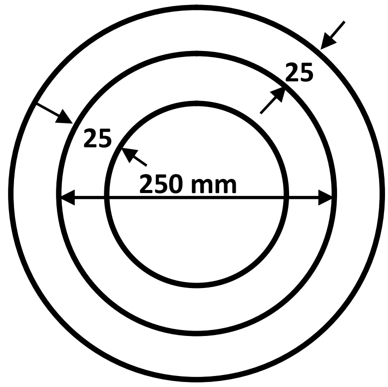
And finally, $\sigma_L = \frac{P_1 R_1{}^2 - P_2 R_2{}^2}{(R_2{}^2 - R_1{}^2)} = \frac{(34*0.08^2 - 10*0.16^2)}{(0.16^2 - 0.08^2)} =$ -1.98 MPa i.e compressive.

Substitute σ_r, σ_H and σ_L in eqn. 3,

$$\delta D = \frac{0.32}{207*10^9}[6 - 0.29(-10) - 0.29(-1.98)]10^6 = 14.7\mu m$$

Example 6-3: - A compound cylinder is formed by shrinking a tube of 250 mm internal diameter and 25 mm wall thickness onto another tube of 250 mm external diameter and 25 mm wall thickness, both tubes being made of the same material. The stress set up at the junction owing to shrinkage is 10 MN/m^2. The compound tube is then subjected to an internal pressure of 80 MN/m^2. Compare the hoop stress distribution now obtained with that of a single cylinder of 300 mm external diameter and 50 mm thickness subjected to the same internal pressure.

A solution is obtained as described before by considering the effects of shrinkage and internal pressure separately and combining the results algebraically.



Shrinkage only - outer tube,

At r = 0.15, $\sigma_r = 0$ and at r = 0.125, σ_r = -10 MN/m^2

$$0 = A - \frac{B}{(0.15^2)} = A - 44.5B$$

$$-10 = A - \frac{B}{(0.125^2)} = A - 64B$$

$$\therefore B = 0.514, \qquad A = 22.85$$

hoop stress at 0.15 m radius= $A + 44.5B = 45.7\ MP$.

hoop stress at 0.125 m radius= $A + 64B = 55.75 MP$.

Shrinkage only- inner tube,

At r = 0.10, $\sigma_r = 0$ and at r = 0.125, σ_r = - 10 MN/m^2

$$0 = A - \frac{B}{(0.1^2)} = A - 100B$$

$$-10 = A - \frac{B}{(0.125^2)} = A - 64B$$

$$\therefore B = -0.278, \qquad A = -27.8$$

hoop stress at 0.125 m radius= $A + 64B = -45.6\ MP$.

hoop stress at 0.10 m radius= $A + 100B = -55.6\ MP$.

Considering internal pressure only (on complete cylinder)

At r = 0.15, $\sigma_r = 0$ and at r = 0.10, σ_r = -80

$$0 = A - \frac{B}{(0.15^2)} = A - 44.5B$$

$$-80 = A - \frac{B}{(0.1^2)} = A - 100B$$

$$\therefore B = 1.44, \qquad A = 64.2$$

At r = 0.15 m, σ_H = A + 44.5B = 128.4 MN/m^2

r= 0.125m, σ_H = A + 64B = 156.4 MN/m^2

r= 0.1m , σ_H = A + 100B = 208.2 MN/m^2

The resultant stresses for combined shrinkage and internal pressure are then:

outer tube: r = 0.15 σ_H = 128.4 +45.7 = 174.1 MN/m^2 .

r = 0.125 $\sigma_H = 156.4 + 55.75 = 212.15\ MN/m^2$.

inner tube: r = 0.125 $\sigma_H = 156.4 - 45.6 = 110.8\ MN/m^2$.

r = 0.1 $\sigma_H = 208.2 - 55.6 = 152.6\ MN/m^2$.

…..…………………….END…………………………

Lecture No. 7

-STRAIN ENERGY-

7-1 Introduction: -

Strain energy is as the energy which is stored within a material when work has been done on the material. Here it is assumed that the material remains elastic whilst work is done on it so that all the energy is recoverable and no permanent deformation occurs due to yielding of the material,

Strain energy U = work done

Thus for a gradually applied load the work done in straining the material will be given by the shaded area under the load-extension graph of Fig.

$$U = \frac{1}{2} P \delta$$

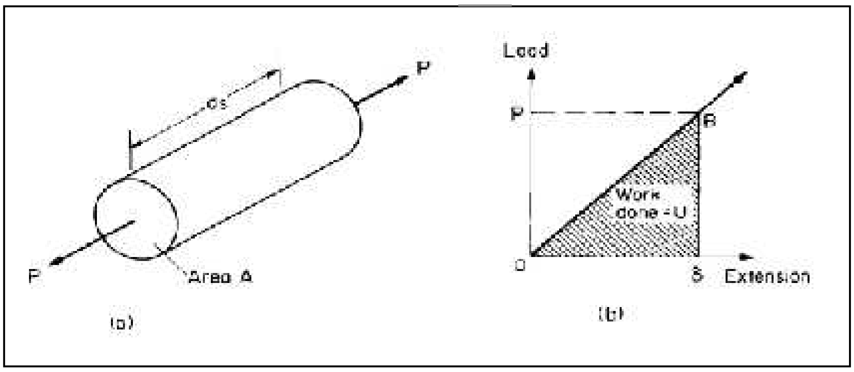


Figure 7.1: - Work done by a gradually applied load.

The unshaded area above the line OB of Fig. 7.1 is called the complementary energy, a quantity which is utilized in some advanced energy methods of solution and is not considered within the terms of reference of this text.

7-2 Strain energy - tension or compression: -

a- Neglecting the weight of the bar: -

Consider a small element of a bar, length ds, shown in Fig. 7.1. If a graph is drawn of load against elastic extension the shaded area under the graph gives the work done and hence the strain energy,

$U = \frac{1}{2} P \delta$ ….(1)

But young modulus $E = \frac{Pds}{A\delta}$ $\therefore\ \delta = \frac{Pds}{AE}$ …. (2)

Now, substituting eqn. (2) in (1)

For bar element, $U = \frac{P^2 ds}{2AE}$

∴ Total strain energy for a bar of length L, $U = \int_o^L \left(\frac{P^2 ds}{2AE}\right)$

$$\boxed{U = \frac{P^2 L}{2AE}} \quad \text{….7.1}$$

b- Including the weight of the bar: -

Consider now a bar of length L mounted vertically, as shown in Fig. 7.2. At any section A B the total load on the section will be the external load P together with the weight of the bar material below AB.

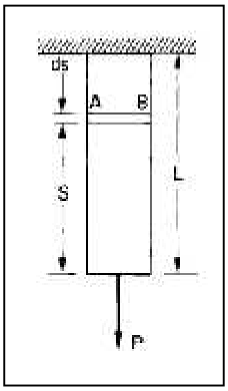


Figure 7.2: - Direct load - tension or compression.

Load on section $A\ B = P \pm \rho gAs$

The positive sign being used when P is tensile and the negative sign when P is compressive. Thus, for a tensile force P the extension of the element ds is given by the definition of Young's modulus E to be

$$\delta = \frac{\sigma ds}{E}$$

$$\delta = \frac{(P \pm \rho gAs)ds}{AE}$$

But work done $= \frac{1}{2}$ load x extension

$$= \frac{1}{2}(P \pm \rho gAs)\frac{(P \pm \rho gAs)ds}{AE}$$

$$= \frac{P^2}{2AE}ds + \frac{P\rho g}{E}sds + \frac{(\rho g)^2 A}{2E}s^2 ds$$

So, the total strain energy or work done is,

$$= \int_0^L \frac{P^2}{2AE}ds + \int_0^L \frac{P\rho g}{E}sds + \int_0^L \frac{(\rho g)^2 A}{2E}s^2 ds$$

$$\boxed{U = \frac{P^2 L}{2AE} + \frac{P\rho gL^2}{2E} + \frac{(\rho g)^2 AL^3}{6E}} \qquad \text{....7.2}$$

7-2 Strain energy-shear: -

Consider the elemental bar now subjected to a shear load Q at one end causing deformation through the angle y (the shear strain) and a shear deflection 6, as shown in Fig. 7.3.

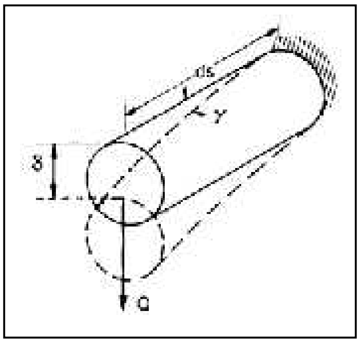


Figure 7.3: - Shear.

Strain energy U = work done = $\frac{1}{2}Q\delta = \frac{1}{2}Q\gamma ds$ …. (1)

But modulus of rigidity $G = \frac{\tau}{\gamma} = \frac{Q}{\gamma A}$

$$\gamma = \frac{Q}{AG} \qquad \text{…. (2)}$$

Substitute eqn. (2) in (1),

$$U = \frac{1}{2}Q\frac{Q}{AG}ds$$

Shear strain energy$= \frac{Q^2}{2AG}ds$

:. Total strain energy resulting from shear$= \int_0^L \frac{Q^2}{2AG}ds$

$$\boxed{U = \frac{Q^2 L}{2AG}}$$

7-3 Strain energy –bending: -

Let the element now be subjected to a constant bending moment M causing it to bend into an arc of radius R and subtending an angle *dΘ* at the center (Fig. 7.4). The beam will also have moved through an angle *dΘ*.

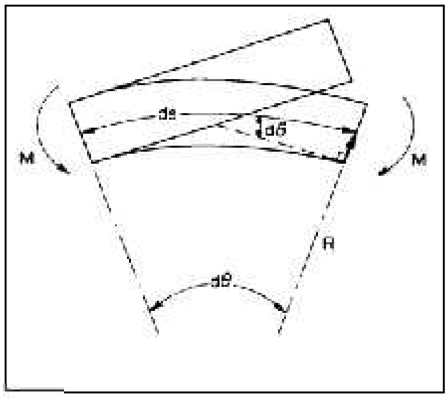


Figure 7.4: - Bending.

Strain energy = work done = $\frac{1}{2}$ moment x angle turned through (in radians)

$$=\frac{1}{2}Md\theta \qquad \text{.... (1)}$$

But $ds = Rd\theta$ and $\frac{M}{I} = \frac{E}{R}$

$$\therefore d\theta = \frac{ds}{R} = \frac{M}{EI}ds \qquad \text{.... (2)}$$

Substitute eqn. (2) in (1),

$\therefore$ Strain energy = $\frac{1}{2}M\frac{M}{EI}ds = \frac{M^2 ds}{2EI}$

Total strain energy resulting from bending, $U = \int_0^L \frac{M^2 ds}{2EI}$

$$\therefore \boxed{U = \frac{M^2 L}{2EI}}$$

7-4 Strain energy – torsion: -

The element is now considered subjected to a torque T as shown in Fig. 7.5, producing an angle of twist $d\Theta$ radians.

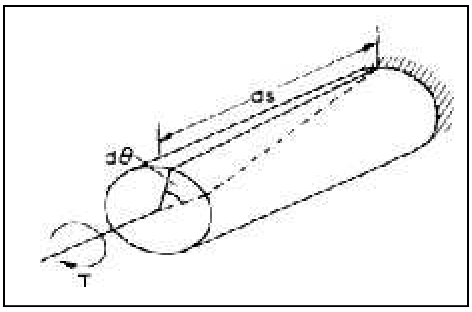


Figure 7.5: - Torsion.

Strain energy = work done = $\frac{1}{2}T\, d\Theta$ .... (1)

But, from the simple torsion theory,

$$\frac{T}{J} = \frac{Gd\theta}{ds} \text{ and } d\theta = \frac{Tds}{GJ} \qquad \text{.... (2)}$$

Substitute eqn. (2) in (1),

$\therefore$ Total strain energy resulting from torsion,

$$U = \int_0^L \frac{T^2}{2GJ} ds \rightarrow \boxed{U = \frac{T^2 L}{2GJ}}$$

Note: - It should be noted that in the four types of loading case considered above the strain energy expressions are all identical in form,

$$\boxed{U = \frac{(unit\ applied\ load)^2 \times L}{2 \times Product\ of\ two\ related\ constants}}$$

7-5 Castigliano's first theorem assumption for deflection: -

If the total strain energy of a body or framework is expressed in terms of the external loads and is partially differentiated with respect to one of the loads the result is the deflection of the point of application of that load and in the direction of that load,

i.e. $a = \frac{\delta U}{\delta P_a}$, $b = \frac{\delta U}{\delta P_b}$ and $c = \frac{\delta U}{\delta P_c}$

Where a, b and c are deflections of a beam under loads P_a, P_b and P_c etc. as shown in fig 7.6.

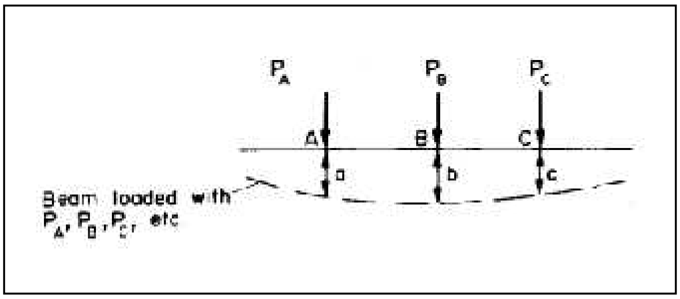


Figure 7.6: - Any beam or structure subjected to a system of applied concentrated loads P_a, P_b, P_c, . . . P_n,, etc.

In most beam applications the strain energy, and hence the deflection, resulting from end loads and shear forces are taken to be negligible in comparison with the strain energy resulting from bending (torsion not normally being present),

$$U = \int \frac{M^2 ds}{2EI}$$

$$\therefore \frac{\partial U}{\partial P} = \frac{\partial U}{\partial M}\frac{\partial M}{\partial P}$$

$$\therefore \frac{\partial U}{\partial P} = \int \frac{2M ds}{2EI}\frac{\partial M}{\partial P} \quad \rightarrow \quad \boxed{\delta = \frac{\partial U}{\partial P} = \int \frac{M}{EI}\frac{\partial M}{\partial P} ds}$$

7-6 Application of Castigliano's theorem to angular movements:

If the total strain energy, expressed in terms of the external moments, be partially differentiated with respect to one of the moments, the result is the angular deflection (in radians) of the point of application of that moment and in its direction,

$$\boxed{\theta = \int \frac{M}{EI}\frac{\partial M}{\partial M_i} ds}$$

Example 7-1: -Determine the diameter of an aluminum shaft which is designed to store the same amount of strain energy per unit volume as a 50mm diameter steel shaft of the same length. Both shafts are subjected to equal compressive axial loads. What will be the ratio of the stresses set up in the two shafts?

$E_{steel} = 200\ GN/m^2$; $E_{aluminum} = 67\ GN/m^2$.

Sol.

Strain energy per unit volume $= \frac{P^2}{2A^2E}$ but $P = \sigma A$

Strain energy per unit volume $U = \frac{\sigma^2}{2E}$

Since the strain energy/unit volume in the two shafts is equal,

$$\frac{{\sigma_A}^2}{2E_A} = \frac{{\sigma_S}^2}{2E_S}$$

$$\frac{\sigma_A{}^2}{\sigma_S{}^2} = \frac{E_A}{E_S} = \frac{67}{200} = \frac{1}{3}$$ approximately.

$$3\sigma_A{}^2 = \sigma_S{}^2 \rightarrow \frac{\sigma_A}{\sigma_S} = \sqrt{\frac{1}{3}}$$

$$3\left(\frac{P_A}{A_A}\right)^2 = \left(\frac{P_S}{A_S}\right)^2$$ but $P_s = P_A = P$ and $A = \frac{\pi}{4}D^2$

$$3\left(\frac{1}{D_A{}^2}\right)^2 = \left(\frac{1}{D_S{}^2}\right)^2 \rightarrow 3D_S{}^4 = D_A{}^4 \rightarrow D_A = \sqrt[4]{3}D_S$$

$$\therefore\ D_A = \sqrt[4]{3}(0.050) = 0.0658\,m$$

Example 7-2: - Two shafts are of the same material, length and weight. One is solid and 100 mm diameter, the other is hollow. If the hollow shaft is to store 25 % more energy than the solid shaft when transmitting torque, what must be its internal and external diameters? Assume the same maximum shear stress applies to both shafts.

Sol.

Let A be the solid shaft and B the hollow shaft. If they are the same weight and the same material their volume must be equal.

$$\frac{\pi}{4}D_A{}^2 = \frac{\pi}{4}\left(D_B{}^2 - d_B{}^2\right)$$

$$D_A{}^2 = \left(D_B{}^2 - d_B{}^2\right) = (0.100^2) = 0.1\,m^2 \qquad \text{.... (1)}$$

Now for the same maximum shear stress,

$$\tau = \frac{TD}{2J}$$

$$\frac{T_A D_A}{2J_A} = \frac{T_B D_B}{2J_B} \rightarrow \frac{T_A}{T_B} = \frac{D_B J_A}{D_A J_B} \qquad \text{.... (2)}$$

But the strain energy of B = 1.25 x strain energy of A.

$$U = \frac{T^2 L}{2GJ}$$

$$\frac{T_B{}^2 L}{2GJ_B} = 1.25\frac{T_A{}^2 L}{2GJ_A} \rightarrow \frac{T_A{}^2}{T_B{}^2} = \frac{J_A}{1.25J_B} \quad \text{.... (3)}$$

Now substitute eqn. (2) in (3),

$$\frac{D_B{}^2}{D_A{}^2} = \frac{J_B}{1.25J_A} \rightarrow \frac{D_B{}^2}{D_A{}^2} = \frac{D_B{}^4 - d_B{}^4}{1.25D_A{}^4}$$

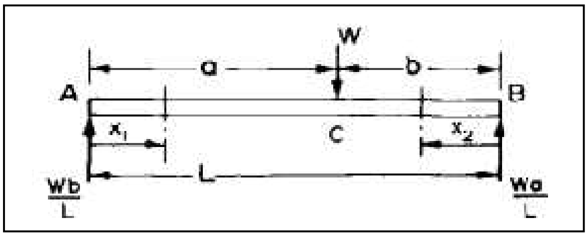
$$\therefore D_B{}^2 = \frac{D_B{}^4 - d_B{}^4}{1.25D_A{}^2} \rightarrow \frac{D_B{}^4 - (D_B{}^2 - 0.1)^2}{1.25D_A{}^2}$$

$\therefore D_B = 115.47\ mm$ Substitute in eqn. (1)

$$\therefore d_B = 57.74\ mm.$$

Example 7-3: - Using Castigliano's first theorem, obtain the expressions for (a) the deflection under a single concentrated load applied to a simply supported beam as shown in Figure below, (b) the deflection at the center of a simply supported beam carrying a uniformly distributed load.

a-



$$\delta = \int_B^A \frac{M}{EI}\frac{\partial M}{\partial W}\,ds$$

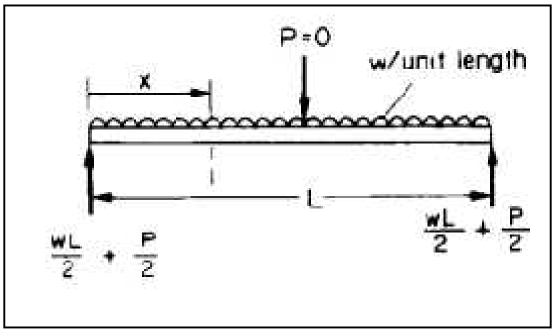
$$= \int_A^C \frac{M}{EI}\frac{\partial M}{\partial W}\,ds + \int_C^B \frac{M}{EI}\frac{\partial M}{\partial W}\,ds$$

$$= \frac{1}{EI}\int_0^a \frac{Wbx_1}{L} \times \frac{bx_1}{L} \times dx_1 + \frac{1}{EI}\int_0^b \frac{Wax_2}{L} \times \frac{ax_2}{L} \times dx_2$$

$$= \frac{Wb^2}{L^2EI}\int_0^a x_1^2\,dx_1 + \frac{Wa^2}{L^2EI}\int_0^b x_2^2\,dx_2$$

$$= \frac{Wb^2a^3}{3L^2EI} + \frac{Wa^2b^3}{3L^2EI} = \frac{Wa^2b^2}{3L^2EI}(a+b) = \frac{Wa^2b^2}{3LEI}$$

b-



$$\delta = \int_0^L \frac{Mm}{EI}\,ds = 2\int_0^{L/2} \frac{Mm}{EI}\,ds$$

$$M = \frac{wL}{2} - \frac{wx^2}{2} \quad \text{and} \quad m = \frac{x}{2}$$

$$\delta = \frac{2}{EI}\int_0^{L/2} \left(\frac{wLx}{2} - \frac{wx^2}{2}\right)\frac{x}{2}\,dx$$

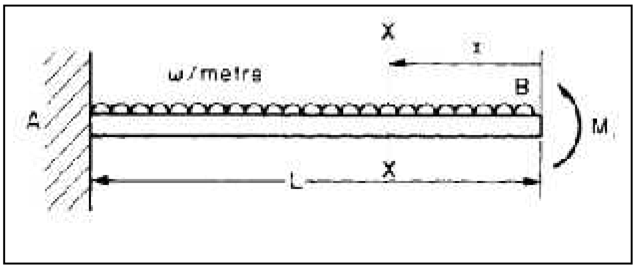
$$= \frac{1}{2EI}\int_0^{L/2} (wLx^2 - wx^3)\,dx$$

$$\delta = \frac{w}{2EI}\left[\frac{Lx^3}{3} - \frac{x^4}{4}\right]_0^{L/2}$$

$$= \frac{wL^4}{2EI}\left[\frac{1}{24} - \frac{1}{64}\right]$$

$$= \frac{wL^4}{2EI}\left[\frac{8-3}{192}\right] = \frac{5WL^4}{384EI}$$

Example 7-4: - Derive the equation for the slope at the free end of a cantilever carrying a uniformly distributed load over its full length.



Sol.

$$M = M_i - \frac{wx^2}{2}$$

$$\frac{\partial M}{\partial M_i} = 1$$

$$\theta = \int_0^L \frac{M}{EI} \cdot \frac{\partial M}{\partial M_i} \cdot dx$$

$$= \frac{1}{EI}\int_0^L \left(M_i - \frac{wx^2}{2}\right)(1)\, dx$$

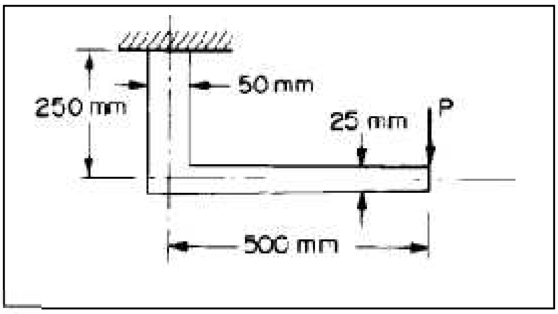
$$\theta = \frac{-w}{2EI}\int_0^L x^2 . dx = \frac{wL^3}{6EI} \textbf{ radian}$$

Example 7-5: -Determine, for the cranked member shown in Figure below:

(a) The magnitude of the force P necessary to produce a vertical movement of P of 25 mm;

(b) The angle, in degrees, by which the tip of the member diverges when the force P is applied.

The member has a uniform width of 50mm throughout. E = 200GN/m^2.



Sol. (a)

Horizontal beam: -

$$M = Px, \frac{\partial M}{\partial P} = x$$

$$\delta = \int \frac{M}{EI}\frac{\delta M}{\delta P}dx$$

$$I_h = \frac{bh^3}{12} = \frac{0.05(0.025)^3}{12} \rightarrow \frac{1}{(EI)_h} = 76.8 * 10^{-6}$$

$$\delta_{horizontal} = 76.9 * 10^{-6} \int_0^{0.5} Px^2 dx$$

$$\delta_{horizontal} = 76.9 * 10^{-6} P \left[\frac{x^3}{3}\right]_0^{0.5} = 3.2 * 10^{-6} P$$

Vertical beam: -

$$M = 0.5P, \frac{\partial M}{\partial P} = 0.5$$

$$I_v = \frac{bh^3}{12} = \frac{0.05(0.05)^3}{12} \rightarrow \frac{1}{(EI)_v} = 9.6 * 10^{-6}$$

$$\delta_{verticlal} = 9.6 * 10^{-6} \int_0^{0.25} 0.5P(0.5)dx$$

$\delta_{verticlal} = 9.6 * 10^{-6}[0.25Px]_0^{0.25} = 600 * 10^{-9}P$

$\delta_{total} = \delta_{horizontal} + \delta_{verticlal}$

$0.025 = 3.2 * 10^{-6}P + 600 * 10^{-6}P$

$P =$ 6.58kN.

Sol. (b)

$\theta = \int \frac{M}{EI}\frac{\delta M}{\delta M_i}dx$

Horizontal beam: -

$M = Px + M_i \, , \frac{\partial M}{\partial M_i} = 1$

$\theta_{horizontal} = 76.9 * 10^{-6} \int_0^{0.5}(Px + M_i)(1)dx$

$\theta_{horizontal} = 76.9 * 10^{-6}\left[\frac{Px^2}{2} + M_i x\right]_0^{0.5} = 76.9 * 10^{-6}\frac{0.5^2 P}{2}$ when $M_i = 0$

But P=6.58kN.

$\theta_{horizontal} = 63.168 * 10^{-3} rad.$

Vertical beam: -

$M = 0.5P + M_i \, , \frac{\partial M}{\partial M_i} = 1$

$\theta_{vertical} = 9.6 * 10^{-6} \int_0^{0.25}(0.5P + M_i)(1)dx$

$\theta_{vertical} = 9.6 * 10^{-6}[0.5Px + M_i x]_0^{0.25}$ but $M_i = 0$ and P=6.58kN.

$\theta_{vertical}$ =7.896*10^{-3} *rad.*

$\theta_{total} = \theta_{horizontal} + \theta_{verticlal}$

$\theta_{total} = 63.168 * 10^{-3}$ +7.896*10^{-3}=0.071 *rad*=4.1°

H.W. A semicircular frame of flexural rigidity (**EI**) is built in at **A** and carries a vertical load **W** at **B** as shown in Figure (1). Calculate the magnitude of horizontal deflection at **B**.

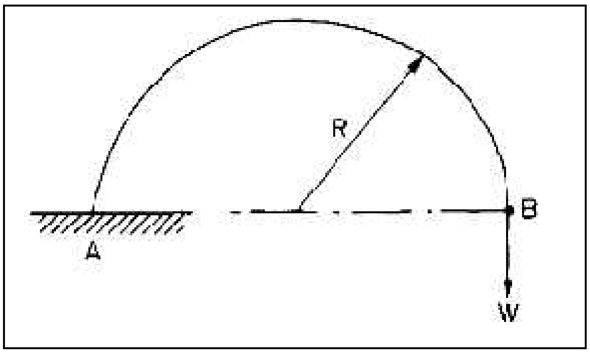


Figure (1)

..…………………….End…………………………

Lecture No. 8

-Struts-

8-1 Introduction: -

Structural members which carry compressive loads may be divided into two broad categories depending on their relative lengths and cross-sectional dimensions. Short, thick members are generally termed columns and these usually fail by crushing when the yield stress of the material in compression is exceeded. Long, slender columns or struts, however, fail by buckling some time before the yield stress in compression is reached. The buckling occurs owing to one or more of the following reasons:

(a) The strut may not be perfectly straight initially;

(b) The load may not be applied exactly along the axis of the strut;

(c) one part of the material may yield in compression more readily than others owing to some lack of uniformity in the material properties throughout the strut.

8-2 Euler's theory: -

a- <u>Strut with pinned ends: -</u>

Consider the axially loaded strut shown in Fig. 8.1 subjected to the crippling load P_e, producing a deflection y at a distance x from one end. Assume that the ends are either pin-jointed or rounded so that there is no moment at either end.

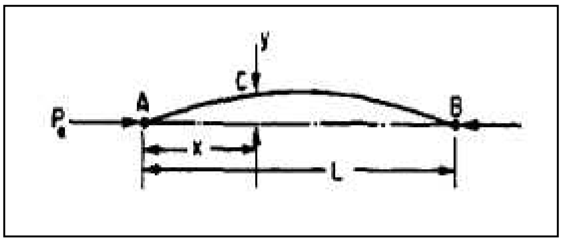


Figure 8.1:- Strut with axial load and pinned ends.

B.M. at C $= EI\frac{d^2y}{dx^2} = -P_e y$

$$EI\frac{d^2y}{dx^2} + P_e y = 0$$

$$\frac{d^2y}{dx^2} + \frac{P_e}{EI}y = 0$$

i.e. in operator form, with D = d / d x ,

$$(D^2 + n^2)y = 0 \;\; where\, n^2 = \frac{P_e}{EI}$$

This is a second-order differential equation which has a solution of the form

$$y = Acosnx + Bsinnx$$

$$y = Acos\sqrt{\frac{P_e}{EI}}x + Bsin\sqrt{\frac{P_e}{EI}}x\ ,$$

B.C.

At x=0, y=0 $\quad \therefore A = 0$

And at x=L, y=0 $\therefore BsinL\sqrt{\frac{P_e}{EI}} = 0$

If B = 0 then y = 0 and the strut has not yet buckled. Thus the solution required is,

$$\therefore sinL\sqrt{\frac{P_e}{EI}} = 0\ ,\ \therefore L\sqrt{\frac{P_e}{EI}} = \pi$$

$$\boxed{\therefore P_e = \frac{\pi^2 EI}{L^2}} \qquad \text{….8.1}$$

It should be noted that other solutions exist for the equation,

$$sinL\sqrt{\frac{P_e}{EI}} = 0\ i.e\ .sinnL = 0$$

The solution chosen of $nL = \pi$ is just one particular solution; the solutions $nL = 2\pi$, $3\pi, 5\pi$ etc. are equally as valid mathematically and they do.

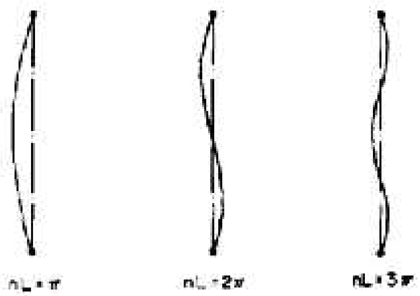


Figure 8.2:- Strut failure modes.

b- <u>One end fixed, the other free:</u> -

Consider now the strut of Fig. 8.3 with the origin at the fixed end.

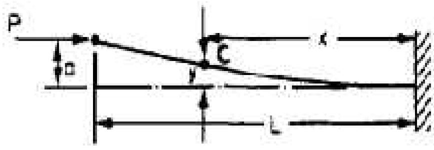


Figure 8.3: - Fixed-free strut.

B.M. at C $= EI\dfrac{d^2y}{dx^2} = +P(a-y)$

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{Pa}{EI} \qquad \text{....8.2}$$

$$(D^2+n^2)y = n^2a$$

$$y = A\cos nx + B\sin nx + (particular\ solution)$$

The particular solution is a particular value of y which satisfies eqn. (8.2), and in this case can be shown to be y = a.

$$y = A\cos nx + B\sin nx + a$$

Now at x= 0, y=0 $\qquad \therefore A = -a$

and at x=0, $\frac{dy}{dx} = 0$ $\qquad \therefore$ B =0

$$\therefore y = -a\cos nx + a$$

But when x=L, y= a

$$a=-a\cos nL+a$$

$$0=\cos nL$$

The fundamental mode of buckling in this case therefore is given when $nL=\frac{\pi}{2}$.

$$L\sqrt{\left(\frac{P}{EI}\right)}=\frac{\pi}{2}$$

$$\boxed{\therefore P_e=\frac{\pi^2 EI}{4L^2}} \quad \text{....8.3}$$

c- Fixed ends: -

Consider the strut of Fig. 8.4 with the origin at the center.

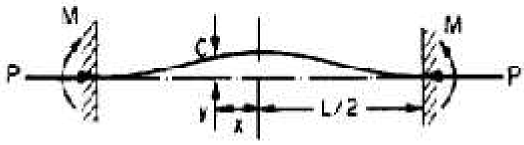


Figure 8.4: - Strut with fixed ends.

In this case the B.M. at C is given by,

B.M. at C $=EI\frac{d^2y}{dx^2}=M-Py$

$$\frac{d^2y}{dx^2}+\frac{Py}{EI}=\frac{M}{EI}$$

$$(D^2+n^2)y=\frac{M}{EI}$$

Here the particular solution is

$$y=\frac{M}{n^2EI}=\frac{M}{P}$$

$$y=A\cos nx+B\sin nx+\frac{M}{P}$$

Now when $x=0,\ \frac{dy}{dx}=0\ \therefore B=0$

And when $x=\frac{1}{2}L,\ y=0\ \therefore A=-\frac{M}{P}\sec\frac{nL}{2}$

$$y = -\frac{M}{P}\sec\frac{nL}{2}\cos nx + \frac{M}{P}$$

But when $x=\frac{1}{2}L, \frac{dy}{dx} = 0$

$$0 = \frac{nM}{P}\sec\frac{nL}{2}\sin\frac{nL}{2}$$

$$0 = \frac{nM}{P}\tan\frac{nL}{2}$$

The fundamental buckling mode is then given when $\frac{nL}{2} = \pi$

$$\frac{L}{2}\sqrt{\frac{P}{EI}} = \pi$$

$$\boxed{\therefore P_e = \frac{4\pi^2 EI}{L^2}} \qquad \text{....8.4}$$

8-3 Comparison of Euler theory with experimental results (see Fig. 8.5)

Between L / k = 40 and L / k = 100 neither the Euler results nor the yield stress are close to the experimental values, each suggesting a critical load which is in excess of that which is actually required for failure - a very unsafe situation! Other formulae have therefore been derived to attempt to obtain closer agreement between the actual failing load and the predicted value in this particular range of slenderness ratio.

(a) Straight-line formula,

$$\mathbf{P = \sigma_y A [1 - n (L / k)]}$$

the value of (**n**) depending on the material used and the end condition.

(b) Johnson parabolic formula,

$$\mathbf{P = \sigma_y A [l - b (L/k)^2]}$$

the value of (**b**) depending also on the end condition.

Neither of the above formulae proved to be very successful, and they were replaced by:

(c) Rankine-Gordon formula,

$$\boxed{\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c}}$$

where $\mathbf{P_e}$ is the Euler buckling load and $\mathbf{P_c}$ is the crushing (compressive yield) load = $\boldsymbol{\sigma_y}$ **A**. This formula has been widely used.

8-4 Euler "validity limit"

From the graph of Fig. 8.5 and the comments above, it is evident that the Euler theory is unsafe for small L/k ratios. It is useful, therefore, to determine the limiting value of L / k below which the Euler theory should not be applied; this is termed the validity limit.

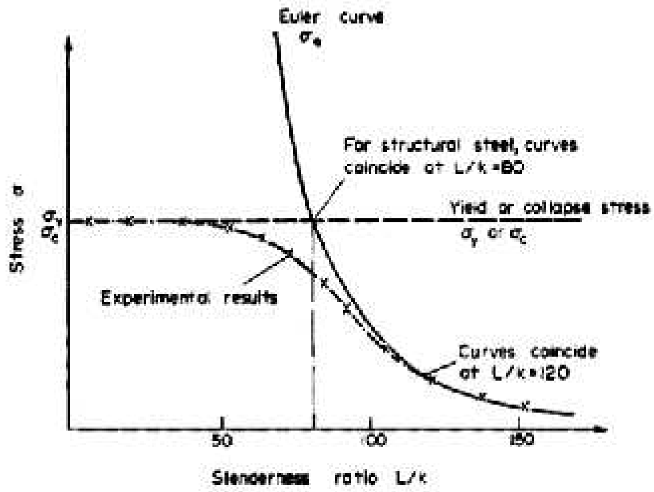


Figure 8.5: - Comparison of Experimental results with Euler curve.

The validity limit is taken to be the point where the Euler σ_e equals the yield or crushing stress σ_y, i.e. the point where the strut load

$$P = \sigma_y A$$

Now the Euler load can be written in the form,

$$P_e = C\frac{\pi^2 EI}{L^2} = C\frac{\pi^2 EAk^2}{L^2}$$

where C is a constant depending on the end condition of the strut.

Therefore in the limiting condition

$$\sigma_y A = C\frac{\pi^2 EAk^2}{L^2}$$

$$\frac{L}{k} = \sqrt{\left(\frac{C\pi^2 E}{\sigma_y}\right)}$$

The value of this expression will vary with the type of end condition; as an example, low carbon steel struts with pinned ends give L / k = 80.

8-5 Rankine or Rankine-Gordon formula,

As stated above, the Rankine formula is a combination of the Euler and crushing loads for a strut

$$\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c}$$

*For very short struts P_e is very large; $1/P_e$ can therefore be neglected and $P_R = P_c$.

*For very long struts P_e is very small and $1/P_e$ is very large so that $1/P_c$ can be neglected. Thus The Rankine formula is therefore valid for extreme values of L/k. It is also found to be fairly accurate for the intermediate values in the range under consideration. Thus, re-writing the formula in terms of stresses,

$$\frac{1}{\sigma A} = \frac{1}{\sigma_e A} + \frac{1}{\sigma_y A}$$

$$\frac{1}{\sigma} = \frac{1}{\sigma_e} + \frac{1}{\sigma_y} = \frac{\sigma_e + \sigma_y}{\sigma_e \sigma_y}$$

$$\sigma = \frac{\sigma_e \sigma_y}{\sigma_e + \sigma_y} = \frac{\sigma_y}{\left[1 + \left(\frac{\sigma_y}{\sigma_e}\right)\right]}$$

For a strut with both **ends pinned**,

$$\sigma_e = \frac{\pi^2 E}{\left(L/_k\right)^2}$$

$$\sigma = \frac{\sigma_y}{1 + \frac{\sigma_y}{\pi^2 E}\left(L/_k\right)^2}$$

Rankine stress $\sigma_R = \frac{\sigma_y}{1+a\left(L/_k\right)^2}$

where a = $\frac{\sigma_y}{\pi^2 E}$, theoretically, but having a value normally found by experiment for various materials. This will take into account other types of end condition.

Therefore Rankine load $\boxed{P_R = \frac{\sigma_y A}{1+a\left(L/_k\right)^2}}$

Example 8-1:- A circular shaft of diameter 60 mm and its length is 1.5m. Given a factor of safety of 3, a compressive yield stress of 300 MN/m^2 and a constant (a) of 1/7500, determine the allowable load which can be carried by shaft according to the Rankine-Gordon formulae.

Sol.

$$I = \frac{\pi D^4}{64} = \frac{\pi(0.060)^4}{64} = 6.361 * 10^{-7} m^4$$

$$A = \frac{\pi D^2}{4} = \frac{\pi(0.060)^2}{4} = 2.827 * 10^{-3} m^2$$

$$\therefore\ k^2 = \frac{I}{A} = \frac{6.361 * 10^{-7}}{2.827 * 10^{-3}} = 2.2497 * 10^{-4} m^2$$

According to Rankine –Gordon formula

$$P_R = \frac{\sigma_y A}{1+a\left(L/_k\right)^2} = \frac{300*10^6*2.827*10^{-3}}{1+\frac{1}{7500}\left(\frac{1.5^2}{2.2497*10^{-4}}\right)} = 363.443 KN$$

With a factor of safety of 3 the maximum permissible load therefore becomes,

$$P_{max} = \frac{363.443}{3} = 121.147 KN.$$

Example 8-2:- In an experiment an alloy rod 1 m long and of 6 mm diameter, when tested as a simply supported beam over a length of 750 mm, was found to have a maximum deflection of 5.8 mm under the action of a central load of 5 N.

a- Find the Euler buckling load when this rod is tested as a strut, pin-jointed and guided at both ends.
b- What will be the central deflection of this strut when the material reaches a yield stress of 240 MN/m^2?

Note: - Take maximum stress $= \frac{P}{A} \pm \frac{My}{I}$ $where\ M = P\delta_{max}$

Sol.

$$I = \frac{\pi D^4}{64} = \frac{\pi(0.006)^4}{64} = 6.361 * 10^{-11} m^4$$

For simply supported beam with a concentrated load W at center,

$$\delta = \frac{WL^3}{48\,EI} = \frac{5(0.75)^3}{48(6.361 * 10^{-11})E} = 0.0058 \rightarrow E = 119.0994\ GN/m^2$$

$$But\ P_e = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(119.0994 * 10^9)(6.361 * 10^{-11})}{1^2} = 74.77N$$

Take maximum stress$= 240 * 10^6 = \frac{P}{A} + \frac{My}{I} = \frac{74.77}{\pi(0.003)^2} + \frac{74.77(\delta)(0.003)}{6.361*10^{-11}}$

$$240 * 10^6 = 2644447.8 + 3.526 * 10^9 \delta$$

$$\therefore\ \delta = 0.0673\ m$$

Example 8-3: - In tests it was found that a tube 2 m long, 50 mm outside diameter and 2 mm thick when used as a pin-jointed strut failed at a load of 43 kN. In a compression test on a short length of this tube failure occurred at a load of 115 kN.

a- Determine whether the value of the critical load obtained agrees with that given by the Euler theory.
b- Find from the test results the value of the constant a in the Rankine-Gordon formula. Assume E = 200 GN/m^2.

Sol.

$$A = \frac{\pi(D^2 - d^2)}{4} = \frac{\pi(0.050^2 - 0.046^2)}{4} = 3.016 * 10^{-4} m^2$$

$$I = \frac{\pi(D^4 - d^4)}{64} = \frac{\pi(0.050^4 - 0.046^4)}{64} = 8.7 * 10^{-8} m^4$$

$$\sigma_y = \frac{P}{A} = \frac{115 * 10^3}{3.016 * 10^{-4}} = 381.308 MP \; .$$

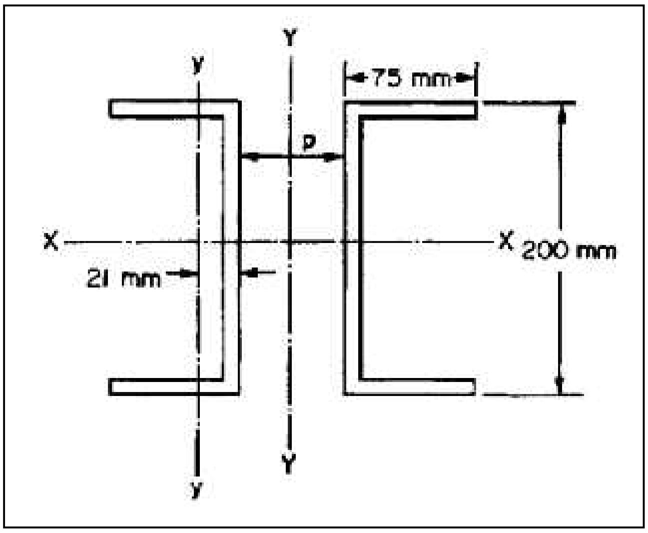
$$P_e = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 (200*10^9)(8.7*10^{-8})}{2^2} = 42.937 kN \quad \text{So, yes}$$

$$a = \frac{\sigma_y}{\pi^2 E} = \frac{381.308*10^6}{\pi^2 (200*10^9)} = \frac{1}{5176.7}$$

Example 8-4: - A stanchion is made from two 200 mm x 75 mm channels placed back to back, as shown in figure below, with suitable diagonal bracing across the flanges. For each channel I_{xx} = $20*10^{-6}$ m^4, I_{yy} = $1.5*10^{-6}$ m^4, the cross-sectional area is $3.5*10^{-3}$ m^2 and the centroid is 21 mm from the back of the web.

At what value of p will the radius of gyration of the whole cross-section be the same about the X and Y axes?

The strut is 6 m long and is pin-ended. Find the Euler load for the strut and discuss briefly the factors which cause the actual failure load of such a strut to be less than the Euler load. E = 210 GN/m^2.



Sol.

For stanchion,

I_{xx}=2($20*10^{-6}$) = $40*10^{-6}$ m^2

$I_{yy} = I_{yy} + Ad^2 = 2[1.5*10^{-6} + 3.5*10^{-3}*(0.021+0.5p)^2]$

$$I = Ak^2 \rightarrow k = \sqrt{\frac{I}{A}}$$

$k_{xx} = k_{yy}$ and $A_{xx} = A_{yy} = A$ $\therefore I_{xx} = I_{yy}$

$\therefore 40*10^{-6} = 2[1.5*10^{-6} + 3.5*10^{-3}*(0.021+0.5p)^2]$

$\therefore$ p=0.1034 m

$$P_e = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(210*10^9)(40*10^{-6})}{6^2} = 2.303\ MN$$

……………….……………………………..End………………………………………

Lecture No. 9

-Introduction to Fracture Mechanics-

9-1 Introduction.

Simple fracture is the separation of a body into two or more pieces in response to an imposed stress that is static (i.e., constant or slowly changing with time) and at temperatures that are low relative to the melting temperature of the material. The applied stress may be tensile, compressive, shear, or torsional.

The study of how materials fracture is known as ***"fracture mechanics"*** and the resistance of a material to fracture is colloquially known as its ***"toughness".***

The theory of fracture mechanics assumes the pre-existence of cracks and develops criteria for the catastrophic growth of these cracks.

9-2 Crack modes: -

In a stressed body, a crack can propagate in a combination of the three opening modes shown in figure 9.1.

I- Mode represents opening in a purely tensile load

II- Mode represents sliding in-plane shear load

III- Mode represents tearing in anti-plane shear load

Note: - The most commonly found failures are due to cracks propagating predominantly in ***mode I***, and for this reason materials are generally characterized by their resistance to fracture in that mode.

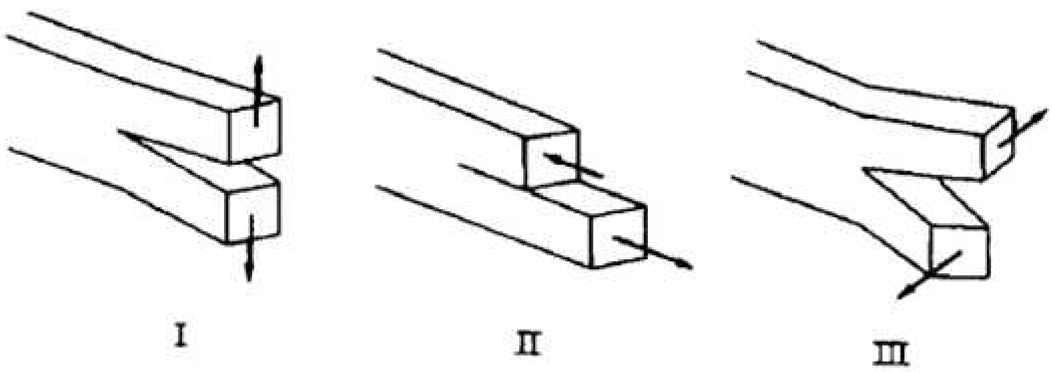


Figure 9.1: - Modes of crack growth: mode I-tensile; mode II-in-plane shear; mode III-anti-plane shear

9-3 Griffith crack theory: -

Griffith noted in 1921 that when a stressed plate of an elastic material containing a crack, the potential energy decreased and the surface energy increased. Potential energy is related to the release of stored energy and the work done by the external loads. The “surface energy” results from the presence of a crack as shown in figure 9.2.

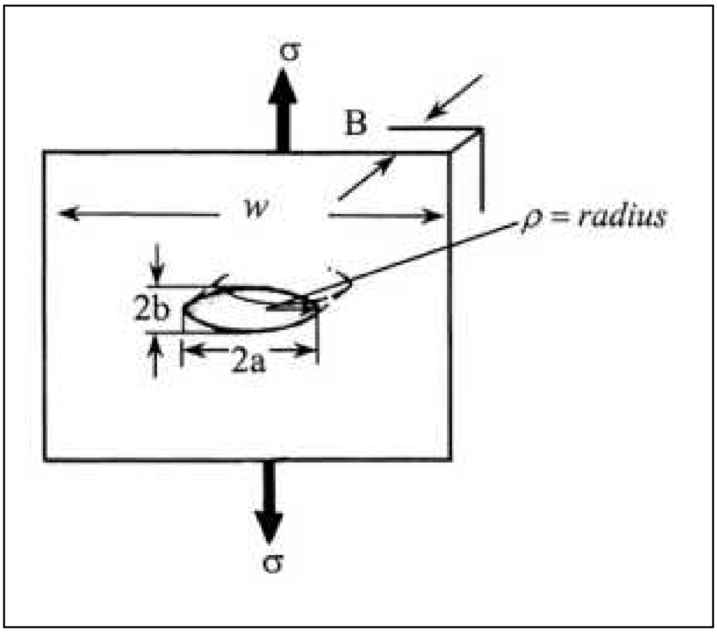


Figure 9.2: - Through thickness crack in a large plate containing an elliptical crack tip radius ρ.

The total potential energy of the system is given by

$$U = U_o - U_a + U_\gamma \qquad \ldots 9.1$$

$$U = U_o - \frac{\pi\beta a^2\sigma^2 B}{E} + 2(2aB\gamma_s) \qquad \ldots 9.2$$

where,

U = Potential energy of the cracked body.

U_o= Potential energy of uncracked body.

U_a= Elastic energy due to the presence of the crack.

U_γ= Elastic-surface energy due to the formation of crack surfaces.

a = One-half crack length.

$2(2aB)$ = Total surface crack area.

γ_s= Specific surface energy.

E = Modulus of elasticity.

σ = applied stress and υ = Poisson's ratio.

β= 1 for plane stress and $\beta = 1 - \upsilon^2$ for plane strain.

The equilibrium condition of eq. (9.2) is defined by the first order partial derive. The equilibrium condition of eq. (9.2) is defined by the first order derivative with respect to crack length. This derivative is of significance because the critical crack size may be predicted very easily. If $\frac{dU}{da} = 0$ the crack size and total surface energy are, respectively

$$a = \frac{(2\gamma_s)E}{\pi\beta\sigma^2} \qquad \ldots 9.3$$

$$(2\gamma_s) = \frac{\pi\beta a\sigma^2}{E} \qquad \ldots 9.4$$

$$G = \frac{\pi a\sigma^2}{E} \qquad \text{in plane stress.}$$

$$G = \frac{\pi a\sigma^2}{E}(1 - \upsilon^2) \qquad \text{in plane strain.}$$

where G is the strain energy release rate. While, Griffith criterion for fracture is:

Where σ_f is the fracture stress, a is half crack length at the beginning of fracture

If $G \geq G_c$, this is the criterion for which the crack will begin to propagate.

Returning to equation (9.3) and rearranging it, we will get a significant expression in linear elastic fracture mechanics (LEFM)

$$\sigma\sqrt{\pi a} = \sqrt{\frac{(2\gamma_s)E}{\beta}} \qquad \ldots 9.5$$

$$\boxed{K_I = \sigma\sqrt{\pi a}} \qquad \ldots 9.6$$

The parameter K_I is called the stress intensity factor which is the crack driving force and its critical value is a material property known as fracture toughness, which in turn, is the resistance force to crack extension.

K_{IC} is the fracture toughness, is important to recognize that fracture parameter K_{IC} has different values when measured under plane stress and plane strain.

Fracture occurs when $K_I \geq K_{IC}$.

9-4 Irwin theory: -

Griffith's criterion is an energy-based theory which ignores the actual stress distribution near the crack tip. In this respect the theory is somewhat inflexible. An alternative treatment of the elastic crack was developed by Irwin, who used a similar mathematical model to that employed by Griffith except in this case the remotely applied stress is biaxial as shown in figure 9.3. Irwin's theory obtained expressions for the stress components near the crack tip. The most elegant expression of the stress field is obtained by relating the Cartesian components of stress to polar coordinates based at the crack tip as shown in figure 9.4.

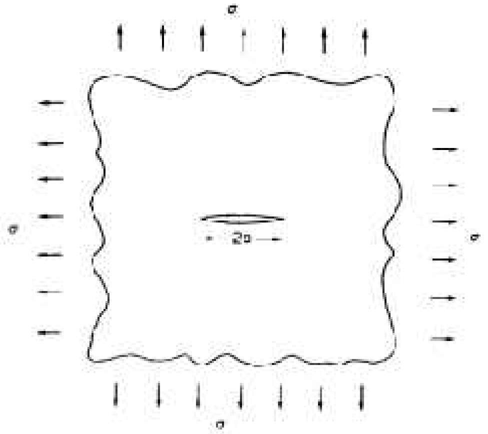


Figure 9.3: - Mathematical model for Irwin's analysis.

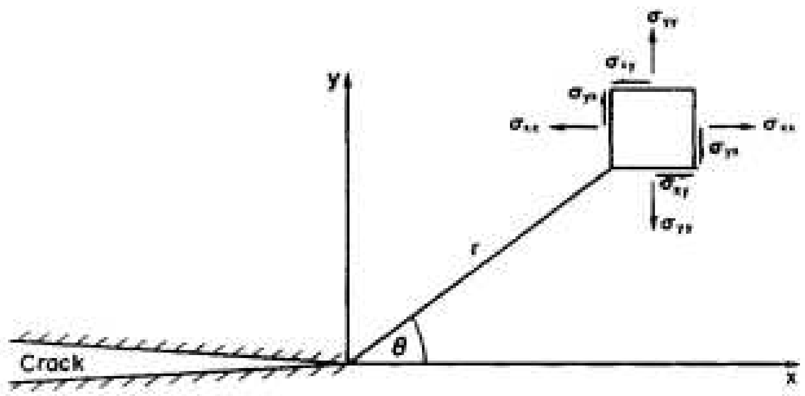


Figure 9.4: - Coordinate system for stress components in Irwin's analysis.

$$\sigma_{yy} = \frac{K}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$
$$\sigma_{xx} = \frac{K}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] \qquad \ldots 9.7$$
$$\tau_{xy} = \frac{K}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

For plane stress, $\sigma_{zz} = 0$

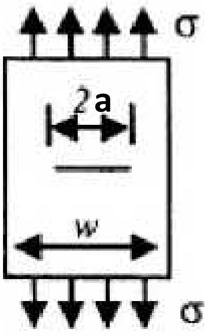
For plane strain, $\epsilon_{zz} = 0 \quad \therefore \quad \sigma_{zz} = \upsilon(\sigma_{xx} + \sigma_{yy})$

9-5 Stress intensity factor for different states: -

a- Through- thickness central crack.

$K_I = \sigma\sqrt{\pi a}$ for infinite plate.

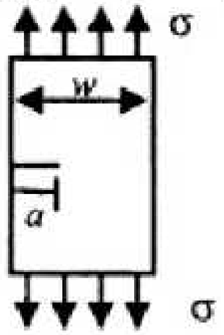
$K_I = \sigma\sqrt{\pi a}\left(\frac{W}{\pi a}\tan\frac{\pi a}{W}\right)^{\frac{1}{2}}$ for plate of finite width, W.



b- Single edge crack.

$K_I = Y\sigma\sqrt{a}$

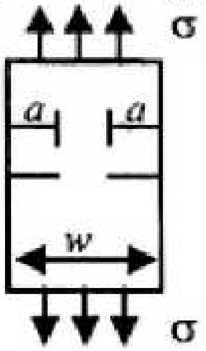
$Y = 1.99 - 0.41\left(\frac{a}{W}\right) + 18.7\left(\frac{a}{W}\right)^2 - 38.48\left(\frac{a}{W}\right)^3 + 53.85\left(\frac{a}{W}\right)^4$



c- Double edge crack.

$K_I = Y\sigma\sqrt{a}$

$Y = 1.99 + 0.76\left(\frac{a}{W}\right) - 8.48\left(\frac{a}{W}\right)^2 + 27.36\left(\frac{a}{W}\right)^3$



Example 9.1: -Steel tie in a girder bridge has a rectangular cross-section 200 mm wide and 20 mm deep. Inspection reveals that a fatigue crack has grown from the shorter edge and in a direction approximately normal to the edge. The crack has grown 23 mm across the width on one face and 25 mm across the width on the opposite face. If K_{Ic} for the material is 55 MPa . $m^{1/2}$ estimate the greatest tension that the tie can withstand. Assume that the expression for K in a SEN specimen is applicable.

Solution

a=(23+25)/2 = 24 mm

a/W = 24/200 = 0.12

from 9.5.b find Compliance function constants, then

$$Y = 1.99 - 0.41\left(\frac{a}{W}\right) + 18.7\left(\frac{a}{W}\right)^2 - 38.48\left(\frac{a}{W}\right)^3 + 53.85\left(\frac{a}{W}\right)^4$$

$$Y = 1.99 - 0.41(0.12) + 18.7(0.12)^2 - 38.48(0.12)^3 + 53.85(0.12)^4$$

$$Y = 2.154$$

At the onset of fracture K = K_{Ic}

$$K_I = Y\sigma\sqrt{a}$$

$$55 * 10^6 = 2.154\frac{P}{0.02 * 0.2}\ \sqrt{0.024}$$

Hence failure load **P** = **659.282 kN**

Example 9.2: - A thin cylinder has a diameter of 1.5 m and a wall thickness of 100 mm. The working internal pressure of the cylinder is 15 MN/m^2 and K_{IC} of the material is 38 $MPa.m^{1/2}$. Estimate the size of the largest flaw that the cylinder can contain. Assume that for this physical configuration ($K = \sigma \sqrt{\pi a}$) and that the flaw is sharp, of length 2a, and perpendicular to the hoop stress.

Solution

$$\sigma = \frac{Pd}{2t} = \frac{15 \times 1.5}{2 \times 0.1} = 112.5 \text{ MN/m}^2$$

$$K = \sigma\sqrt{\pi a}$$

$$38 = 112.5\sqrt{\pi a}$$

$$a = \mathbf{64.3 \text{ mm}}$$

H.W.

A photoelastic model shows that a fringe has a maximum distance of 2.2 mm from the crack tip. If the fringe constant is 11 N/mm^2/fringe-mm and the model thickness is 5 mm, determine the value of the stress-intensity factor under this applied load.

Ans. 1.293 MPa.m$^{1/2}$

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